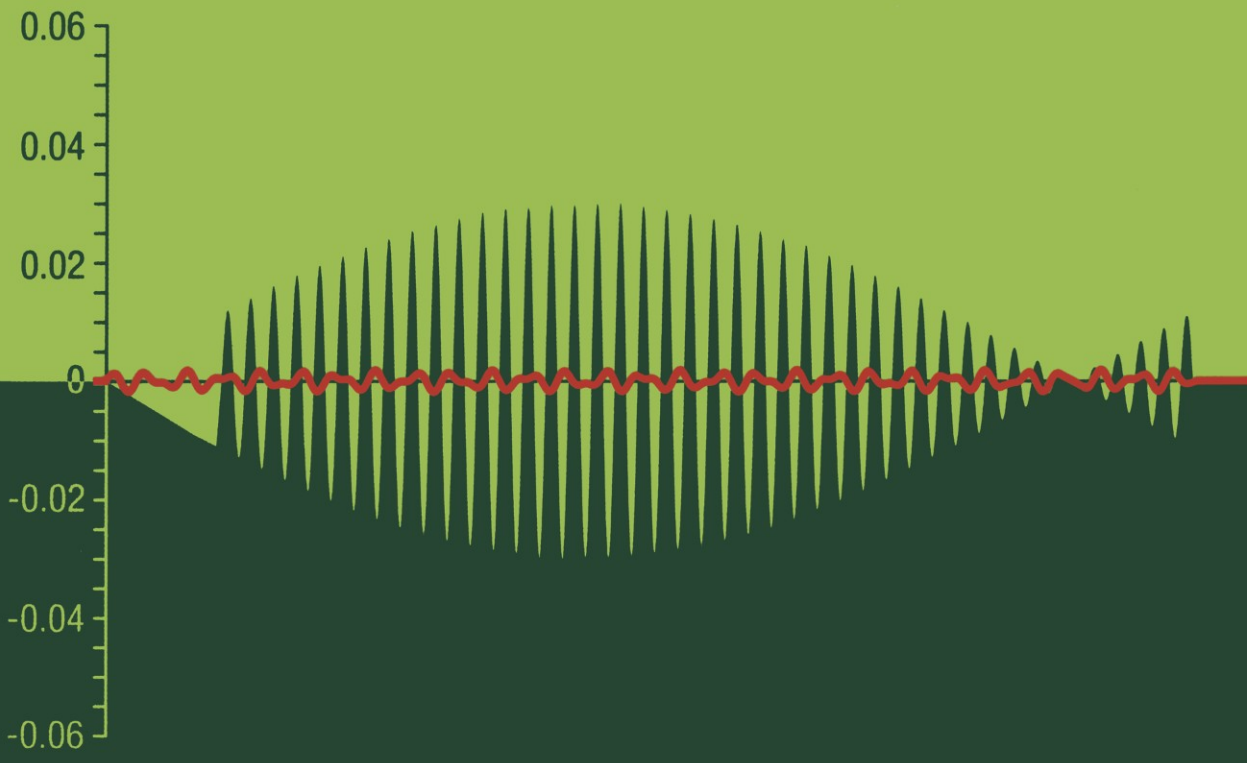
Urs Graf

Applied Laplace Transforms and z-Transforms for Scientists and Engineers

A Computational Approach using a *Mathematica* Package



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A Computational Approach using a *Mathematica* Package

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Preface

The theory of Laplace transformation is an important part of the mathematical background required for engineers, physicists and mathematicians. Laplace transformation methods provide easy and effective techniques for solving many problems arising in various fields of science and engineering, especially for solving differential equations. What the Laplace transformation does in the field of differential equations, the z-transformation achieves for difference equations. The two theories are parallel and have many analogies. Laplace and z-transformations are also referred to as operational calculus, but this notion is also used in a more restricted sense to denote the operational calculus of Mikusinski. This book does not use the operational calculus of Mikusinski, whose approach is based on abstract algebra and is not readily accessible to engineers and scientists.

The symbolic computation capability of *Mathematica* can now be used in favor of the Laplace and z-transformations. The first version of the *Mathematica* Package LaplaceAndzTransforms developed by the author appeared ten years ago. The Package computes not only Laplace and z-transforms but also includes many routines from various domains of applications. Upon loading the Package, about one hundred and fifty new commands are added to the built-in commands of *Mathematica.* The code is placed in front of the already built-in code of Laplace and z-transformations of *Mathematica* so that built-in functions not covered by the Package remain available. The Package substantially enhances the Laplace and z-transformation facilities of *Mathematica.*

The book is mainly designed for readers working in the field of applications. Thus the exposition does not emphasize sophisticated proofs. It can be used as a textbook or as a supplement to a current standard text in a formal course on Laplace transform and z-transform theory and their applications. The first two chapters are a straightforward introduction to Laplace and z-transformation without mentioning the Package. Thus readers who already know Laplace and z-transformation may omit the first two chapters and start with Chapters 3 and 4, which introduce the Package and show how to make use of it. These two chapters have the nature of a manual. The next two chapters are written in a mixed style. They introduce subjects generally considered to be on a higher level, followed by the description of the Package commands that can be helpful for these advanced subjects. The Package is not only intended to overcome cumbersome computations by hand; the aim of the project was rather to extend the scope of the method of Laplace and z-transforms to problems that can no longer be computed by hand. Chapters 8, 9, and 10 illustrate the use of the Package on problems from the fields of electricity, mechanics and control engineering. Chapter 11 introduces Duhamel's formulas, Green's functions and the finite Fourier transforms. The last Chapter 12 explains several numerical procedures for numerically computing an inverse Laplace transform. The CD in the back cover of the book contains the Package and the reference guide with all commands of the Package. When working with the Package, you may need to consult this guide, given the complex structure of many commands.

Chapter 1 introduces the ordinary or one-sided Laplace transform with its operational properties. Efros' theorem (a generalization of the convolution property) merits perhaps special mention. It is rarely exposed in an elementary introduction and may be omitted during the first reading. Some important special functions are then introduced. The inverse Laplace transform is mainly applied to rational image functions and the complex inversion formula is just mentioned. Complex variable theory is not needed. It follows an introduction of the two-sided

Laplace transform. The theory of Laplace transformation and its applications need the Dirac impulse with its principal properties. These are introduced in a heuristic manner by taking advantage of the two-sided Laplace transform. The first chapter closes with the application of the Laplace transformation on linear ordinary differential equations. The concepts of stability and of transfer functions of linear time invariant systems are also introduced. Chapter 2 parallels Chapter 1 for the z-transformation. The discrete Laplace transform of a sequence is introduced as well, and a short excursion to another transform, called the MacLaurin transform follows. Some relationships between the z-transform, the discrete Laplace transform, the Laplace transform and the MacLaurin transform are exhibited. They are followed by the application of the z-transform to solve difference and systems of difference equations. The z-transfer function and the stability of a discrete system are introduced. It is the author's intention that the first two chapters make no use of and are not related to the Package.

Chapter 3 introduces the Package LaplaceAndzTransforms. It has a manual character and is the first chapter to begin with for a reader already familiar with the theory of Laplace transforms. The material covers the various elementary subjects treated in the first chapter and shows how the Package can be used to do the computations. A novice can study it in parallel with the first chapter. Chapter 4 parallels Chapter 3, but for the z-transformation.

Chapter 5 on automatic control systems is inserted before the method of Laplace and z-transforms are extended to more advanced problems. Laplace and z-transforms are widely applied in linear control theory. The Laplace transformation is the principal mathematical tool for analog automatic linear control systems, while the z-transformation applies to sampled-data systems. It should be noted that the Laplace transformation is not only a useful tool for the more classical approach to automatic control via the concept of transfer functions, but it is as well for the more modern state-variables approach. This chapter is not intended for learning automatic control theory, but the various commands of the Package in this field are clearly explained and illustrated with examples. A control engineer will find in Chapters 1 to 5, together with Chapter 9, the relevant subjects for his or her domain.

Chapter 6 carries on with the theory of Laplace transformation and deals with some more advanced subjects. The reader is now assumed to know some concepts of complex-variables theory. The complex inversion formula is introduced and the calculus of residues is applied to compute inverse Laplace transforms. There follows the technique of translating and deforming the vertical integration path in the complex inversion integral. Asymptotics, especially in connection with Laplace transformation, deserves a better understanding than it is now the case in the mathematical education of engineers and scientists. In many cases where it is hard or impossible to find a closed form expression for an inverse Laplace transform, it is still possible to compute an asymptotic expansion for the original, valid for large (or small) values of the time t. The computations can be cumbersome when done by hand, but the Package has routines to do it. The chapter closes with an exposition of anomalous systems of ordinary differential equations. The fact that an anomalous system of linear ordinary differential equations with given initial conditions has no solution in a strictly classical sense, is still widely ignored. The computed solutions will only satisfy all given initial conditions if certain compatibility conditions are satisfied. This may be considered a detail but, unfortunately, most systems of differential equations that model real physical systems are anomalous.

Chapter 7 introduces the so-called advanced z-transform. In contrast to the ordinary z-transform, it acts on continuously varying time functions, and not on sequences as the ordinary z-transform does. The advanced z-transform is applied to sampled-data systems, when a more complete description is required. It can also be used to express in closed form the output of a stable linear system with a periodic input. Another application is to compute the solution of difference equations with a continuous variable. Next we learn how to obtain the ordinary or

the advanced z-transform from a Laplace image function. Computations involving advanced z-transforms by hand are often laborious. The Package helps to do most computations by the computer.

Chapter 8 illustrates the use of the Package by computing the various solutions of the telegrapher partial differential equation of transmission lines in electricity. Examples of asymptotic expansions of the solution can be found here. Chapter 9 displays the Package by means of several classical control problems. Chapter 10 uses the Package to compute the solution of several boundary value problems in connection with the heat conduction or diffusion equation. Samples of problems of waves and vibrations in elastic solids follow.

In Chapter 11, further techniques for solving partial differential equations with the method of Laplace transformation are presented. The formulas of Duhamel and Green are developed and the Finite Fourier Transform is introduced.

The use of Laplace transformation often comprises a hard step: the inverse Laplace transform. In fact, a large part of the theory and a still larger part of the Package code is concerned with the computation of an inverse Laplace transform. A comprehensive table of correspondences is built into the Package. For many problems, a closed form expression or an infinite series solution for the original function can be obtained. Sometimes an asymptotic expansion of the solution gives a valuable partial answer. Despite of all these analytical tools, every so often the problem arises that one finds the Laplace transform of the solution, but one cannot analytically find its inverse. Numerical methods for the computation of an inverse Laplace transform can then be helpful. This is the topic of Chapter 12. Unfortunately, this is not a so-called well-posed numerical problem. The inverse Laplace transform is an unbounded operator which gives rise to a numerically ill-conditioned problem. Nevertheless, in the last forty years several numerical algorithms were developed, which for many problems gives astonishingly good results; for others, they may fail. Chapter 12 presents several of these procedures with a short description. The intention is to give just an idea of how the algorithm works.

There is no software without bugs. The software `LaplaceAndzTransforms` on the CD delivered with the book, and which has absorbed a far bigger part of work and effort than the book itself, makes no exception. Therefore, please note the Disclaimer below. The author would appreciate it if bugs or other suggestions about the Package are conveyed to him at the e-mail address below. The internet site below gives information on future versions of the Package.

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Chapter 1
Laplace Transformation

1.1 The One-Sided Laplace Transform

■ 1.1.1 Definition and Elementary Properties

Let $f(t)$ be a real or complex-valued function defined on the positive part $\mathbb{R}_+$ of the real axis. The Laplace transform of $f(t)$ is defined as the function $F(s)$

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt, \tag{1.1}$$

provided that the integral exists. We also say, $f(t)$ is the *original function* or for short the *original* and $F(s)$ is its *image function* or its *image*. For a given image function $F(s)$, we call $f(t)$ the inverse Laplace transform of $F(s)$. It is not necessary but we prefer to extend always the domain of an original function $f(t)$ to the left by defining $f(t) = 0$ for $t < 0$.

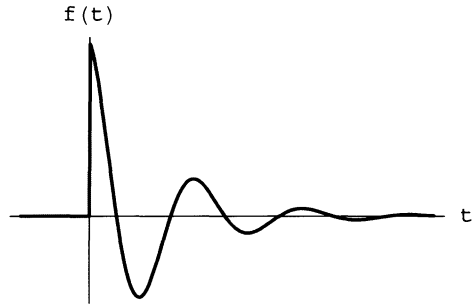


Fig.1.1: Typical original function

Thus, an original function is a complex-valued function of a real variable vanishing on the negative part of the real axis.The graph of a typical original function is shown in Fig.1.1.

A simple but important example of an original is the unit-step function or the Heaviside function $u(t)$. It is defined as

$$u(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

The shifted unit-step $u(t-a)$ is the original that vanishes for $t < a$ and takes on the value 1 for $t \geq a$. The shift a is assumed to be positive. We obtain this shifted unit-step function by translating the graph of $u(t)$ by a units to the right. Let us now compute the Laplace transform of the unit-step function:

$$\int_0^\infty e^{-st}\,1\,dt = \int_0^\infty e^{-st}\,dt = \left(\frac{e^{-st}}{-s}\right)_{t=0}^{t\to\infty} = 0 - \frac{1}{-s} = \frac{1}{s}$$

provided that $s > 0$, or that the real part $\Re s$ of s is positive, if s varies over the complex plane. A pair of an original function and its image function is called a correspondence, denoted by $f(t) \circ\!-\!\bullet\, F(s)$. Thus we have just established the correspondence

$$u(t) \circ\!-\!\bullet\, \frac{1}{s}.$$

As a second example, let us compute the Laplace transform of the shifted unit-step function $u(t-a)$:

$$\int_0^\infty e^{-st}\,u(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \left(\frac{e^{-st}}{-s}\right)_{t=a}^{t\to\infty} = 0 - \frac{e^{-sa}}{-s} = \frac{e^{-sa}}{s},$$

provided that $\Re s > 0$. Thus the correspondence is given by

$$u(t-a) \circ\!-\!\bullet\, \frac{e^{-sa}}{s}.$$

The Laplace transform of the original function $f(t) = u(t)\sin\omega t$ is

$$F(s) := \int_0^\infty e^{-st}\sin\omega t\,dt = \left(\frac{e^{-st}}{\omega^2 + s^2}(s\sin\omega t - \omega\cos\omega t)\right)_{t=0}^{t\to\infty} = \frac{\omega}{\omega^2 + s^2},$$

provided that $\Re s > 0$. We have established the correspondence

$$\sin\omega t \circ\!-\!\bullet\, \frac{\omega}{\omega^2 + s^2}.$$

For the original function $f(t) = u(t)\,e^{at}$, we find

$$F(s) := \int_0^\infty e^{-st}\,e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left(\frac{e^{-(s-a)t}}{-(s-a)}\right)_{t=0}^{t\to\infty} = \frac{1}{s-a},$$

provided that $\Re(s-a) > 0$, i.e. $\Re s > \Re a$. Thus we have the correspondence

$$e^{at} \circ\!-\!\bullet\, \frac{1}{s-a}.$$

We have made use of two conventions: First, we have not written the factor $u(t)$ on the left-hand side of the last correspondences, second, we have not restricted the corresponding image function to $\Re s > 0$ or $\Re s > \Re a$. Thus, when in the following we speak of the originals $\sin\omega t$, t, t^n, e^{at}, ... we in fact always mean $u(t)\sin\omega t$, $u(t)\,t$, $u(t)\,t^n$, $u(t)\,e^{at}$, ... but we omit the factor $u(t)$ which has the effect that the expression in consideration is replaced by 0 for negative values of t. Due to this convention the originals $u(t)$ and 1 are the same, the unit-step function. The reason for the second convention is that the domain of an image function, if it exists, is always a right half-plane of $\mathbb{C}$ of the form $\Re s > x_0$. The real number x_0 is called the *abscissa of convergence* of the Laplace integral (1.1). Thus, the first three examples above have the abscissa of convergence $x_0 = 0$, while in the fourth example $x_0 = a$. For the unilateral or one-sided Laplace transformation, we seldom have to worry about the abscissa of convergence.

For most functions encountered in applications a Laplace transform exists. We can make the following statement:

Proposition. *If the original $f(t)$ satisfies $|f(t)| < M e^{ct}$ for $t > 0$, for some real number c and some positive number M, then the abscissa of convergence is smaller than or equal to c, i.e. the image function $F(s)$ of $f(t)$ is defined on the right half-plane $\Re s > c$.*

This is seen as follows:

$$-\left|\int_0^\infty e^{-st} f(t)\,dt\right| \le F(s) := \int_0^\infty e^{-st} f(t)\,dt \le \left|\int_0^\infty e^{-st} f(t)\,dt\right| .$$

If $\Re s > c$, we can write

$$\left|\int_0^\infty e^{-st} f(t)\,dt\right| \le \int_0^\infty \left|e^{-st} f(t)\right| dt = \int_0^\infty |e^{-st}|\,|f(t)|\,dt$$

$$= \int_0^\infty e^{-\Re s\, t}\,|f(t)|\,dt < \int_0^\infty e^{-\Re s\, t} M e^{ct}\,dt$$

$$= M\int_0^\infty e^{-(\Re s - c)t}\,dt = \left(\frac{M e^{-(\Re s - c)}}{-(\Re s - c)}\right)_{t=0}^{t\to\infty} = \frac{M}{\Re s - c} . \quad \square$$

Furthermore, we see that in the right half-plane of convergence $\Re s > c$ we have the estimate

$$\left|F(s)\right| < \frac{M}{\Re s - c}$$

which implies that $F(s) \longrightarrow 0$ for $\Re s \longrightarrow \infty$.

A function satisfying the above proposition is called a function of *bounded exponential growth*. Hence functions of bounded exponential growth have a Laplace transform. Conversely, functions need not be of bounded exponential growth in order to have a Laplace transform. A simple example of a function that does not have a Laplace transform is $f(t) = e^{t^2}$.

Besides the correspondence sign $f(t) \circ\!-\!\bullet\, F(s)$, we also use the following notations:

$$F = \mathcal{L}f, \quad f = \mathcal{L}^{-1} F .$$

But more often we shall explicitly indicate the original function variable and the image function variable by writing

$$F(s) = \mathcal{L}[f(t)](s), \quad F(s) = \mathcal{L}f(s), \quad f(t) = \mathcal{L}^{-1}[F(s)](t), \quad f(t) = \mathcal{L}^{-1} F(t).$$

Problem 1.1. Compute the Laplace transforms of the given original functions and determine their abscissa of convergence.

a) t b) t^2 c) t^n (n positive integer) d) $\cos \omega t$ e) $u(t) - u(t-1)$

■ 1.1.2 Operational Properties

1.1.2.1 Linearity

The Laplace transformation is a linear operator. The sum of two original functions corresponds to the sum of their image functions. If an original function is multiplied by a constant, then the image is also multiplied by this constant. This is because the defining integral has these properties, which can be combined into one statement.

Proposition. *The Laplace transform of a linear combination of originals is the linear combination of the corresponding images:*

$$\mathcal{L}\left[\sum_{k=1}^{n} c_k f_k(t)\right](s) = \sum_{k=1}^{n} c_k \mathcal{L}[f_k(t)](s). \tag{1.2}$$

By making use of the correspondence $t^n \circ\!-\!\bullet\, n!/s^{n+1}$ it is easy to compute the image of a polynomial by use of linearity:

$$\sum_{k=0}^{n} c_k t^k \circ\!-\!\bullet \sum_{k=0}^{n} c_k \frac{k!}{s^{k+1}}.$$

Problem 1.2. Compute the image functions of

a) $\cosh a t$ b) $\sinh a t$ c) $3u(t) - \cos t + t^3 - 3e^{-2t}$

Problem 1.3. The inverse Laplace transform of a given image function is unique. Find the inverse Laplace transform of the following image functions:

$$\frac{5s-3}{s^2+4}, \quad \frac{as+b}{s^2+\omega^2}, \quad \frac{5s-3}{s^2-4}, \quad \frac{as+b}{s^2-\omega^2}.$$

1.1.2.2 The Integration Rule

If $f(t) \circ\!-\!\bullet\, F(s)$, what is the image function of $g(t) := \int_0^t f(x)\,dx$? We can write

$$G(s) := \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^\infty e^{-st} \int_0^t f(x)\,dx\,dt = \int_0^\infty f(x) \int_x^\infty e^{-st}\,dt\,dx$$

by interchanging the order of integration in the double integral (draw the 2-dimensional integration domain). If $\Re s > 0$, we can evaluate the inner integral and write

$$\int_0^\infty f(x) \int_x^\infty e^{-st}\,dt\,dx$$

$$= \int_0^\infty f(x) \frac{e^{-sx}}{s}\,dx = \frac{1}{s} \int_0^\infty f(x) e^{-sx}\,dx = \frac{F(s)}{s}.$$

Hence we have obtained the integration rule.

Proposition. $f(t) \circ\!-\!\bullet\, F(s)$ *implies*

$$\int_0^t f(x)\,dx \circ\!-\!\bullet\, \frac{1}{s} F(s) \qquad (\Re s > 0). \tag{1.3}$$

This says that integrating an original function from 0 to t amounts to dividing the image by s. By iterating this rule, we obtain

$$\int_0^t \ldots \int_0^{x_3} \int_0^{x_2} f(x_1)\,dx_1\,dx_2 \ldots dx_n \circ\!-\!\bullet\, \frac{1}{s^n} F(s).$$

Consider the following line of thought:

$$u(t) = 1 \circ\!-\!\bullet\, \frac{1}{s} \Longrightarrow \int_0^t 1\,dx = t \circ\!-\!\bullet\, \frac{1}{s^2},$$

$$t \circ\!-\!\bullet\, \frac{1}{s^2} \Longrightarrow \int_0^t x\,dx = \frac{t^2}{2} \circ\!-\!\bullet\, \frac{1}{s^3},$$

$$t^2 \circ\!-\!\bullet\, \frac{2}{s^3} \Longrightarrow \int_0^t x^2\,dx = \frac{t^3}{3} \circ\!-\!\bullet\, \frac{2}{s^4} \Longrightarrow t^3 \circ\!-\!\bullet\, \frac{3!}{s^4}.$$

In general,

$$t^n \circ\!-\!\bullet\, \frac{n!}{s^{n+1}}.$$

Problem 1.4. Use the integration rule and the result of Problem 1.3 to find the inverse Laplace transforms of the image functions

$$\frac{5s - 3}{s(s^2 + 4)}, \quad \frac{as + b}{s(s^2 + \omega^2)}.$$

1.1.2.3 Differention Rules

It can be shown that an original function $f(t)$ that is differentiable for $t > 0$, with a Laplace transformable derivative $f'(t)$, is itself Laplace transformable. How does the Laplace transform of $f'(t)$ relate to $F(s) = \mathcal{L} f(s)$? Before finding the differentiation rule, let us note the following: If $f(t)$ is assumed to be differentiable for $t > 0$, then $f(t)$ is also continuous for $t > 0$. The differentiation rule as stated below is only true if $f(t)$ is continuous for all $t > 0$. This means that the original $f(t)$ can only have a discontinuity at $t = 0$.

What is the image of $f'(t)$ if the image of $f(t)$ is $F(s)$? Suppose that the derivative of $f(t)$ has a Laplace transform, then

$$\int_0^t f'(x)\,dx = f(t) - f(0+) \circ\!-\!\bullet\, F(s) - \frac{f(0+)}{s}.$$

But the Laplace transform of the left-hand side, by the integration rule, equals $G(s)/s$, if $G(s)$ is the image of $g(t) := f'(t)$. Hence

$$\frac{G(s)}{s} \circ\!-\!\bullet\, F(s) - \frac{f(0+)}{s},$$

$$G(s) = \mathcal{L}[f'(t)](s) = sF(s) - f(0+).$$

Remark. We needed to write $f(0+)$ rather then $f(0)$, because an original function can have a discontinuity at $t = 0$. Remember that $f(t)$ vanishes for negative t, which implies that $f(0-) = 0$, while $f(0+)$ can be non-zero.

The fundamental differentiation rule given by the implication

$$f(t) \circ\!-\!\bullet\, F(s) \Longrightarrow f'(t) \circ\!-\!\bullet\, sF(s) - f(0+) \tag{1.4}$$

can be iterated, provided that for higher derivatives of $f(t)$ the Laplace transforms exist.

$$\begin{aligned} &f'(t) \circ\!-\!\bullet\, sF(s) - f(0+) \\ &\Longrightarrow f''(t) \circ\!-\!\bullet\, s[sF(s) - f(0+)] - f'(0+) \\ &= s^2 F(s) - sf(0+) - f'(0+), \end{aligned}$$

$$\begin{aligned} &f''(t) \circ\!-\!\bullet\, s^2 F(s) - sf(0+) - f'(0+) \\ &\Longrightarrow f^{(3)}(t) \circ\!-\!\bullet\, s[s^2 F(s) - sf(0+) - f'(0+)] - f''(0+) \\ &= s^3 F(s) - s^2 f(0+) - sf'(0+) - f''(0+). \end{aligned}$$

In general, we have the following result:

Proposition. *If* $f(t) \circ\!-\!\bullet\, F(s)$, *then*

$$f^{(n)}(t) \circ\!-\!\bullet\, s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0+). \tag{1.5}$$

Example. Find the solution of the initial value problem

$$\begin{aligned} &\frac{d^2 y}{dt^2} + \omega^2 y(t) = 0, \quad t > 0, \\ &y(0+) = a, \quad y'(0+) = b. \end{aligned}$$

We assume that the unknown function $y(t)$ has a Laplace transform $Y(s)$. The differentiation rule together with the linearity imply that the image equation of the differential equation is

$$s^2 Y(s) - sa - b + \omega^2 Y(s) = 0.$$

It is easy to solve this equation for the unknown Laplace transform $Y(s)$, namely

$$Y(s) = \frac{as + b}{s^2 + \omega^2} = a\frac{s}{s^2 + \omega^2} + \frac{b}{\omega}\frac{\omega}{s^2 + \omega^2}.$$

By use of linearity again, and two known correspondences, we obtain at once the solution

$$y(t) = a\cos\omega t + \frac{b}{\omega}\sin\omega t.$$

Problem 1.5. Find the solution of the initial value problem

$$\frac{d^2 y}{dt^2} - a^2 y(t) = 0, \quad t > 0,$$
$$y(0+) = A, \quad y'(0+) = B.$$

Problem 1.6. Solve the initial value problem

$$\frac{d^2 y}{dt^2} + \omega^2 y(t) = 1, \quad t > 0,$$
$$y(0+) = 0, \quad y'(0+) = 0.$$

1.1.2.4 The Image Shift Rule

Knowing the correspondence $f(t) \circ\!-\!\bullet F(s)$, one may want to find the original function corresponding to $F(s-a)$. The answer is surprisingly simple. Consider the following:

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt \Longrightarrow F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$
$$= \int_0^\infty e^{-st} \{e^{at} f(t)\}\,dt.$$

The last expression on the right-hand side is the Laplace transform of $e^{at} f(t)$. Thus, we proved the

Proposition. *For any complex number a it holds that*

$$f(t) \circ\!-\!\bullet F(s) \Longleftrightarrow e^{at} f(t) \circ\!-\!\bullet F(s-a) \ . \tag{1.6}$$

If in an image function s is replaced by $s-a$, then the corresponding original is multiplied by the exponential factor e^{at}. Clearly, we can also write

$$f(t) \circ\!-\!\bullet F(s) \Longleftrightarrow e^{-at} f(t) \circ\!-\!\bullet F(s+a).$$

This simple but important rule has many applications. For example, let us compute the original function that corresponds to

$$F(s) = \frac{as+b}{s^2 + 2\kappa s + \omega^2}.$$

By completing the square, we can write

$$\frac{as+b}{s^2 + 2\kappa s + \omega^2} = \frac{a(s+\kappa) + b - a\kappa}{(s+\kappa)^2 + \omega^2 - \kappa^2}.$$

According to whether the *discriminant* $\omega^2 - \kappa^2$ is positive, zero, or negative, we distinguish the following cases:

(i) The discriminant is positive. We set $\Omega^2 := \omega^2 - \kappa^2$ and write

$$\frac{as+b}{s^2 + 2\kappa s + \omega^2} = a\,\frac{(s+\kappa)}{(s+\kappa)^2 + \Omega^2} + \frac{b - a\kappa}{\Omega}\,\frac{\Omega}{(s+\kappa)^2 + \Omega^2}.$$

We apply the image shift rule, linearity, and the (known) correspondences

$$\frac{s}{s^2+\Omega^2} \blacksquare\!-\!\circ\, \cos \Omega t, \quad \frac{\Omega}{s^2+\Omega^2} \blacksquare\!-\!\circ\, \sin \Omega t$$

to obtain

$$\frac{as+b}{s^2+2\kappa s+\omega^2} \blacksquare\!-\!\circ\, e^{-\kappa t}\left\{a \cos \Omega t + \frac{b-a\kappa}{\Omega} \sin \Omega t\right\}$$

where $\Omega = \sqrt{\omega^2-\kappa^2}$. If $\kappa > 0$, then the above original represents a *damped harmonic oscillation.*

(ii) The discriminant is zero. Then we write

$$\frac{as+b}{s^2+2\kappa s+\omega^2} = a\frac{1}{(s+\kappa)} + (b-a\kappa)\frac{1}{(s+\kappa)^2}.$$

By applying the image shift rule again, linearity, and the correspondences

$$\frac{1}{s} \blacksquare\!-\!\circ\, 1 \ (= u(t)), \quad \frac{1}{s^2} \blacksquare\!-\!\circ\, t$$

we obtain

$$\frac{as+b}{s^2+2\kappa s+\omega^2} \blacksquare\!-\!\circ\, e^{-\kappa t}\{a + (b-a\kappa)t\}.$$

If $\kappa > 0$, then the original on the right-hand side represents a *critically damped harmonic oscillation.*

(iii) The discriminant is negative. By setting $\Omega^2 := \kappa^2 - \omega^2$, which is now positive, we obtain

$$\frac{as+b}{s^2+2\kappa s+\omega^2} = a\frac{(s+\kappa)}{(s+\kappa)^2-\Omega^2} + \frac{b-a\kappa}{\Omega}\frac{\Omega}{(s+\kappa)^2-\Omega^2}.$$

Thus

$$\frac{as+b}{s^2+2\kappa s+\omega^2} \blacksquare\!-\!\circ\, e^{-\kappa t}\left\{a \cosh \Omega t + \frac{b-a\kappa}{\Omega} \sinh \Omega t\right\}.$$

The right-hand side can also be written as

$$\frac{1}{2}\left(a+\frac{b-a\kappa}{\Omega}\right)e^{-(\kappa-\Omega)t} + \frac{1}{2}\left(a-\frac{b-a\kappa}{\Omega}\right)e^{-(\kappa+\Omega)t}$$

with $\Omega = \sqrt{\kappa^2-\omega^2}$. This is left to the reader as an exercise. In the case of $\kappa > 0$, the above original represents an *overdamped harmonic oscillation.*

Problem 1.7. Show that the following initial value problem,

$$\frac{d^2y}{dt^2} + 2\kappa\frac{dy}{dt} + \omega^2 y(t) = 0, \quad t > 0, \ \kappa > 0,$$
$$y(0+) = a,$$
$$y'(0+) = b,$$

models a damped, a critically damped, or an overdamped harmonic oscillation, according to whether the discriminant $\omega^2 - \kappa^2$ is positive, zero or negative.

1.1.2.5 Original Shift Rules

There are two rules according to whether a given original function is shifted to the right or to the left. We first consider the case where the graph of an original is shifted to the right. Remember that a given functional expression $f(t)$ defines an original function that is explicitly written as $u(t)\,f(t)$. If the graph of this original is shifted by c units to the right, where c is a positive constant, then the shifted original is represented by $u(t-c)\,f(t-c)$. What is the Laplace transform of this shifted original? Because

$$u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}, \quad u(t-c)\,f(t-c) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c, \end{cases}$$

$$u(t-c)\,f(t-c) \circ\!-\!\bullet \int_c^{\infty} e^{-st} f(t-c)\,dt$$

$$= \int_0^{\infty} e^{-s(t+c)} f(t)\,dt = e^{-cs} \int_0^{\infty} e^{-st} f(t)\,dt = e^{-cs} F(s).$$

Hence we obtain the *First Original Shift Rule:*

Proposition. *If* $c > 0$ *and* $f(t) = u(t)\,f(t) \circ\!-\!\bullet F(s)$, *then*

$$u(t-c)\,f(t-c) \circ\!-\!\bullet e^{-cs} F(s). \tag{1.7}$$

An original function that is shifted c units to the right has an image function that is multiplied by the factor e^{-cs}.

The graphs in Fig.1.2 and Fig.1.3 are meant to clarify what shifting the original by $c = 5$ units to the right means:

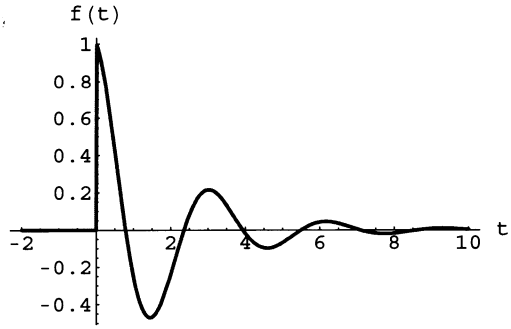


Fig.1.2: A typical original

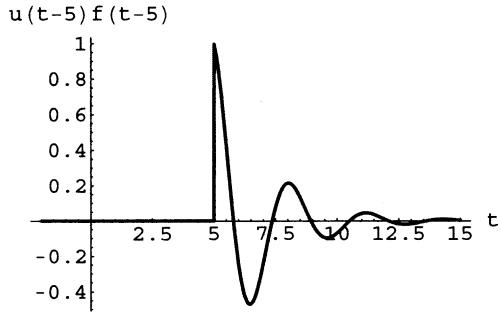


Fig.1.3: The above original shifted to right

Let us proceed to the second original shift rule. If a given original function $f(t) := u(t)\,f(t)$ is shifted by $c > 0$ units to the left, the shifted function would be represented by $u(t+c)$ times $f(t+c)$. But this, in general, is not an original function (it could have non-zero function values on the interval $[-c,\ 0]$). We need the expression $u(t)\,u(t+c)\,f(t+c) = u(t)\,f(t+c)$, which again is an original function. The last expression, when applying our convention, is simply written as $f(t+c)$. Its Laplace transform is given by

$$
\begin{aligned}
& f(t+c) \circ\!-\!\bullet \int_0^\infty e^{-st} f(t+c)\,dt \\
&= \int_c^\infty e^{-s(t-c)} f(t)\,dt = e^{cs} \int_c^\infty e^{-st} f(t)\,dt \\
&= e^{cs}\left[\int_0^\infty e^{-st} f(t)\,dt - \int_0^c e^{-st} f(t)\,dt\right] \\
&= e^{cs}\left[F(s) - \int_0^c e^{-st} f(t)\,dt\right].
\end{aligned}
$$

This is the *Second Original Shift Rule:*

Proposition. *If $c > 0$, then $f(t) \circ\!-\!\bullet F(s)$ implies*

$$
\begin{aligned}
& f(t+c) := u(t)\,f(t+c) \\
& \circ\!-\!\bullet\ e^{cs}\left[F(s) - \int_0^c e^{-st} f(t)\,dt\right].
\end{aligned}
\tag{1.8}
$$

The graph of $f(t+c) := u(t)\,f(t+c)$ is obtained by translating the graph of the original $f(t)$ by c units to the left and then truncating the part of the graph that is over the negative real axis.

Definition. The right-most expression in (1.8)

$$
F_c(s) = F(s, c) := \int_0^c e^{-st} f(t)\,dt \tag{1.9}
$$

is called the *finite Laplace transform* of $f(t)$.

Problem 1.8. Compute the Laplace transforms of

$$u(t-c)\cos(t-c) \text{ and } \cos(t-c).$$

Problem 1.9. Compute the Laplace transform of

$$u(t)\cos(t+c) = \cos(t+c)$$

in two different ways: (i) By use of the second original shift rule; (ii) by use of the identity $\cos(t+c) = \cos c \cos t - \sin c \sin t$.

1.1.2.6 The Similarity Rule

Suppose you know the correspondence $f(t) \circ\!-\!\bullet\, F(s)$. What is the image function of $f(ct)$, where c is a positive constant? We readily obtain

$$f(ct) \circ\!-\!\bullet \int_0^\infty e^{-st} f(ct)\,dt$$
$$= \frac{1}{c}\int_0^\infty e^{-(s/c)t} f(t)\,dt = \frac{1}{c}F\left(\frac{s}{c}\right)$$

by a simple change of variable. This is the similarity or scaling rule:

Proposition. *If $f(t) \circ\!-\!\bullet\, F(s)$, then, with $c > 0$,*

$$f(ct) \circ\!-\!\bullet \frac{1}{c}F\left(\frac{s}{c}\right). \tag{1.10}$$

Replacing c by $1/c$ and using the linearity property yields

$$F(s) \bullet\!-\!\circ f(t) \Longrightarrow F(cs) \bullet\!-\!\circ \frac{1}{c} f\left(\frac{t}{c}\right). \tag{1.11}$$

What is the original corresponding to $F((s-b)/c)$? By the similarity rule,

$$F\left(\frac{s}{c}\right) \bullet\!-\!\circ\, c f(ct)$$

and by the image shift rule, we finally obtain

$$F\left(\frac{s-b}{c}\right) \bullet\!-\!\circ c\, e^{bt} f(ct), \quad c > 0.$$

Similarly, we find

$$F(cs+b) \bullet\!-\!\circ \frac{1}{c} e^{-(b/c)t} f\left(\frac{t}{c}\right), \quad c > 0.$$

1.1.2.7 The t-Multiplication Rule

We know that the domain of the image function $F(s)$ of a given original $f(t)$ is always some half-plane of the form $\Re s > x_0$. The higher derivatives of $F(s)$ may be found by differentiating under the integral sign, i.e.,

$$\frac{d^n F}{ds^n} = \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt$$
$$= \int_0^\infty \frac{\partial^n}{\partial s^n} e^{-st} f(t)\, dt = \int_0^\infty e^{-st}(-t)^n f(t)\, dt.$$

On the right-hand side we have the Laplace transform of $(-t)^n f(t)$. This yields the t-multiplication rule.

Proposition. *If* $f(t) \circ\!-\!\bullet F(s)$, *then*

$$t^n f(t) \circ\!-\!\bullet (-1)^n \frac{d^n F}{ds^n}. \tag{1.12}$$

We also can speak of the image-differentiation rule:

$$F(s) \bullet\!-\!\circ f(t) \Longrightarrow F^{(n)}(s) \bullet\!-\!\circ (-1)^n t^n f(t). \tag{1.13}$$

Example.

$$\sin \omega t \circ\!-\!\bullet \frac{\omega}{s^2 + \omega^2}$$
$$\Longrightarrow t \sin \omega t \circ\!-\!\bullet -\frac{d}{ds} \frac{\omega}{s^2 + \omega^2} = \frac{2\omega s}{(s^2 + \omega^2)^2}.$$

Problems 1.10. Compute the image functions of $t \cos \omega t$, $t^2 \sin \omega t$ $t^2 \cos \omega t$, $t^n e^{at}$.

1.1.2.8 The Image Integration Rule

Multiplying an original function by t amounts to differentiating its image function and multiplying by -1. Thus, we may expect that dividing an original by t amounts to integrating the corresponding original. Under certain assumptions this is true. First, $f(t)$ being an original function does not necessarily imply that $f(t)/t$ is an original, too. The reason is that the Laplace integral of $f(t)/t$ need not converge; for example take $f(t) = u(t)/t$: the Laplace integral does not converge at its lower limit. In order to avoid this, we require that $|f(t)|/t$ is integrable on any finite interval $[0, b]$. This is satisfied if $\lim_{t \to 0+} f(t)/t$ exists, i.e. if it is finite, and for example for $f(t) = t^\alpha$, $\mathrm{R}\alpha > 0$, since $f(t)/t = t^{\alpha - 1}$ is integrable on $[0, b]$. Let us now prove the following rule:

Proposition. *If* $f(t) \circ\!-\!\bullet F(s)$ *and*

$$\int_0^b \frac{|f(t)|}{t}\, dt < \infty \quad \textit{for any } b > 0, \textit{ then}$$

$$\frac{f(t)}{t} \circ\!-\!\bullet \int_s^\infty F(p)\, dp \qquad (\mathrm{Re}\, s > 0). \tag{1.14}$$

Proof. Consider

$$\int_s^\infty F(p)\,dp = \int_s^\infty \int_0^\infty e^{-pt} f(t)\,dt\,dp$$
$$= \int_0^\infty \int_s^\infty e^{-pt} f(t)\,dp\,dt$$

where the order of integration is changed. Furthermore,

$$\int_0^\infty \int_s^\infty e^{-pt} f(t)\,dp\,dt$$
$$= \int_0^\infty f(t) \int_s^\infty e^{-pt}\,dp\,dt = \int_0^\infty f(t)\left(\frac{e^{-pt}}{-t}\right)_s^\infty dt$$
$$= \int_0^\infty f(t)\,\frac{e^{-st}}{t}\,dt = \int_0^\infty e^{-st}\,\frac{f(t)}{t}\,dt \;\square\!-\!\!-\!\square\; \frac{f(t)}{t}. \;\square$$

For example, take $f(t) = \sin t$. Clearly $f(t)/t$ exists if $t \to 0+$. Applying the above, we obtain

$$\frac{\sin t}{t} \;\square\!-\!\!-\!\square \int_s^\infty \frac{1}{p^2+1}\,dp = \int_{1/s}^0 \frac{1}{\left(\frac{1}{p}\right)^2+1}\,\frac{-1}{p^2}\,dp$$
$$= \int_0^{1/s} \frac{1}{p^2+1}\,dp = (\arctan p)_0^{1/s} = \arctan\frac{1}{s}.$$

Problems 1.11. Find the Laplace transforms of

$$\frac{1-e^{-t}}{t} \quad \text{and} \quad \frac{1-\cos t}{t}.$$

Problem 1.12. If $\sqrt{t} \;\square\!-\!\!-\!\square\; \sqrt{\pi}\,/\left(2s\sqrt{s}\right)$, find the Laplace transform of $1/\sqrt{t}$.

1.1.2.9 The Convolution of Originals and the Convolution Rule

Two original functions can be combined by forming the so-called convolution or convolution product.

Definition. Let $f(t)$ and $g(t)$ be two original functions, then *the convolution of f and g*, denoted by $f * g$, is the original defined for $t > 0$ by

$$(f*g)(t) := \int_0^t f(t-x)\,g(x)\,dx. \tag{1.15}$$

The convolution product is *commutative, distributive* and *associative*. The commutative law is seen as follows:

$$(g*f)(t) := \int_0^t g(t-x)\,f(x)\,dx = \int_t^0 g(y)\,f(t-y)\,(-dy)$$

by a change of variables $y = t - x$. The latter integral can be written as

$$\int_t^0 g(y)\, f(t-y)\, (-\mathbf{d}y)$$

$$= \int_0^t g(y)\, f(t-y)\, \mathbf{d}y = \int_0^t f(t-y)\, g(y)\, \mathbf{d}y = (f*g)(t).$$

This proves the commutative law $g*f = f*g$.

The proof of the distributive law $f*(g+h) = f*g + f*h$ is readily done and is left as an exercise to the reader.

The convolution of three originals is defined by

$$(f*g*h)(t) := \int\int_{0\le x_1+x_2\le t} f(t-x_1-x_2)\, g(x_1)\, h(x_2)\, \mathbf{d}x_1\, \mathbf{d}x_2,$$

and the convolution of four originals is

$$(f*g*h*k)(t) :=$$
$$\int\int\int_{0\le x_1+x_2+x_3\le t} f(t-x_1-x_2-x_3)\, g(x_1)\, h(x_2)\, k(x_3)\, \mathbf{d}x_1\, \mathbf{d}x_2\, \mathbf{d}x_3,$$

and so on.

Let us prove that $f*g*h = (f*g)*h$, establishing the associative law:

$$(f*g*h)(t) := \int\int_{0\le x_1+x_2\le t} f(t-x_1-x_2)\, g(x_1)\, h(x_2)\, \mathbf{d}x_1\, \mathbf{d}x_2$$

$$= \int_0^t \int_0^{t-x_2} f(t-x_2-x_1)\, g(x_1)\, \mathbf{d}x_1\, h(x_2)\, \mathbf{d}x_2$$

$$= \int_0^t (f*g)(t-x_2)\, h(x_2)\, \mathbf{d}x_2 = ((f*g)*h)(t)\,. \quad \square$$

One of the corner stones of the method of Laplace transformation is hidden in this operation of convolution, namely that the image function of the convolution of two originals is the ordinary product of the two image functions! This is stated in the

Convolution Theorem. *If* $f(t) \circ\!-\!\bullet F(s)$ *and* $g(t) \circ\!-\!\bullet G(s)$, *then*

$$(f*g)(t) := \int_0^t f(t-x)\, g(x)\, \mathbf{d}x \circ\!-\!\bullet F(s)\, G(s). \tag{1.16}$$

The convolution product of originals corresponds to the ordinary product of their images.

Consequently, whenever we encounter products of image functions, we form the convolutions of the corresponding original functions.

Proof. $F(s)\, G(s) =$

$$\int_0^\infty e^{-st} f(t)\, \mathbf{d}t \int_0^\infty e^{-st} g(t)\, \mathbf{d}t = \int\int_D e^{-s(x+y)} f(x)\, g(y)\, \mathbf{d}x\, \mathbf{d}y,$$

where the 2-dimensional integration domain D is the infinite square that coincides with the first quadrant. By a change of variables

$$(x, y) \bullet\!\!-\!\!\!\rightarrow (t, \tau);\ t = x + y,\ \tau = y \Longleftrightarrow x = t - \tau,\ y = \tau$$

with $dx\,dy = |\,\partial(x, y)/\partial(t, \tau)\,|\,dt\,d\tau = dt\,d\tau$:

$$\int\int_D e^{-s(x+y)} f(x)\,g(y)\,dx\,dy = \int\int_{D'} e^{-st} f(t-\tau)\,g(\tau)\,dt\,d\tau$$

where the domain $D = \{(x, y) \mid 0 \le x < \infty,\ 0 \le y < \infty\}$ is transformed into the domain $D' = \{(t, \tau) \mid 0 \le t < \infty,\ 0 \le \tau \le t\}$. Therefore

$$\int\int_{D'} e^{-st} f(t-\tau)\,g(\tau)\,dt\,d\tau$$
$$= \int_0^\infty e^{-st} \int_0^t f(t-\tau)\,g(\tau)\,d\tau\,dt \bullet\!\!-\!\!\circ (f * g)(t).\ \square$$

As a special case, if $f(t) = u(t) = 1$, we have the integration rule, found earlier:

$$(1 * g)(t) = \int_0^t g(x)\,dx \circ\!\!-\!\!\bullet \frac{1}{s}\,G(s).$$

As another example, we compute the general solution of the initial value problem

$$\frac{d^2 y}{dt^2} + \omega^2\,y(t) = f(t), \qquad t > 0,$$
$$y(0+) = a,$$
$$y'(0+) = b.$$

We assume that the right-hand side $f(t)$ has the Laplace transform $F(s)$. If $y(t) \circ\!\!-\!\!\bullet Y(s)$ the image equation of the given differential equation is

$$s^2\,Y(s) - s\,a - b + \omega^2\,Y(s) = F(s).$$

Solving for the unknown $Y(s)$, yields

$$Y(s) = \frac{F(s) + a\,s + b}{s^2 + \omega^2} = \frac{1}{\omega}\,\frac{\omega}{s^2 + \omega^2}\,F(s) + a\,\frac{s}{s^2 + \omega^2} + \frac{b}{\omega}\,\frac{\omega}{s^2 + \omega^2}.$$

The first term on the right-hand side is the product of two image functions, thus we apply the convolution rule to obtain

$$y(t) = \frac{1}{\omega}\sin t * f(t) + a\cos\omega t + \frac{b}{\omega}\sin\omega t$$
$$= \frac{1}{\omega}\int_0^t \sin(t-x)\,f(x)\,dx + a\cos\omega t + \frac{b}{\omega}\sin\omega t.$$

A pleasant application of the convolution theorem is the computation of the *volume of an n-dimensional sphere:*

Let $x_1, \ldots, x_n$ refer to an orthogonal cartesian coordinate system in n-dimensional space. The volume $V_n(R)$ of the n-dimensional sphere of radius R is given by

$$V_n(R) = \int\limits_{x_1^2 + \cdots + x_n^2 \le R^2} \cdots \int d x_1 \ldots d x_n \, .$$

On account of the symmetry with respect to the center of the sphere we may also write

$$V_n(R) = 2^n \int\limits_D \cdots \int d x_1 \cdots d x_n$$

where the coordinates are restricted to positive values, i.e. the integration domain D is specified by

$$x_1 \ge 0, \ \ldots, \ x_n \ge 0, \ x_1^2 + \cdots + x_n^2 \le R^2 .$$

A change of variables $x_k = \sqrt{u_k}$, $d x_k = 1/2 \sqrt{u_k}\, d u_k$, yields

$$V_n(R) = \int\limits_{0 \le u_1 + \cdots + u_n \le R^2} \cdots \int \frac{1}{\sqrt{u_1} \cdots \sqrt{u_n}} d u_1 \cdots d u_n$$

$$= \int\limits_{0 \le u_1 + \cdots + u_n \le R^2} \cdots \int u(R^2 - u_1 - \cdots - u_n) \frac{1}{\sqrt{u_1}} \cdots \frac{1}{\sqrt{u_n}} d u_1 \cdots d u_n$$

which is the $(n+1)$-fold convolution product

$$\left(u(t) * \frac{1}{\sqrt{t}} * \cdots * \frac{1}{\sqrt{t}} \right)_{t = R^2}$$

($u(t)$ denotes the unit step-function). The convolution theorem implies (also see Section 1.1.3.2)

$$V_n(R) \circ\!-\!\bullet \ \frac{1}{s} \frac{\sqrt{\pi}}{\sqrt{s}} \cdots \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\pi^{n/2}}{s^{n/2+1}} .$$

By taking the inverse Laplace transform, we obtain

$$V_n(R) = \mathcal{L}^{-1}\left[\frac{\pi^{n/2}}{s^{n/2+1}} \right](R^2)$$

$$= \pi^{n/2} \left(\frac{t^{n/2}}{\Gamma(n/2+1)} \right)_{t = R^2} = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n .$$

Hence, in one, two, three, and four dimensions:

$$V_1(R) = \frac{\sqrt{\pi}}{\sqrt{\pi}/2} R = 2R, \ V_2(R) = \pi R^2,$$

$$V_3(R) = \frac{\pi^{3/2}}{3\sqrt{\pi}/4} R^3 = \frac{4\pi}{3} R^3, \ V_4(R) = \frac{\pi^2}{2} R^4 .$$

Observe that $\partial V_n(R) / \partial R$ yields the area of an n-dimensional sphere.

1.1.2.10 Efros' Theorem

Efros' theorem is a generalization of the convolution theorem that paves the way for many interesting new correspondences. In order to establish this important theorem, we assume $f(t) \circ\!-\!\bullet\ F(s)$ and that also a correspondence of the type

$$G(s)\, e^{-q(s)\tau} \bullet\!-\!\circ\ g(t, \tau)$$

holds, where the Laplace transform is taken with respect to t and τ is a parameter. From $f(t)$ and $g(t, \tau)$ we form the new original

$$h(t) = \int_0^\infty g(t, \tau)\, f(\tau)\, d\tau$$

and compute its image

$$\begin{aligned} H(s) &:= \int_0^\infty e^{-st}\, h(t)\, dt \\ &= \int_0^\infty e^{-st} \int_0^\infty g(t, \tau)\, f(\tau)\, d\tau\, dt \\ &= \int_0^\infty \int_0^\infty e^{-st}\, g(t, \tau)\, f(\tau)\, d\tau\, dt. \end{aligned}$$

Interchanging the order of integration yields

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-st}\, g(t, \tau)\, f(\tau)\, dt\, d\tau \\ &= \int_0^\infty f(\tau) \left\{ \int_0^\infty e^{-st}\, g(t, \tau)\, dt \right\} d\tau \\ &= \int_0^\infty f(\tau)\, G(s)\, e^{-q(s)\tau}\, d\tau \\ &= G(s) \int_0^\infty f(\tau)\, e^{-q(s)\tau}\, d\tau \ = \ G(s)\, F[q(s)]\ . \end{aligned}$$

This results in $H(s) = F[q(s)]\, G(s)$. We have established

Efros' Theorem. *Let* $F(s) \bullet\!-\!\circ\ f(t)$ *and* $G(s)\, \mathrm{Exp}[-q(s)\tau] \bullet\!-\!\circ\ g(t, \tau)$. *Then*

$$F[q(s)]\, G(s) \bullet\!-\!\circ \int_0^\infty g(t, \tau)\, f(\tau)\, d\tau\,. \tag{1.17}$$

For $q(s) = s$, and by the original translation rule, we obtain

$$G(s)\, e^{-q(s)\tau} = G(s)\, e^{-s\tau} \bullet\!-\!\circ\ u(t-\tau)\, g(t-\tau) = g(t, \tau),$$

where $g(t)$ is the original function of $G(s)$. In this special case, Efros' theorem yields

$$F[q(s)]\,G(s) = F(s)\,G(s) \;\blacksquare\!-\!\square\; \int_0^\infty u(t-\tau)\,g(t-\tau)\,f(\tau)\,d\tau$$

$$= \int_0^t g(t-\tau)\,f(\tau)\,d\tau = (g*f)(t) = (f*g)(t)$$

which is the convolution theorem from before.

From Efros' theorem we will obtain interesting and useful special cases. By making use of the correspondence

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}} \;\square\!-\!\blacksquare\; \frac{1}{\sqrt{\pi t}}\,e^{-a^2/(4t)}$$

in Efros' theorem, we have $G(s) = 1/\sqrt{s}$, $q(s) = \sqrt{s}$, and $\tau = a$. Thus

$$g(t,\tau) = \frac{1}{\sqrt{\pi t}}\,e^{-\tau^2/(4t)}$$

and finally, if $F(s) \;\blacksquare\!-\!\square\; f(t)$,

$$\frac{F(\sqrt{s})}{\sqrt{s}} \;\blacksquare\!-\!\square\; \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) f(\tau)\,d\tau. \tag{1.18}$$

By taking $q(s) = s + 1/s$ and $G(s) = 1/s$ we have first

$$G(s)\,e^{-q(s)\tau} = \frac{1}{s}\,e^{-(s+1/s)\tau} = e^{-\tau s}\,\frac{e^{-\tau/s}}{s}.$$

Then by use of the correspondence

$$\frac{e^{-\tau/s}}{s} \;\blacksquare\!-\!\square\; J_0\left(2\sqrt{\tau t}\right), \tag{1.19}$$

where $J_0(t)$ is the Bessel Function of order zero, we conclude, by use of the original shift rule, that $g(t,\tau) = u(t-\tau)\,J_0\left(2\sqrt{\tau(t-\tau)}\right)$. Therefore, by Efros' theorem, if $F(s) \;\blacksquare\!-\!\square\; f(t)$,

$$\frac{F(s+\frac{1}{s})}{s} \;\blacksquare\!-\!\square\; \int_0^t J_0\left(2\sqrt{\tau(t-\tau)}\right) f(\tau)\,d\tau. \tag{1.20}$$

Example. When $F(s) = 1/\sqrt{s} \;\blacksquare\!-\!\square\; 1/\sqrt{\pi t} = f(t)$, we find

$$\frac{1}{s\sqrt{s+1/s}} \;\blacksquare\!-\!\square\; \frac{1}{\sqrt{\pi}} \int_0^t \frac{J_0\left[2\sqrt{\tau(t-\tau)}\right]}{\sqrt{\tau}}\,d\tau.$$

The left-hand side equals $1/\left(\sqrt{s}\,\sqrt{s^2+1}\right)$ which, by the convolution theorem, gives

$$\frac{1}{\sqrt{s}}\,\frac{1}{\sqrt{s^2+1}} \;\blacksquare\!-\!\square\; \frac{1}{\sqrt{\pi}} \int_0^t \frac{J_0(t-\tau)}{\sqrt{\tau}}\,d\tau$$

where the correspondence $1/\sqrt{s^2+1}$ ■ – ▫ $J_0(t)$ is used. We established, rather suprisingly, that

$$\int_0^t \frac{J_0\left[2\sqrt{\tau(t-\tau)}\right]}{\sqrt{\tau}}\,d\tau = \int_0^t \frac{J_0(t-\tau)}{\sqrt{\tau}}\,d\tau.$$

If $q(s) = 1/s$ and $G(s) = 1/s$, then correspondence (1.19) yields $g(t, \tau) = J_0\left(2\sqrt{\tau t}\right)$ and Efros' theorem implies, if $F(s)$ ■ – ▫ $f(t)$,

$$\frac{1}{s}F\left(\frac{1}{s}\right) ■ – ▫ \int_0^\infty J_0\left(2\sqrt{\tau t}\right) f(\tau)\,d\tau. \tag{1.21}$$

If $q(s) = \ln s$, $G(s) = 1/s$, then

$$G(s)\,e^{-q(s)\tau} = \frac{1}{s^{\tau+1}} ■ – ▫ g(t, \tau) = \frac{t^\tau}{\Gamma(\tau+1)}.$$

Therefore, by Efros' theorem, if $F(s)$ ■ – ▫ $f(t)$,

$$\frac{F(\ln s)}{s} ■ – ▫ \int_0^\infty \frac{t^\tau}{\Gamma(\tau+1)} f(\tau)\,d\tau. \tag{1.22}$$

Example. By use of $F(s) = 1/s$ ■ – ▫ 1 we find

$$\frac{1}{s\ln s} ■ – ▫ \int_0^\infty \frac{t^\tau}{\Gamma(\tau+1)}\,d\tau =: \nu(t);$$

$\nu(t)$ is called a function of the Volterra type.

1.1.2.11 The Initial and Final Value Theorem

One of the advantages of using the Laplace transformation to solve differential equations is that the asymptotic behavior of the image function for $s \to \infty$ and for $s \to 0$ is related to the asymptotic behavior of its original function for $t \to 0+$ and for $t \to \infty$, respectively. The asymptotic relations between an image function and its original is important, but not on an elementary level and will be treated in Section 6.2. Here we shall introduce two simple theorems which are basic instruments in the method of Laplace transformation.

The Initial Value Theorem. *If the limit* $\lim_{t\to 0+} f(t) = f(0+)$ *exists, it can be computed by*

$$f(0+) = \lim_{s\to\infty} s\,F(s), \tag{1.23}$$

where $F(s)$ *is the Laplace transform of* $f(t)$.

The simple fact that $s\,F(s)$ has a limit, when s tends to infinity, does not imply that $f(0+)$ exists. This is readily seen in the following correspondence:

$$\frac{1}{\sqrt{\pi t}}\cos\left(\frac{1}{t}\right) ▫ – ■ \frac{1}{\sqrt{s}}\,e^{-\sqrt{2s}}\cos\sqrt{2s}.$$

A simple proof for the case that $f'(t)$ has an absolutely convergent Laplace integral, which means that there exists an x_0 such that

$$\int_0^\infty e^{-x_0 t} \, | f'(t) | \, dt < \infty ,$$

is as follows: By the differentiation rule, we obtain

$$\int_0^\infty e^{-st} f'(t) \, dt = sF(s) - f(0+).$$

The above assumption allows interchanging the limit and the integral

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t) \, dt = \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t) \, dt$$
$$= \int_0^\infty 0 \, f'(t) \, ds = 0 = \lim_{s\to\infty} sF(s) - f(0+)$$
$$\Longrightarrow \lim_{s\to\infty} sF(s) = f(0+) . \; \square$$

The Final Value Theorem. *If the two limits below exist, then they are equal:*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{1.24}$$

Under the same assumption as for the Initial Value Theorem, the following steps are allowed:

$$\lim_{s\to 0} sF(s) - f(0+) = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t) \, dt = \lim_{s\to 0} \lim_{b\to\infty} \int_0^b e^{-st} f'(t) \, dt$$
$$= \lim_{b\to\infty} \int_0^b \lim_{s\to 0} e^{-st} f'(t) \, dt = \lim_{b\to\infty} \int_0^b f'(t) \, dt = \lim_{b\to\infty} \{ f(b) - f(0+) \}$$
$$= \lim_{t\to\infty} f(t) - f(0+) \Longrightarrow \lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) . \; \square$$

Again, note that if one or both of the limits do not exist, the theorem does not hold. As a counterexample, take $f(t) = \sin t$.

1.1.2.12 The Expansion Theorem of the First Kind

This theorem can be used to transform back a series of inverse powers of s. We assume that $F(z)$ is a holomorphic function at $z = 0$ with $F(0) = 0$. This means that for some $R > 0$ the convergent power series

$$F(z) = \sum_{k=1}^{\infty} c_k z^k, \quad | z | < R$$

holds. Due to Cauchy's inequality (a result from complex function theory), we have the estimate

$$| c_k | < \frac{M}{R^k}.$$

Let us consider the power series

$$f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1}.$$

We have the estimate

$$|f(t)| \leq \sum_{k=1}^{\infty} \frac{|c_k|}{(k-1)!} |t|^{k-1} < \sum_{k=1}^{\infty} \frac{M}{(k-1)!\,R^k} |t|^{k-1}$$

$$= \frac{M}{R} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\frac{|t|}{R}\right)^{k-1} = \frac{M}{R} \exp\left(\frac{|t|}{R}\right) = \frac{M}{R} \exp\left(\frac{t}{R}\right)$$

since t is positive. This means that the power series for $f(t)$ converges for all complex t and moreover that $f(t)$ is of bounded exponential growth for $t \to \infty$. Thus $u(t) f(t)$ is an original function. The above estimate allows a term by term integration:

$$f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1}$$

$$\Longrightarrow e^{-st} f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} e^{-st} t^{k-1}$$

$$\Longrightarrow \int_0^{\infty} e^{-st} f(t)\, dt = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} \int_0^{\infty} e^{-st} t^{k-1}\, dt$$

$$= \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} \frac{(k-1)!}{s^k} = \sum_{k=1}^{\infty} \frac{c_k}{s^k} = F(s).$$

We have thus proved the

Expansion Theorem of the First Kind. *If $F(s)$ is holomorphic at $s = \infty$ with $\lim_{s\to\infty} F(s) = 0$ and has the series representation*

$$F(s) = \sum_{k=1}^{\infty} \frac{c_k}{s^k}, \qquad |s| > 1/R,$$

then the entire function

$$f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1},$$

multiplied by $u(t)$, is the inverse Laplace transform of $F(s)$.

Example.

$$F(s) = \frac{1}{s^{n+1}} e^{-1/s} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s^{k+n+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k+n}}{(k+n)!}.$$

Since the Bessel function $J_n(z)$ is defined as

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\frac{z}{2})^{2k+n}}{(k+n)!},$$

we have

$$t^{n/2} J_n(2\sqrt{t}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(t^{1/2})^{2k+2n}}{(k+n)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k+n}}{(k+n)!}.$$

Finally,

$$t^{n/2} J_n(2\sqrt{t}) \square - \blacksquare \frac{1}{s^{n+1}} e^{-1/s}.$$

Problems 1.13. Assuming $F(s) \blacksquare - \square f(t)$ use the correspondence

$$e^{-a\sqrt{s}} \blacksquare - \square \frac{a}{2t\sqrt{\pi t}} \exp\left(-\frac{a^2}{4t}\right), \quad a > 0$$

to establish

$$F(\sqrt{s}) \blacksquare - \square \frac{1}{2t\sqrt{\pi t}} \int_0^{\infty} \tau \exp\left(-\frac{\tau^2}{4t}\right) f(\tau)\, d\tau.$$

Problem 1.14. Use the result of Problem 1.11 to show

$$F(\sqrt{s}) \blacksquare - \square \frac{f(0+)}{\sqrt{\pi t}} + \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \exp\left(-\frac{\tau^2}{4t}\right) f'(\tau)\, d\tau.$$

Problem 1.15. Establish

$$\int_0^{\infty} \exp[-(at^2 + 2bt + c)]\, dt$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - ac}{a}\right) \operatorname{erfc}\left(\frac{b}{\sqrt{a}}\right), \quad a > 0$$

where $\operatorname{erfc}(x) := 2/\sqrt{\pi} \int_x^{\infty} e^{-t^2}\, dt$ is the Complementary Error Function.

Problem 1.16. Use the formula of Problem 13 and the formulas derived from Efros' theorem to establish the following correspondences:

$$\frac{1}{\sqrt{s}\,(\sqrt{s}+b)} \blacksquare - \square e^{b^2 t} \operatorname{erfc}(b\sqrt{t}),$$

$$\frac{1}{\sqrt{s}+b} \blacksquare - \square \frac{1}{\sqrt{\pi t}} - b\, e^{b^2 t} \operatorname{erfc}(b\sqrt{t}).$$

Problem 1.17. For $t > 0$, prove

$$\cos t = \int_0^{\infty} J_0(2\sqrt{\tau t}) \sin\tau\, d\tau, \qquad \sin t = \int_0^{\infty} J_0(2\sqrt{\tau t}) \cos\tau\, d\tau.$$

■ 1.1.3 Some Special Functions and their Laplace Transforms

1.1.3.1 Periodic Original Functions

An original function $f(t)$ (remember $f(t) = 0$ for $t < 0$) is *periodic* if there exists a positive constant T such that $f(t+T) = f(t)\ \forall\, t > 0$. The smallest positive constant T that satisfies the above relation is called the fundamental period of $f(t)$. Observe that this defining relation immediately implies

$$f(t+T) = f(t) = f(t + kT)$$

for all positive t and any positive integer k.

The Laplace transform of a periodic original has a remarkable form:

$$f(t) = f(t+T) \circ\!-\!\bullet F(s) := \int_0^\infty e^{-st} f(t)\,dt$$

$$= \sum_{k=0}^{\infty} \int_k^{kT} e^{-st} f(t)\,dt = \sum_{k=0}^{\infty} \int_0^T e^{-s(t+kT)} f(t+kT)\,dt$$

$$= \sum_{k=0}^{\infty} e^{-skT} \int_0^T e^{-st} f(t+kT)\,dt = \sum_{k=0}^{\infty} e^{-skT} \int_0^T e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt \sum_{k=0}^{\infty} e^{-skT}.$$

Adding up the infinite geometric series yields

$$\sum_{k=0}^{\infty} e^{-skT} = \sum_{k=0}^{\infty} (e^{-sT})^k = \frac{1}{1 - e^{-sT}} \quad (\Re s > 0).$$

Thus we have established the general expression for the Laplace transform of a periodic original.

Proposition. *The Laplace transform $F(s)$ of a periodic original $f(t) = f(t+T)$ is given by*

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt = \frac{F_T(s)}{1 - e^{-sT}}. \tag{1.25}$$

Without proof we state the fact that a finite Laplace transform

$$F_T(s) := \int_0^T e^{-st} f(t)\,dt$$

is always an entire holomorphic function (a function of the complex variable s without singularities in the finite complex plane). Hence the only singularities of $F(s)$ are the zeros of the denominator which are all simple and given by $s_k = 2\pi k i / T$ with k running through all integers. Hence we see that the Laplace transform of a periodic original is a meromorphic

function whose poles are all simple and evenly spaced along the imaginary axis. A meromorphic function is a function of a complex variable whose only singularities are poles.

A periodic function often has a symmetry. Periodic functions may be even or odd functions or having a symmetry of the third kind.

Definitions. A periodic original $f(t)$ with period T is *even (odd)* if the two relations

$$(i)\ f(t+T) = f(t) \quad \forall\, t > 0,$$
$$(ii)\ f(t) = f(T-t) \quad (f(t) = -f(T-t), \quad 0 \le t \le T)$$

hold. A periodic original $f(t)$ with period T has a *symmetry of the third kind* if the three relations

$$(i)\ f(t+T) = f(t) \quad \forall\, t > 0,$$
$$(ii)\ f(t) = f(T/2 - t), \quad 0 \le t \le T/2,$$
$$(iii)\ f(t) = -f(T-t), \quad 0 \le t \le T$$

hold. A periodic original with a symmetry of the third kind is a special case of an odd periodic original.

The graphs in Figs. 1.4, 1.5, 1.6, 1.7 may illustrate the various types of periodic original functions.

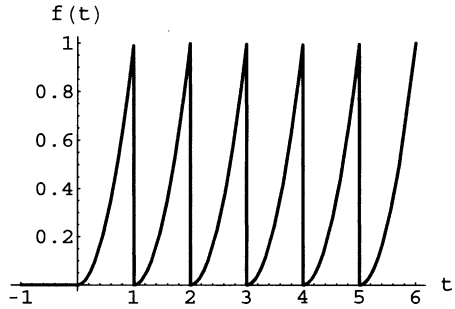


Fig.1.4: A periodic original of period T = 1.

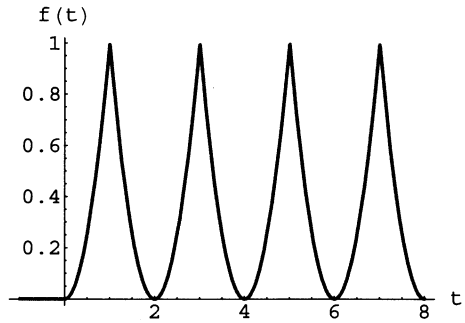


Fig.1.5: An even periodic original function of period T = 2.

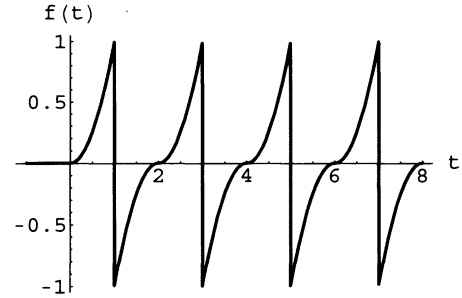


Fig.1.6: An odd periodic original function of period T = 2.

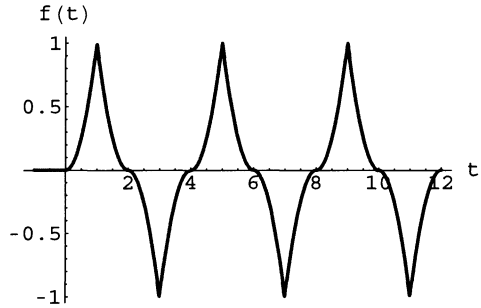


Fig.1.7: A periodic function with a symmetry of the third kind.

Example. Compute the Laplace transform of the periodic original with period 2π and given by $f(t) = t^2$ for $0 < t < 2\pi$.

$$F(s, 2\pi) = \int_0^{2\pi} e^{-st} t^2 \, dt = \frac{2}{s^3} - \frac{e^{-2\pi s}(4\pi^2 s^2 + 4\pi s + 2)}{s^3}.$$

We obtain

$$F(s) = \frac{2 - e^{-2\pi s}(4\pi^2 s^2 + 4\pi s + 2)}{s^3(1 - e^{-2\pi s})}.$$

At the first glance, it seems that

$$F(s, 2\pi) = \frac{2 - e^{-2\pi s}(4\pi^2 s^2 + 4\pi s + 2)}{s^3}$$

has a singularity at $s = 0$ and is not an entire function as stated above (finite Laplace transforms are always entire functions). But the singularity is removable, because the numerator can be expanded as follows:

$$2 - e^{-2\pi s}(4\pi^2 s^2 + 4\pi s + 2) = \frac{8\pi^3}{3} s^3 - 4\pi^4 s^4 + \frac{16\pi^5}{5} s^5 + \cdots.$$

1.1.3.2 The Gamma Function

We define the gamma function by Euler's integral, i.e. by

$$\Gamma(z) := \int_0^\infty t^{z-1}\, e^{-t}\, dt \qquad (\Re z > 0).$$

Integration by parts immediately yields the important recursion formula

$$z\,\Gamma(z) = \Gamma(z+1).$$

Because

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$$

recursion implies that,

$$\Gamma(2) = 1\,\Gamma(1) = 1,\ \Gamma(3) = 2\,\Gamma(2) = 2.1 = 2,\ \Gamma(4) = 3\,\Gamma(3) = 3.2 = 3!,$$

hence, for any positive integer n,

$$\Gamma(n+1) = n!$$

This shows that the gamma function can be regarded as a continuous interpolation of the discrete factorial function $f(k) := (k-1)!$ $(k = 1, 2, 3, \ldots)$ (see Fig. 1.8).

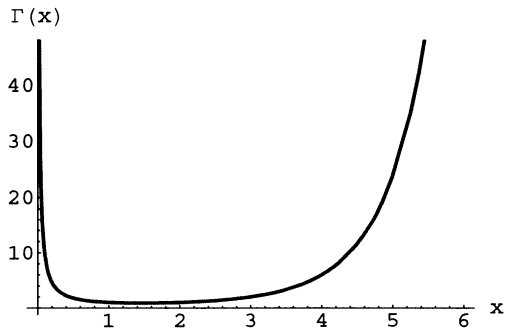


Fig.1.8: Graph of the gamma function.

Note that, by a change of variable $t = x^2$, one obtains

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt = 2\int_0^\infty e^{-x^2}\, dt = 2\,\frac{\sqrt{\pi}}{2} = \sqrt{\pi}\,.$$

From this and the recursion formula, we have

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\frac{\sqrt{\pi}}{2},\ \text{etc.}$$

In Euler's integral we compute, by a change of variables $t = s\,x$, where s is fixed with $\Re s > 0$,

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha\, e^{-t}\, dt = s\int_0^\infty (s\,x)^\alpha\, e^{-s\,x}\, dx = s^{\alpha+1}\int_0^\infty x^\alpha\, e^{-s\,x}\, dx\,.$$

The integral on the right-hand side is the Laplace transform of t^α. We assumed that $\Re\alpha > -1$, so the Laplace integral converges at its lower integration limit 0, otherwise there is no restriction on α. Thus, we have established the important correspondence

$$t^{\alpha} \circ\!-\!\bullet \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \Re\alpha > -1. \tag{1.26}$$

This formula generalizes the known correspondence

$$t^{n} \circ\!-\!\bullet \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

for real or complex exponents. For $\alpha = -1/2$ and $1/2$ we have the correspondences

$$\frac{1}{\sqrt{t}} \circ\!-\!\bullet \frac{\sqrt{\pi}}{\sqrt{s}} \quad \text{and} \quad \sqrt{t} \circ\!-\!\bullet \frac{\sqrt{\pi}}{2}\,\frac{1}{s\sqrt{s}}.$$

By applying the image shift rule to the first correspondence, we obtain

$$\frac{e^{i\omega t}}{\sqrt{t}} \circ\!-\!\bullet \frac{\sqrt{\pi}}{\sqrt{s-i\omega}}, \quad \frac{e^{-i\omega t}}{\sqrt{t}} \circ\!-\!\bullet \frac{\sqrt{\pi}}{\sqrt{s+i\omega}}.$$

Then, by linearity,

$$\frac{e^{i\omega t} + e^{-i\omega t}}{\sqrt{t}} \circ\!-\!\bullet \frac{\sqrt{\pi}}{\sqrt{s-i\omega}} + \frac{\sqrt{\pi}}{\sqrt{s+i\omega}}$$

$$\Longrightarrow \frac{\cos\omega t}{\sqrt{t}} \circ\!-\!\bullet \frac{\sqrt{\pi}}{2}\,\frac{\sqrt{s+i\omega}+\sqrt{s-i\omega}}{\sqrt{s^2+\omega^2}}$$

$$= \sqrt{\frac{\pi}{2}}\,\frac{\sqrt{\sqrt{s^2+\omega^2}+s}}{\sqrt{s^2+\omega^2}}.$$

Similarly, we obtain

$$\frac{\sin\omega t}{\sqrt{t}} \circ\!-\!\bullet \sqrt{\frac{\pi}{2}}\,\frac{\sqrt{\sqrt{s^2+\omega^2}-s}}{\sqrt{s^2+\omega^2}}.$$

1.1.3.3 Bessel Functions

Let us start with the second order linear homogeneous differential equation called the *Bessel Differential Equation*

$$t^2 w''(t) + t w'(t) + (t^2 - n^2) w(t) = 0, \quad n \in \mathbb{Z}.$$

Any solution of this differential equation is called a *cylinder function.* Because the differential equation is linear and homogenous, any linear combination of solutions is again a solution. In order to find a solution we first set $w(t) = t^{-n} y(t)$. The Bessel Functions are defined as the particular solutions $w(t) =: J_n(t)$ that satisfy the relation $n\,y(0+) = 0$. Let us now find the differential equation that is satisfied by $y(t)$. We have

$$t w'(t) = -n t^{-n} y(t) + t^{-n+1} y'(t),$$

$$t^2 w''(t) = n(n+1) t^{-n} y(t) - 2n t^{-n+1} y'(t) + t^{-n+2} y''(t).$$

Therefore,

$$t^2 w''(t) + t w'(t) + (t^2 - n^2) w(t)$$
$$= t^{-n+1} \{t y'(t) + (-2n+1) y'(t) + t y(t)\} = 0.$$

We find that $y(t)$ satisfies the differential equation

$$t y''(t) - (2n-1) y'(t) + t y(t) = 0.$$

This differential equation is solved by means of Laplace transformation. By applying the known operational properties:

$$t y''(t) \circ\!-\!\bullet -\frac{d}{ds}\{s^2 Y(s) - s y(0+) - y'(0+)\}$$
$$= -2s Y(s) - s^2 Y'(s) + y(0+),$$

$$-(2n-1) y'(t) \circ\!-\!\bullet -(2n-1)\{s Y(s) - y(0+)\}$$
$$= -(2n-1) s Y(s) + (2n-1) y(0+),$$

$$t y(t) \circ\!-\!\bullet -Y'(s),$$

$$t y''(t) - (2n-1) y'(t) + t y(t)$$
$$= -(s^2+1) Y'(s) - (2n+1) s Y(s) + 2n y(0+) = 0.$$

Finally, the assumption $n\, y(0+) = 0$ yields

$$Y'(s) + (2n+1)\frac{s}{(s^2+1)} Y(s) = 0.$$

This linear first order differential equation is the image equation for the Laplace transform $Y(s)$ of $y(t)$. If we had tried to directly solve the Bessel differential equation by means of Laplace transformation, we would have ended up with another second order differential equation with variable coefficients, because the coefficients of Bessel's differential equation contain second powers of t. By contrast, the above first order linear differential equation is readily solved. The solution is

$$Y(s) = c \exp\left[-\left(n+\frac{1}{2}\right)\ln(s^2+1)\right] = c\,(s^2+1)^{-n-1/2}$$

where c is an arbitrary constant.

For $n = 0$, we have

$$Y(s) = \frac{c}{\sqrt{s^2+1}}.$$

Thus, the Laplace transform of the Bessel Function of order 0 is given by

$$J_0(t) \circ\!-\!\bullet \frac{c}{\sqrt{s^2+1}}.$$

The constant factor c will be determined later. Let us now find a power series for $y(t)$. By use of the binomial series we obtain

$$Y(s) = c\,s^{-2n-1}\left(1+\frac{1}{s^2}\right)^{-n-1/2} = c\,s^{-2n-1}\sum_{k=0}^{\infty}\binom{-n-1/2}{k}s^{-2k}$$

$$= c\sum_{k=0}^{\infty}\binom{-n-1/2}{k}s^{-2(n+k)-1}.$$

The binomial coefficient can be transformed as follows:

$$\binom{-n-1/2}{k} = \frac{(-n-1/2)(-n-3/2)\ \ldots\ (-n-k+1/2)}{k!}$$

$$= (-1)^k\,\frac{(2n+1)(2n+3)\ \ldots\ (2n+2k-1)}{2^k\,k!}$$

$$= \frac{(-1)^k\,(2n+1)(2n+2)\ldots(2n+2k-1)(2n+2k)}{2^k\,k!\,(2n+2)(2n+4)\ \ldots(2n+2k)}$$

$$= \frac{(-1)^k\,(2n+1)(2n+2)\ldots(2n+2k-1)(2n+2k)}{2^k\,k!\,(2^k\,(n+1)(n+2)\ \ldots(n+k))}$$

$$= \frac{(-1)^k}{2^k\,k!}\,\frac{(2n+2k)!\,n!}{(2n)!\,2^k\,(n+k)!}$$

$$= \frac{(-1)^k}{2^{2k}\,k!}\,\frac{\Gamma(2n+2k+1)\,\Gamma(n+1)}{\Gamma(2n+1)\,\Gamma(n+k+1)}.$$

Hence we have

$$Y(s) = \frac{c}{(s^2+1)^{n+1/2}}$$

$$= c\,\frac{\Gamma(n+1)}{\Gamma(2n+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\,k!}\,\frac{\Gamma(2n+2k+1)}{\Gamma(n+k+1)}\,\frac{1}{s^{2(n+k)+1}}.$$

By taking the inverse Laplace transform term by term (expansion theorem of the first kind), we obtain

$$y(t) = c\,\frac{\Gamma(n+1)\,2^n}{\Gamma(2n+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+n}\,k!}\,\frac{\Gamma(2n+2k+1)}{\Gamma(n+k+1)}\,\frac{t^{2(n+k)}}{\Gamma(2n+2k+1)}.$$

The function $y(t)$, being the solution of a homogeneous linear differential equation, is only determined up to a constant factor. We now choose c such that

$$c\,\frac{\Gamma(n+1)\,2^n}{\Gamma(2n+1)} = 1.$$

Finally, this gives the power series of the Bessel Function of order n

$$J_n(t) = t^{-n}\,y(t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{t}{2}\right)^{2k+n}. \tag{1.27}$$

Also, we obtained the correspondence

$$t^n J_n(t) \;\square\!-\!\blacksquare\; \frac{\Gamma(2n+1)}{2^n\,\Gamma(n+1)}\,\frac{1}{(s^2+1)^{n+1/2}}$$
$$= \frac{\Gamma(n+\frac{1}{2})\,2^n}{\sqrt{\pi}}\,\frac{1}{(s^2+1)^{n+1/2}}. \tag{1.28}$$

The latter two formulas were derived for a positive integer n, but it can be shown that they are still true if n is replaced by a complex number ν, provided $\Re\nu > -1/2$. The Bessel functions describe some damped oscillations as is shown in the graphs of Figs.1.9 and 1.10.

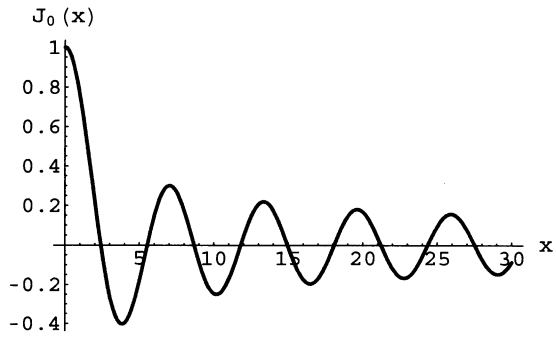


Fig.1.9: Bessel Function of order zero.

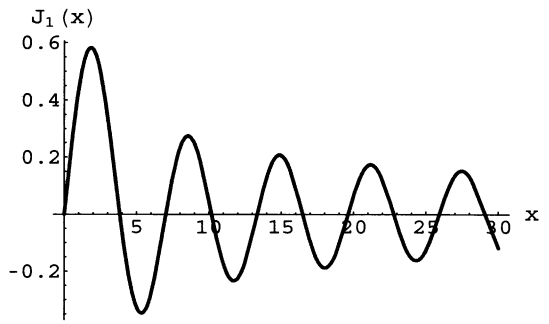


Fig.1.10: Bessel Function of order 1.

Later in Section 6.2, it will be shown that for large values of t the following asymptotic expression holds:

$$J_n(t) \approx \sqrt{\frac{2}{\pi t}}\,\cos\Big(t - \frac{\pi n}{2} - \frac{\pi}{4}\Big), \qquad t \longrightarrow \infty.$$

If $n = 0$, we have $c = 1$. Thus,

$$J_0(t) \;\square\!-\!\blacksquare\; \frac{1}{\sqrt{s^2+1}}. \tag{1.29}$$

If $n = 1$, we also have $c = 1$. Moreover $\lim_{t\to 0+} y(t)/t = J_1(0+) = 0$ exists, hence the image integration rule can be applied:

$$J_1(t) = \frac{y(t)}{t} \circ\!-\!\bullet \int_s^\infty \frac{1}{(s^2+1)^{1+1/2}}\, dp$$

$$= \left(\frac{p}{\sqrt{p^2+1}}\right)_s^\infty = 1 - \frac{s}{\sqrt{s^2+1}}$$

hence

$$J_1(t) \circ\!-\!\bullet \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}. \tag{1.30}$$

The Bessel Functions $J_n(t)$ satisfy the recursion relation

$$J_{n-1}(t) - J_{n+1}(t) = 2J_n'(t)$$

which follows from (1. 27). Because$J_n(0+) = 0$ for all positive integers n, we obtain, by the differentiation rule, the recursion relation for the image functions

$$\mathcal{L}J_{n-1}(s) - \mathcal{L}J_{n+1}(s) = 2s\,\mathcal{L}J_n(s),$$

that is

$$\mathcal{L}J_{n+1}(s) = \mathcal{L}J_{n-1}(s) - 2s\,\mathcal{L}J_n(s), \quad (n = 1, 2, 3, \ldots).$$

This can be used to find

$$\mathcal{L}J_2(s) = \mathcal{L}J_0(s) - 2s\,\mathcal{L}J_1(s) = \frac{1}{\sqrt{s^2+1}} - \frac{2s\left(\sqrt{s^2+1}-s\right)}{\sqrt{s^2+1}}$$

$$= \frac{1 - 2s\sqrt{s^2+1} - 2s^2}{\sqrt{s^2+1}} = \frac{\left[\sqrt{s^2+1}-s\right]^2}{\sqrt{s^2+1}},$$

hence

$$J_2(t) \circ\!-\!\bullet \frac{\left[\sqrt{s^2+1}-s\right]^2}{\sqrt{s^2+1}}. \tag{1.31}$$

Repeating this process

$$\mathcal{L}J_3(s) = \mathcal{L}J_1(s) - 2s\,\mathcal{L}J_2(s) = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} - 2s\,\frac{\left[\sqrt{s^2+1}-s\right]^2}{\sqrt{s^2+1}}$$

$$= \frac{\left(\sqrt{s^2+1}-s\right)\left(1 - 2s\sqrt{s^2+1} - 2s^2\right)}{\sqrt{s^2+1}} = \frac{\left[\sqrt{s^2+1}-s\right]^3}{\sqrt{s^2+1}}$$

and so on. The general result is

$$J_n(t) \;\square\!-\!\blacksquare\; \frac{\left[\sqrt{s^2+1}-s\right]^n}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}\left[\sqrt{s^2+1}+s\right]^n}. \tag{1.32}$$

We conclude this section by deriving a representation for the Bessel function as a Laplace integral. This is done by use of (1.28) and the convolution theorem. We write for $\Re n > -1/2$,

$$\frac{\sqrt{\pi}\, t^n J_n(t)}{2^n\, \Gamma(n+\frac{1}{2})} \;\square\!-\!\blacksquare\; \frac{1}{(s^2+1)^{n+1/2}} = \frac{1}{(s+i)^{n+1/2}} \frac{1}{(s-i)^{n+1/2}}.$$

Therefore, the correspondence

$$\frac{1}{(s \pm i)^{n+1/2}} \;\square\!-\!\blacksquare\; \frac{t^{n-1/2}}{\Gamma(n+1/2)} e^{\mp it}$$

and the convolution theorem yield

$$\frac{\sqrt{\pi}\, t^n J_n(t)}{2^n\, \Gamma(n+\frac{1}{2})} = \frac{1}{\Gamma^2(n+\frac{1}{2})} \int_0^t (t-x)^{n-1/2} e^{-i(t-x)} x^{n-1/2} e^{ix}\, dx.$$

By substituting $x = tu$, we have

$$\sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right) 2^{-n} e^{it} t^n J_n(t) = \int_0^t e^{2ix}(t-x)^{n-1/2} x^{n-1/2}\, dx$$

$$= t^{2n} \int_0^1 e^{2itu}(1-u)^{n-1/2} u^{n-1/2}\, du.$$

By another change of variables $u = (1-\omega)/2$, $1-u = (1+\omega)/2$, we obtain

$$\sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right) 2^{-n} e^{it} t^{-n} J_n(t) = \frac{e^{it}}{2^{2n}} \int_{-1}^1 e^{-it\omega}(1-\omega^2)^{n-1/2}\, dv$$

and, finally,

$$\sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right) 2^n\; t^{-n} J_n(t) = \int_{-1}^1 e^{it\omega}(1-\omega^2)^{n-1/2}\, d\omega \tag{1.33}$$

which is the representation of the function in the left-hand side as a finite Fourier integral. Yet another substitution $\omega = \tau - 1$, $1-\omega^2 = 2\tau - \tau^2$ and by setting $t = s\exp(i\pi/2)$, yields

$$\sqrt{\pi}\, \Gamma\left(n+\frac{1}{2}\right) 2^n\; s^{-n} e^{-n\pi/2} J_n(s e^{i\pi/2}) = e^s \int_0^2 e^{-st}[\tau(2-\tau)]^{n-1/2}\, d\tau.$$

This can also be regarded as

$$\frac{u(2-t)\,[\tau(2-\tau)]^{n-1/2}}{\sqrt{\pi}\, \Gamma(n+\frac{1}{2})\, 2^n} \;\square\!-\!\blacksquare\; s^{-n} e^{-s} e^{-ni\pi/2} J_n(s e^{i\pi/2}). \tag{1.34}$$

Since $e^{-n\pi/2} J_n(s\, e^{i\pi/2}) =: I_n(s)$ for $-\pi < \arg s < \pi/2$, we also have

$$\frac{u(2-t)}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})\, 2^n}\ [t(2-t)]^{n-\frac{1}{2}} \circ\!-\!\bullet\ s^{-n} e^{-s} I_n(s) \tag{1.35}$$

with the modified Bessel function $I_n(s)$.

Problem 1.18. Prove that

$$\frac{d}{dt}\left(\frac{J_n(t)}{t^n}\right) = -\frac{J_{n+1}(t)}{t^n}, \quad \frac{d}{dt}\,(t^n J_n(t)) = t^n J_{n-1}(t)\,.$$

Problem 1.19. Prove that

$$J_n'(t) = \frac{n}{t} J_n(t) - J_{n+1}(t), \quad J_n'(t) = J_{n-1}(t) - \frac{n}{t} J_n(t)\,.$$

Problem 1.20. Show the two recursion relations

$$J_{n-1}(t) + J_{n+1}(t) = \frac{2n}{t} J_n(t), \quad J_{n-1}(t) - J_{n+1}(t) = 2 J_n'(t)\,.$$

Problem 1.21. Show that $J_0'(t) = -J_1(t)$.

Problem 1.21a. Use (1.32) for non integer values of n, to show that

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\,\sin t \circ\!-\!\bullet\ \frac{\sqrt{\sqrt{s^2+1}-s}}{\sqrt{s^2+1}}$$

$$J_{-1/2}(t) = \sqrt{\frac{2}{\pi t}}\,\cos t \circ\!-\!\bullet\ \frac{\sqrt{\sqrt{s^2+1}+s}}{\sqrt{s^2+1}}$$

and compare these relations with the result in Section 1.1.3.2.

Problem 1.22. Use the power series expansion of $\exp[-b/s]/s^{\alpha+1}$, the term by term Laplace inversion, and (1.27) to show that

$$\frac{1}{s^{\alpha+1}}\, e^{-b/s} \bullet\!-\!\circ \left(\frac{t}{b}\right)^{\alpha/2} J_\alpha\!\left(2\sqrt{b\,t}\right), \quad (b>0,\ \alpha>-1)$$

Problem 1.23. Show that

$$\frac{1}{\sqrt{s}}\, e^{-b/s} \bullet\!-\!\circ \frac{\cos\!\left(2\sqrt{b\,t}\right)}{\sqrt{\pi t}}, \quad b>0.$$

1.1.3.4 The Error Function and Related Correspondences

The *Error Function* and the *Complementary Error Function* are defined as

$$\operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du\,,$$

$$\operatorname{erfc}(t) := \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\, du .$$

Because

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2}\, du = 1,$$

we have the identity $\operatorname{erfc}(t) = 1 - \operatorname{erf}(t)$. The following important integral is related to the Complementary Error Function:

$$\int_0^\infty \exp[-(a t^2 + 2 b t + c)]\, dt$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - a c}{a}\right) \operatorname{erfc}\left(\frac{b}{\sqrt{a}}\right), \quad a > 0$$

This follows from the definition of erfc by a change of variables. For $b = s/2$, $c = 0$, we obtain

$$\int_0^\infty e^{-s t} \exp[-a t^2]\, dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right) \operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right)$$

with $a > 0$, hence

$$\exp[-a t^2] \circ\!-\!\bullet \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right) \operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right), \quad a > 0 .$$

In Problem 16 we found the correspondences

$$\frac{1}{\sqrt{s}\,(\sqrt{s} + b)} \bullet\!-\!\circ e^{b^2 t} \operatorname{erfc}(b\sqrt{t}),$$

$$\frac{1}{\sqrt{s} + b} \bullet\!-\!\circ \frac{1}{\sqrt{\pi t}} - b\, e^{b^2 t} \operatorname{erfc}(b\sqrt{t}) .$$

We wish to use (1.18) to find the inverse transform of $\exp[-a\sqrt{s}]/s$. The correspondence $F(s) = \exp[-a s]/s \bullet\!-\!\circ f(t) = u(t - a)$ implies

$$\frac{F[\sqrt{s}]}{\sqrt{s}} = \frac{\exp[-a\sqrt{s}]}{\sqrt{s}\,\sqrt{s}}$$

$$= \frac{\exp[-a\sqrt{s}]}{s} \bullet\!-\!\circ \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) u(\tau - a)\, d\tau$$

$$= \frac{1}{\sqrt{\pi t}} \int_a^\infty \exp\left(-\frac{\tau^2}{4t}\right) d\tau$$

$$= \frac{2\sqrt{t}}{\sqrt{\pi t}} \int_{a/(2\sqrt{t})}^\infty \exp(-u^2)\, du = \operatorname{erfc}\left[\frac{a}{2\sqrt{t}}\right].$$

Finally,

$$\frac{\exp\left[-a\sqrt{s}\right]}{s} \bullet\!-\!\circ\, \operatorname{erfc}\left[\frac{a}{2\sqrt{t}}\right].$$

Problem 1.24. Use the formula of Problem 13 to show that

$$\frac{\exp\left[-a\sqrt{s}\right]}{\sqrt{s}} \bullet\!-\!\circ\, \frac{\exp[-a^2/(4t)]}{\sqrt{\pi t}}.$$

■ 1.1.4 Inverse Laplace Transforms

1.1.4.1 The Inverse Laplace Transform of Rational Image Functions

The inverse Laplace transform of a rational image function is of great importance because rational image functions often occur when ordinary differential equations are solved by means of Laplace transformation. A rational image function is the quotient of two polynomials $Q(s)$ and $P(s)$. Let the degree of the numerator polynomial $Q(s)$ be m and the degree of the denominator polynomial $P(s)$ be n. A rational function is called a *proper rational function* if $m < n$. Any rational function $F(s)$ with $m \geq n$ can be written uniquely as the sum of a polynomial and a proper rational function:

$$F(s) = a_k s^k + a_{k-1} s^{k-1} + \cdots + a_0 + \frac{Q(s)}{P(s)}$$
$$(k = m - n, \text{ degree } Q < \text{degree } P).$$

Due to the linearity property, we shall be able to find the inverse Laplace transform of any rational image function, provided we can transform back a power s^k, the constant 1, and a proper rational image function. We have already noted that the originals corresponding to a power s^k, including the case $k = 0$, cannot be regarded as ordinary functions. The concept of a function must be extended in order to prepare the field for polynomial image functions. In this section we shall only treat the problem of finding the inverse Laplace transform of a proper rational image function. This will be accomplished by means of a partial fraction decomposition of the proper rational function. For doing so, the given denominator polynomial $P(s)$ needs to factor completely into linear factors:

$$P(s) = (s-\alpha_1)^{\nu_1} (s-\alpha_2)^{\nu_2} \cdots (s-\alpha_p)^{\nu_p} \tag{1.36}$$

where $\alpha_1, \ldots, \alpha_p$ are the p mutually different complex zeros or roots of $P(s)$, i.e. $\alpha_1, \ldots, \alpha_p$ are all the mutually different solutions of the algebraic equation $P(s) = 0$. We also assume that the leading coefficient of the denominator polynomial $P(s)$ is 1. The Fundamental Theorem of Algebra ensures that any polynomial can be written uniquely in the form (1.36). However, if the degree of the polynomial is higher than four, the zeros can generally only be determined numerically. But even in the case of degree three or four, closed formulas for computing the zeros are unhandy, so that they are rarely used. The positive integer exponent ν_i associated with the complex zero α_i is called the *multiplicity of the zero* α_i. If the multiplicity is 1, we call it a *simple* zero. Clearly,

$$\nu_1 + \nu_2 + \ldots + \nu_p = n = \text{degree of } P.$$

Most polynomials encountered in applications have real coefficients. We call a polynomial with real coefficients a *real polynomial* (even if its variable s varies over the complex plane). In general, a real polynomial can have real and complex zeros. But the nonreal complex zeros

always occur in pairs, i.e. if α_i is a nonreal zero then its complex conjugate $\overline{\alpha_i}$ is also a zero. Thus the decomposition into linear factors of a real polynomial can be written as

$$\begin{aligned} P(s) &= (s-\xi_1)^{\nu_1}(s-\xi_2)^{\nu_2}\dots(s-\xi_r)^{\nu_r} \\ &[(s-\alpha_1)(s-\overline{\alpha_1})]^{\mu_1}[(s-\alpha_2)(s-\overline{\alpha_2})]^{\mu_2} \\ &\dots[(s-\alpha_c)(s-\overline{\alpha_c})]^{\mu_c} \end{aligned} \tag{1.37}$$

where $\xi_1, \dots, \xi_r$ are the r mutually different real zeros and $\alpha_1, \dots, \alpha_c$ the c mutually different pairs of nonreal zeros. Clearly,

$$\nu_1 + \nu_2 + \cdots + \nu_r + 2(\mu_1 + \mu_2 + \cdots + \mu_c) = n = \text{degree of } P.$$

We shall state, without proof, the theorem of partial fraction decomposition.

Proposition. *Any proper rational function*

$$F(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-\alpha_1)^{\nu_1}(s-\alpha_2)^{\nu_2}\dots(s-\alpha_p)^{\nu_p}}$$

can be written as

$$\sum_{k=1}^{p}\left[\frac{a_{k1}}{(s-\alpha_k)^{\nu_k}} + \frac{a_{k2}}{(s-\alpha_k)^{\nu_k-1}} + \cdots + \frac{a_{k\nu_1}}{(s-\alpha_k)}\right] \tag{1.38}$$

where the complex coefficients a_{ki} are uniquely determined by $F(s)$. In the case of a real proper rational function (a quotient of two real polynomials), the decomposition into partial fractions takes the form:

$$\begin{aligned} F(s) &= \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-\xi_1)^{\nu_1}\dots(s-\xi_r)^{\nu_r}} \\ &\times \frac{1}{[(s-\alpha_1)(s-\overline{\alpha_1})]^{\mu_1}\dots[(s-\alpha_c)(s-\overline{\alpha_c})]^{\mu_c}} \\ &= \sum_{k=1}^{r}\left[\frac{b_{k1}}{(s-\xi_k)^{\nu_k}} + \frac{b_{k2}}{(s-\xi_k)^{\nu_k-1}} + \cdots + \frac{b_{k\nu_1}}{(s-\xi_k)}\right] \\ &+ \sum_{k=1}^{c}\left[\left\{\frac{a_{k1}}{(s-\alpha_k)^{\nu_k}} + \frac{\overline{a_{k1}}}{(s-\overline{\alpha_k})^{\nu_k}}\right\} + \right. \\ &\qquad \left. \cdots + \left\{\frac{a_{k\nu_1}}{(s-\alpha_k)} + \frac{\overline{a_{k\nu_1}}}{(s-\overline{\alpha_k})}\right\}\right]. \end{aligned} \tag{1.39}$$

Note that the coefficients associated with conjugate zeros are also conjugate and the coefficients associated with real zeros are real. Next we wish to compute the coefficients a_{ki} in (1.38). Let us consider a fixed index $m \in \{1, 2, \dots, p\}$. The reduced rational function

$$F_m(s) := F(s)(s-\alpha_m)^{\nu_m} = \frac{Q(s)(s-\alpha_m)^{\nu_m}}{P(s)} = \frac{Q(s)}{P_m(s)},$$

where

$$P_m(s) := (s-\alpha_1)^{\nu_1}\dots(s-\alpha_{m-1})^{\nu_{m-1}}(s-\alpha_{m+1})^{\nu_{m+1}}\dots(s-\alpha_p)^{\nu_p},$$

obtained from $F(s)$ by omitting the factor $(s-\alpha_m)^{\nu_m}$ in the denominator, is holomorphic at $s=\alpha_m$ and, therefore, can be expanded into a Taylor series about $s=\alpha_m$.

$$F_m(s) = c_{m1} + c_{m2}(s-\alpha_m) + c_{m3}(s-\alpha_m)^2 + \cdots \tag{1.40}$$

with

$$c_{mi} = \frac{1}{(i-1)!}\left(\frac{d^{i-1}}{ds^{i-1}} F_m(s)\right)_{s=\alpha_m}, \quad i = 1, 2, 3, \ldots$$

(1.38) yields

$$\begin{aligned} F_m(s) &= \frac{Q(s)(s-\alpha_m)^{\nu_m}}{P(s)} \\ &= a_{m1} + a_{m2}(s-\alpha_m) + \cdots + a_{m\nu_1}(s-\alpha_m)^{\nu_m-1} + \\ &\quad (s-\alpha_m)^{\nu_m} \sum_{k=1,(k\neq m)}^{p} \left[\frac{a_{k1}}{(s-\alpha_k)^{\nu_k}} + \frac{a_{k2}}{(s-\alpha_k)^{\nu_k-1}} + \cdots + \frac{a_{k\nu_1}}{(s-\alpha_k)}\right]. \end{aligned}$$

The last term on the right-hand side is of the form $(s-\alpha_m)^{\nu_m} h(s)$, where the sum $h(s)$ is a holomorphic function at $s=\alpha_m$. Thus

$$\begin{aligned} &(s-\alpha_m)^{\nu_m} h(s) \\ &= (s-\alpha_m)^{\nu_m} \{h_0 + h_1(s-\alpha_m) + h_2(s-\alpha_m)^2 + \cdots\} \\ &= h_0(s-\alpha_m)^{\nu_m} + h_1(s-\alpha_m)^{\nu_m+1} + \cdots. \end{aligned}$$

Therefore, the Taylor expansion of $F_m(s)$ is

$$\begin{aligned} F_m(s) = {} & a_{m1} + a_{m2}(s-\alpha_m) + \cdots + a_{m\nu_1}(s-\alpha_m)^{\nu_m-1} + \\ & + h_0(s-\alpha_m)^{\nu_m} + h_1(s-\alpha_m)^{\nu_m+1} + \cdots. \end{aligned} \tag{1.41}$$

The Taylor expansion is unique. By comparing the expansion (1.40) and (1.41) we see that

$$a_{mi} = \frac{1}{(i-1)!}\left(\frac{d^{i-1}}{ds^{i-1}} F_m(s)\right)_{s=\alpha_m}, \quad i = 0, 1, 2, \ldots \nu_m - 1. \tag{1.42}$$

The first three coefficients in the partial fraction decomposition (1.38) are

$$a_{m1} = F_m(\alpha_m) = \frac{Q(\alpha_m)}{P_m(\alpha_m)},$$

$$a_{m2} = F_m'(\alpha_m) = \frac{d}{ds}\left(\frac{Q(s)}{P_m(s)}\right)_{s=a_m},$$

$$a_{m3} = \frac{F_m'(\alpha_m)}{2} = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{Q(s)}{P_m(s)}\right)_{s=a_m},$$

and so forth.

In the case of a real rational function (1.42) implies

$$\frac{1}{(i-1)!}\left(\frac{d^{i-1}}{ds^{i-1}}F_m(s)\right)_{s=\overline{\alpha_m}} = \overline{\frac{1}{(i-1)!}\left(\frac{d^{i-1}}{ds^{i-1}}F_m(s)\right)_{s=\alpha_m}} = \overline{a_{mi}}$$

yielding the representation (1.39). This means that in the case of a pair of nonreal complex zeros, only one coefficient a_{mi} has to be computed, while the coefficient corresponding to the conjugate zero, is simply the conjugate complex value of a_{mi}. Remember that this is only true for real rational functions.

The partial fraction decompositions (1.38) and (1.39) indicate that all that is needed to compute the inverse Laplace transform of a given proper rational image function, are the following two correspondences:

$$\frac{1}{(s-\alpha)^k} \blacksquare\!-\!\square\ ?\ , \quad \frac{a}{(s-\alpha)^k} + \frac{\overline{a}}{(s-\overline{\alpha})^k} \blacksquare\!-\!\square\ ?$$

In the first correspondence the zero α can be real or nonreal, in the second one, we assume that α is nonreal. By use of

$$\frac{1}{s^k} \blacksquare\!-\!\square\ \frac{t^{k-1}}{(k-1)!} \quad (k \in \mathbb{N})$$

and the image shift rule, we immediately obtain

$$\frac{1}{(s-\alpha)^k} \blacksquare\!-\!\square\ \frac{t^{k-1}\,e^{\alpha t}}{(k-1)!} \quad (k \in \mathbb{N}) \tag{1.43}$$

which is correct for real or nonreal α. A real partial fraction

$$\frac{a}{(s-\xi)^k} \quad (a \in \mathbb{R},\ \xi \in \mathbb{R})$$

will always produce a real-valued original function:

$$\frac{a}{(s-\xi)^k} \blacksquare\!-\!\square\ a\,\frac{t^{k-1}\,e^{\xi t}}{(k-1)!} \in \mathbb{R}.$$

The pair

$$\frac{a}{(s-\alpha)^k} + \frac{\overline{a}}{(s-\overline{\alpha})^k}$$

will also produce a real-valued original. We set for the nonreal zero

$$\alpha = \kappa + i\omega \quad (\kappa \in \mathbb{R},\ \omega > 0)$$

i.e. κ is the real part and ω is the imaginary part of the nonreal zero α. The complex coefficient a can be represented in cartesian or in polar coordinates:

$$a = A + Bi = re^{i\varphi}\ ,$$
$$(A, B \in \mathbb{R},\ r = |a| > 0,\ \varphi = \arg a \in \mathbb{R})$$

$$\left(r = \sqrt{A^2 + B^2},\ A = r\cos\varphi,\ B = r\sin\varphi\right)$$

We now have

$$\frac{a}{(s-\alpha)^k} \blacksquare\!-\!\square\; a\,\frac{t^{k-1}\,e^{\alpha t}}{(k-1)!} = (A + Bi)\,\frac{t^{k-1}\,e^{(\kappa+i\omega)t}}{(k-1)!}$$

$$= (A + Bi)\,\frac{t^{k-1}\,e^{\kappa t}\,(\cos\omega t + i\sin\omega t)}{(k-1)!}$$

and similarly

$$\frac{\overline{a}}{(s-\overline{\alpha})^k} \blacksquare\!-\!\square\; \overline{a}\,\frac{t^{k-1}\,e^{\overline{\alpha} t}}{(k-1)!} = (A - Bi)\,\frac{t^{k-1}\,e^{(\kappa-i\omega)t}}{(k-1)!}$$

$$= (A - Bi)\,\frac{t^{k-1}\,e^{\kappa t}\,(\cos\omega t - i\sin\omega t)}{(k-1)!}.$$

Thus, by linearity, we obtain the desired correspondence

$$\frac{a}{(s-\alpha)^k} + \frac{\overline{a}}{(s-\overline{\alpha})^k} \blacksquare\!-\!\square\; 2\,\frac{t^{k-1}\,e^{\kappa t}}{(k-1)!}\,\{A\cos\omega t - B\sin\omega t\}$$
$$= 2\,\frac{t^{k-1}\,e^{\kappa t}}{(k-1)!}\,r\cos(\omega t + \varphi). \tag{1.44}$$

For the special case of a pair of simple zeros one obtains

$$\frac{a}{s-\alpha} + \frac{\overline{a}}{s-\overline{\alpha}} \blacksquare\!-\!\square\; 2\,e^{\kappa t}\,\{A\cos\omega t - B\sin\omega t\}$$
$$= 2\,e^{\kappa t}\,r\cos(\omega t + \varphi) \tag{1.45}$$

with no powers of t. If the zero has a higher multiplicity, powers of t come up in the corresponding original. A pair of simple and nonreal complex conjugate zeros of the denominator polynomial yields an increasing harmonic oscillation, a stationary harmonic oscillation, or a damped harmonic oscillation, depending on the common real part κ of the zero: positive, zero, or negative. In any case, a real or a nonreal zero α of the denominator polynomial will produce an exponentially decaying term for $t \to \infty$, if and only if, the real part of this zero is strictly negative, i.e. is located in the negative complex half-plane. A pair of complex conjugate nonreal poles located on the imaginary axis will produce a stationary harmonic oscillation with a constant amplitude if the poles are simple. If the poles have a higher multiplicity, the oscillation will have an amplitude which grows up with a power of t. A pole located in the right complex half-plane will always produce a term which is exponentially growing for $t \to \infty$.

Example.

$$F(s) = \frac{s+3}{s^2\,(s+1)\,(s^2+2s+5)} = \frac{s+3}{s^2\,(s+1)\,[s-(-1+2i)]\,[s-(-1-2i)]}.$$

This is a proper real rational image function where $s = 0$ is a pole of order 2 (a zero of multiplicity 2 of the denominator polynomial), $s = -1$ is a simple pole, and $s = -1 \mp 2i$ is a pair of complex conjugate poles. The partial fraction decomposition has the following form.

$$F(s) = \frac{a_{11}}{s^2} + \frac{a_{12}}{s} + \frac{a_2}{s+1} + \frac{a_{31}}{[s-(-1+2i)]} + \frac{\overline{a_{31}}}{[s-(-1-2i)]},$$

$$F_1(s) = \frac{s+3}{(s+1)(s^2+2s+5)},$$
$$a_{11} = F_1(0) = 3/5.\ a_{12} = F_1'(0) = -16/25,$$

$$F_2(s) = \frac{s+3}{s^2(s^2+2s+5)},$$
$$a_2 = F_2(-1) = 1/2,$$

$$F_3(s) = \frac{s+3}{s^2(s+1)[s-(-1+2i)]},$$
$$a_{31} = F_3(-1-2i) = (7-i)/100 \implies \overline{a_{31}} = (7+i)/100.$$

Finally,

$$f(t) = \frac{3}{5}t - \frac{16}{25} + \frac{1}{2}e^{-t} + 2e^{-t}\left[\frac{7}{100}\cos 2t + \frac{1}{100}\sin 2t\right].$$

Problem 1.25. Find the original corresponding to the rational image functions

$$\frac{1}{s(s+1)^3(s^2+2s+5)}, \quad \frac{1}{s(s+1)(s^2+2s+5)^2}.$$

An important special case is the situation where all zeros (real or nonreal) of the denominator polynomial are simple. The rational image function can then be written as

$$F(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)}. \tag{1.46}$$

The decomposition into partial fractions is simply

$$F(s) = \sum_{m=1}^{n} \frac{a_m}{s-\alpha_m}. \tag{1.47}$$

The reduced rational function $F_m(s)$ becomes

$$F_m(s) := F(s)(s-a_m) = \frac{Q(s)(s-\alpha_m)}{P(s)} = \frac{Q(s)}{\frac{P(s)}{(s-\alpha_m)}} = \frac{Q(s)}{\frac{P(s)-P(\alpha_m)}{(s-\alpha_m)}}$$

since $P(\alpha_m) = 0$. But

$$a_m = F_m(\alpha_m) = \lim_{s\to\alpha_m} F_m(s) = \lim_{s\to\alpha_m} \frac{Q(s)}{\frac{P(s)-P(\alpha_m)}{(s-\alpha_m)}} = \frac{Q(\alpha_m)}{P'(\alpha_m)}.$$

Therefore

$$F(s) = \sum_{m=1}^{n} \frac{Q(\alpha_m)}{P'(\alpha_m)} \frac{1}{s-\alpha_m} \bullet\!\!-\!\!\circ\ f(t) = \sum_{m=1}^{n} \frac{Q(\alpha_m)}{P'(\alpha_m)} e^{\alpha_m t}. \tag{1.48}$$

1.1.4.2 Inversion Formulas

The Laplace transform $F(s)$ of the original $f(t)$ is defined by

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

The question arises whether there is a similar formula which gives the original $f(t)$ as a function of its image $F(s)$. There are even several. The most important is the *Complex Inversion Formula.* Another one is the *Post–Widder Inversion Formula* which looks like this:

If $f(t) \circ\!-\!\bullet\, F(s)$, then

$$f(t) = \lim_{n\to\infty} \left\{ (-1)^n \frac{s^{n+1}}{n!} F^{(n)}(s) \right\}_{s=(n+1)/t}.$$

In the case where $f(t)$ has a discontinuity of the first kind at t, the left-hand member has to be replaced by $\{f(t+0) + f(t-0)\}/2$.

Example.

$$F(s) = \frac{k!}{s^{k+1}}, \quad k \in \mathbb{N}.$$

We have

$$(-1)^n F^{(n)}(s) = \frac{(k+n)!}{s^{k+n+1}}, \quad (-1)^n \frac{s^{n+1}}{n!} F^{(n)}(s) = \frac{(k+n)!}{s^k\, n!}$$

and then

$$\left\{ (-1)^n \frac{s^{n+1}}{n!} F^{(n)}(s) \right\}_{s=(n+1)/t} = \frac{(k+n)!}{(n+1)^k\, n!} t^k$$

$$= \frac{(n+1)(n+2)\ldots(n+k)}{(n+1)^k} t^k$$

$$= \left(1 + \frac{1}{n+1}\right)\left(1 + \frac{2}{n+1}\right)\ldots\left(1 + \frac{k-1}{n+1}\right) t^k \longrightarrow t^k = f(t) \quad (n \to \infty).$$

For analytical work the Post–Widder Inversion formula had little impact despite the fact that a natural probabilistic interpretation can be given to it. Today however, this inversion formula is the starting point for an important method for the numerical inversion of a Laplace transform, the Post–Widder–Stehfestalgorithm (see Chapter 11).

The most important inversion formula is the Complex Inversion Formula. It is based on *Fourier's Integral Theorem* which belongs to the realm of Fourier series and Fourier transforms. The Complex Inversion Formula is

$$f(t) = \frac{1}{2\pi i}\, pv \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds, \tag{1.49}$$

where the symbol pv stands for Cauchy's principal value of the integral. This means

$$pv \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds := \lim_{b\to\infty} \int_{c-ib}^{c+ib} e^{ts} F(s)\, ds.$$

The integration contour is the vertical path $s = c + iy$, $-\infty < y < \infty$ which must be wholly located in the right half-plane of convergence of the given image function. In the case where $f(t)$ has a discontinuity of the first kind at t, the left-hand member has also to be replaced by

$$\frac{f(t+0)+f(t-0)}{2}.$$

The Complex Inversion Formula is seldom used directly. The formula serves more as a starting point for various procedures based on methods from the domain of functions of a complex variable. Since this first chapter is being thought of as a straightforward introduction to the method of Laplace transformation, we do not go further at this point and postpone the methods based on the Complex Inversion Formula to a later chapter (see Section 6.1).

1.2 The Two-Sided Laplace Transform

The two-sided Laplace transform is a generalization of the ordinary or one-sided Laplace transform. The two-sided Laplace transformation is extensively covered in [Van der Pol B. and Bremmer H. (1987)]. Also see [Doetsch G. I (1971), II (1972), III (1973)].

■ 1.2.1 The Two-Sided Laplace Integral

Remember that an original function is a function defined on $\mathbb{R}$ and which is vanishing on the negative part of the real axis. Explicitly an original can be written in the form $u(t)\,f(t)$ where $u(t)$ is the unit-step function and $f(t)$ is a functional expression. In the context of ordinary (one-sided) Laplace transformation, we adopted the convention that we drop the factor $u(t)$ and write simply $f(t)$ for the original $u(t)\,f(t)$. This convention is now lifted. In the context of two-sided Laplace transformation, a given function $f(t)$ is defined on $\mathbb{R}$ and does not necessarily vanish on the negative part of the real axis. If we refer to an original function, we will always denote it explicitly by $u(t)\,f(t)$ when working in the field of two-side Laplace transformation.

Definition. If $f(t)$ is a complex-valued function defined on the entire real axis, we call the function of the complex variable s,

$$F(s) := \int_{-\infty}^{\infty} e^{-st} f(t)\,dt \tag{1.50}$$

the *two-sided Laplace transform of* $f(t)$, provided the integral exists. We use the following notations:

$$f(t) \circ\!\!-\!\!\bullet\, F(s),\quad F(s) = \mathcal{L}_2[f(t)](s),\quad f(t) = \mathcal{L}_2^{-1}[F(s)](t).$$

It is clear that the two-sided Laplace transform of an original $u(t)\,f(t)$ is the ordinary Laplace transform of $f(t)$:

$$F(s) := \int_{-\infty}^{\infty} e^{-st} u(t)\,f(t)\,dt = \int_{0}^{\infty} e^{-st} f(t)\,dt,$$

or $\mathcal{L}_2[u(t)\,f(t)](s) = \mathcal{L}[f(t)](s)$.

We can write

$$F(s) := \int_{-\infty}^{\infty} e^{-st} f(t)\,dt$$

$$= \int_{-\infty}^{0} e^{-st} f(t)\,dt + \int_{0}^{\infty} e^{-st} f(t)\,dt.$$

The first integral on the right-hand side can be expressed as

$$\int_{-\infty}^{0} e^{-st} f(t)\,dt = \int_{\infty}^{0} e^{-s(-t)} f(-t)\,d(-t) = \int_{0}^{\infty} e^{st} f(-t)\,dt.$$

If we introduce the two ordinary (one-sided) Laplace transforms of $f(t)$ and $f(-t)$,

$$F_+(s) := \int_{0}^{\infty} e^{-st} f(t)\,dt, \qquad F_-(s) := \int_{0}^{\infty} e^{-st} f(-t)\,dt,$$

respectively, we can write

$$F(s) = F_-(-s) + F_+(s). \tag{1.51}$$

Assume that $F_-(s)$ has the right half-plane of convergence $\Re s > -c_-$, i.e. the ordinary Laplace integral exists and defines a holomorphic function in this half-plane. This implies that the domain of $F_-(-s)$ is the left half-plane $\Re s < c_-$ Assume further that $F_+(s)$ has the right half-plane of convergence $\Re s > c_+$. If $c_+ < c_-$ the two half-planes have a non-void intersection which is a vertical strip in the complex plane. In this case, and only in this one, the two-sided Laplace transform

$$F(s) = F_-(-s) + F_+(s)$$

exists for $c_+ < \Re s < c_-$ and is a holomorphic function in this vertical strip of the complex plane. The vertical strip $c_+ < \Re s < c_-$ is called the *strip of convergence of the two-sided Laplace transform* $F(s)$. If $c_+ \geq c_-$ there is no two-sided Laplace transform of $f(t)$. If $c_+ = -\infty$ and $c_- = +\infty$, i.e. $-c_- = -\infty$, the strip of convergence becomes the whole complex plane. When working in the context of the ordinary Laplace transformation it is seldom necessary to keep track of the various right half-planes of convergence which can come up in a problem. The reason is that an intersection of infinite right half-planes is again an infinite right half-plane. In the context of two-sided Laplace transformations however the situation is not so comfortable. We have always to keep track of the various strips of convergence in a problem. First, because intersections of strips of convergence can become void and second, as we will seen later, the same functional expression $F(s)$ can be associated to different strips of convergence and can thus correspond to different functions $f(t)$.

Example. The function $f(t) = \exp(-a t^2)$ is non-zero on the entire real axis. For $a > 0$ we can compute the two-sided Laplace transform by use of (1.50). We obtain

$$e^{-at^2} \circ\!\!-\!\!\bullet \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right), \quad -\infty < \Re s < \infty.$$

We can also use (1.51). Since

$$u(t) \exp[-a t^2] \circ\!\!-\!\!\bullet \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right) \operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right) = F_+(s)$$

$$u(t)\exp[-a(-t)^2] \circ\!-\!\bullet\ \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{s^2}{4a}\right)\operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right) = F_-(s)$$

thus $F_+(s) = F_-(s)$ with $c_+ = -\infty$ and $-c_- = -\infty$. Therefore we have for $c_+ < \Re s < c_-$,

$$\begin{aligned} F(s) &= F_-(-s) + F_+(s) \\ &= \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{s^2}{4a}\right)\left\{\operatorname{erfc}\left(\frac{-s}{2\sqrt{a}}\right) + \operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right)\right\} \\ &= \sqrt{\frac{\pi}{a}}\exp\left(\frac{s^2}{4a}\right) \end{aligned}$$

because the term in the curly braces is equal to 2, which results directly from the definition of the Complementary Error Function. This is an example where the domain of a two-sided Laplace transform is the entire complex plane.

Example. The following function also is non-zero on the entire real axis.

$$f(t) = e^{-a|t|} = \begin{cases} e^{-at}, & t > 0 \\ e^{at}, & t < 0 \end{cases} \qquad (a > 0),$$

$$u(t)\,f(t) \circ\!-\!\bullet\ \frac{1}{s+a} = F_+(s), \quad \Re s > -a,$$

$$u(t)\,f(-t) \circ\!-\!\bullet\ \frac{1}{s+a} = F_-(s), \quad \Re s > -a,$$

$$F(s) = F_-(-s) + F_+(s) = \frac{1}{-s+a} + \frac{1}{s+a} = -\frac{2a}{s^2 - a^2}, \quad -a < \Re s < a.$$

Therefore,

$$e^{-a|t|} \circ\!\!-\!\!\bullet\ -\frac{2a}{s^2 - a^2}, \quad -a < \Re s < a.$$

This is an example of a two-sided Laplace transform with a strip of convergence of finite width.

Example. The following function also is non-zero on the entire real axis and has the Gamma function as its two-sided image.

$$f(t) = \exp[-e^{-t}],$$

$$\exp[-e^{-t}] \circ\!\!-\!\!\bullet\ F(s) = \int_{-\infty}^{\infty} e^{-st}\exp[-e^{-t}]\,dt.$$

A change of variables $x = e^{-t} \Longleftrightarrow t = -\ln x$ gives

$$\int_{-\infty}^{\infty} e^{-st}\exp[-e^{-t}]\,dt = \int_0^{\infty} e^{-x}\,x^{s-1}\,dx = \Gamma(s), \quad \Re s > 0.$$

Therefore,

$$\exp[-e^{-t}] \circ\!\!-\!\!\bullet\ \Gamma(s), \quad \Re s > 0.$$

This is an example where the strip of convergence is a right half-plane.

Two-sided Laplace transforms and Mellin transforms (also see [Doetsch G. I (1971), II (1972)]): The two-sided Laplace transform and the *Mellin transform* are closely related. Let us make the change of variables $t = -\ln x$ in the two-sided Laplace integral.

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\,dt = \int_{\infty}^{0} e^{s \ln x} f(-\ln x)\,\frac{-1}{x}\,dx$$
$$= \int_{0}^{\infty} x^{s-1} f(-\ln x)\,dx.$$

This is the Mellin transform of the function $g(x) := f(-\ln x)$. An introduction to Mellin transforms can be found in [Davies B. (1978)]. If we denote a Mellin correspondence by

$$f(x) \overset{M}{\to} F(z) := \int_{0}^{\infty} x^{z-1} f(x)\,dx := M[f(x)](z),$$

we can say

$$f(t) \circ\!\!-\!\!\bullet\, F(s) \Longrightarrow g(x) := f(-\ln x) \overset{M}{\to} G(z) = F(z),$$
$$f(x) \overset{M}{\to} F(z) \Longrightarrow g(t) := f(e^{-t}) \circ\!\!-\!\!\bullet\, G(s) = F(s).$$

For the Mellin transformation there exists large tables of correspondences as it is the case for the ordinary Laplace transform. These tables can sometimes be used to obtain two-sided Laplace transform correspondences. For example the Mellin correspondence

$$\frac{1}{1+x} \overset{M}{\to} \frac{\pi}{\sin \pi z}, \quad 0 < \Re z < 1$$

yields immediately the two-sided Laplace transform correspondences

$$\frac{1}{1+e^{-t}} \circ\!\!-\!\!\bullet\, \frac{\pi}{\sin \pi s}, \quad 0 < \Re s < 1.$$

The two-sided Laplace transform of a right-sided function: We call a function $f(t)$ defined on the entire real axis a *right-sided function,* if there is an x such that $f(t)$ is vanishing for all $t < x$. The interval $[x, \infty)$ is called the support of $f(t)$. An original function is a special case of a right-sided function where $x = 0$. We can represent a right-sided function by $u(t-x)\,f(t)$. The two-sided Laplace transform of a right-sided function in the case where $x < 0$ is:

$$u(t-x)\,f(t) \circ\!\!-\!\!\bullet\, F(s) := \int_{-\infty}^{\infty} e^{-st}\,u(t-x)\,f(t)\,dt$$
$$= \int_{x}^{\infty} e^{-st} f(t)\,dt = \int_{x}^{0} e^{-st} f(t)\,dt + \int_{0}^{\infty} e^{-st} f(t)\,dt$$
$$= \int_{-x}^{0} e^{st} f(-t)\,d(-t) + \mathcal{L}[f(t)](s) = \int_{0}^{-x} e^{st} f(-t)\,dt + \mathcal{L}[f(t)](s)$$
$$= F_{-}(-s, -x) + \mathcal{L}[f(t)](s)$$

where

$$F_-(s, b) := \int_0^b e^{-st} f(-t)\, dt \qquad (b > 0)$$

is the finite Laplace transform of $f(-t)$. Finally, for $x < 0$,

$$u(t-x) f(t) \circ\!\!-\!\!\bullet \mathcal{L}[f(t)](s) + F_-(-s, -x) . \tag{1.52}$$

Because an ordinary Laplace transform has a right half-plane of convergence and since a finite Laplace integral is always an entire function, the strip of convergence of a two-sided image of a right-sided function is always a right half-plane.

Example. $f(t) = t^n, \quad -\infty < t < \infty.$

$$F_+(s) = \frac{n!}{s^{n+1}}, \quad \Re s > 0,$$

$$F_-(s) = (-1)^n \frac{n!}{s^{n+1}}, \quad \Re s > 0,$$

$$F(s) = F_-(-s) + F_+(s) = (-1)^n \frac{n!}{(-s)^{n+1}} + \frac{n!}{s^{n+1}} = 0$$

but $\Re s < 0$ and $\Re s > 0$! Thus there is no strip of convergence and the two-sided Laplace transform of a power or of a polynomial in t does not exist.

After these few examples we define now a class of functions for which the two-sided Laplace integral will be convergent, even absolutely convergent.

Definition. A function $f(t)$ is of *bounded exponential growth as* $t \to \infty$, if some constant σ' exists such that $\exp(-\sigma' t)\,|f(t)|$ is bounded for all t greater than some finite positive number T_1, i.e., for some positive constant M_1,

$$|f(t)| \leq M_1 e^{\sigma' t} \quad \forall\, t > T_1 > 0 . \tag{1.53}$$

Let σ_1 be the greatest lower bound of all constants σ' which satisfy (1. 53). This means that for every σ' slightly greater than σ_1 (1. 53) is satisfied, but for a σ' smaller than σ_1 (1. 53) no longer holds. We call σ_1 the *growth index of* $f(t)$ *for* $t \to \infty$.

Similarly, a function $f(t)$ is of *bounded exponential growth as* $t \to -\infty$, if some constant σ'' exists such that $\exp(-\sigma'' t)\,|f(t)|$ is bounded for all t smaller than some finite negative number T_2, i.e.,

$$|f(t)| \leq M_2 e^{\sigma'' t} \quad \forall\, t < T_2 < 0. \tag{1.54}$$

Let σ_2 be the least upper bound of all constants σ'' which satisfy (1. 54). Thus for every σ'' slightly smaller than σ_2, (1.54) is satisfied, but for a σ'' greater than σ_2, (1.54) no longer holds. We call σ_2 the *growth index of* $f(t)$ for $t \to -\infty$.

An original function $u(t) f(t)$ is always of bounded exponential growth as $t \to -\infty$ with $\sigma_2 = +\infty$.

We say that a function $f(t)$ is of *two-sided bounded exponential growth,* if it is either of bounded exponential growth as $t \to -\infty$ and as $t \to \infty$. Thus, two growth indices σ_1 and σ_2 are associated to a function of two-sided bounded exponential growth. We can have three

cases: $\sigma_2 < \sigma_1$, $\sigma_2 = \sigma_1$, $\sigma_1 < \sigma_2$. In order to be able to form a two-sided Laplace transform we need the last one of the three cases. But before we show this, let us take advantage of the fact that we are here a little bit more theoretical and let us state another assumption that is in general always made in the background of the theory of Laplace transformation.

Generally, it is always assumed that the function $f(t)$ in a Laplace integral is absolutely integrable over any finite interval, i.e.,

$$\int_a^b |f(t)|\,dt < \infty \qquad (-\infty < a < b < \infty).$$

It is beyond the scope of this book to bother about this assumption. It is a general technical hypothesis needed for the establishment of a logically correct theory of Laplace transformation.

Proposition. *A function $f(t)$ of two-sided bounded exponential growth, for which the relation $\sigma_1 < \sigma_2$ is satisfied, is two-sided Laplace transformable. The image function $F(s)$ has the strip of convergence $\sigma_1 < \Re s < \sigma_2$. Here σ_1 is the growth index of $f(t)$ for $t \to \infty$, and σ_2 is the growth index of $f(t)$ for $t \to -\infty$. Moreover, the Laplace integral is absolutely and uniformly convergent in any sub-strip $\sigma_1 < \alpha \le \Re s \le \beta < \sigma_2$.*

Proof. Let $\sigma_1 < \alpha \le \Re s \le \beta < \sigma_2$ and let α' and β' such that $\sigma_1 < \alpha' < \alpha$ and $\beta < \beta' < \sigma_2$. We then write

$$\int_{-\infty}^{\infty} e^{-st} f(t)\,dt = \int_{-\infty}^{T_2} e^{-st} f(t)\,dt + \int_{T_2}^{T_1} e^{-st} f(t)\,dt + \int_{T_1}^{\infty} e^{-st} f(t)\,dt.$$

The integral in the middle is a finite Laplace integral which exists and represents an entire holomorphic function of s in the entire complex plane. For the first integral we have the estimate

$$\begin{aligned}
&\int_{-\infty}^{T_2} |e^{-st} f(t)|\,dt \\
&= \int_{-\infty}^{T_2} e^{-\Re s\, t}\, |f(t)|\,dt \le \int_{-\infty}^{T_2} e^{-\Re s\, t}\, M_2 e^{\beta' t}\,dt \\
&= M_2 \int_{-\infty}^{T_2} e^{-(\Re s - \beta')t}\,dt = M_2 \left(\frac{e^{-(\Re s - \beta')t}}{-(\Re s - \beta')} \right)_{t \to -\infty}^{T_2} \\
&= M_2 \frac{e^{(\beta' - \Re s) T_2} - 0}{\beta' - \Re s} \le \frac{M_2}{\beta' - \beta}\,.
\end{aligned}$$

This estimate is valid for all s with $\Re s \le \beta < \sigma_2$ (remember $T_2 < 0$). This shows that the first integral has the left half-plane $\Re s \le \beta$ as strip of uniform and absolute convergence. The estimate for the last integral is similar:

$$\begin{aligned}
&\int_{T_1}^{\infty} |e^{-st} f(t)|\,dt \\
&= \int_{T_1}^{\infty} e^{-\Re s\, t}\, |f(t)|\,dt \le \int_{T_1}^{\infty} e^{-\Re s\, t}\, M_1 e^{\alpha' t}\,dt
\end{aligned}$$

$$= M_1 \int_{T_1}^{\infty} e^{-(\Re s - \alpha')t}\, dt = M_1 \left(\frac{e^{-(\Re s - \alpha')t}}{-(\Re s - a')} \right)_{T_1}^{\infty}$$

$$= M_1 \frac{0 - e^{-(\Re s - \alpha')T_1}}{-(\Re s - \alpha')} \leq \frac{M_1}{\alpha - \alpha'} .$$

This estimate is valid for all s with $\Re s \geq \alpha > \sigma_1$ (remember $T_1 > 0$). This shows that the last integral has the right half-plane $\Re s \geq \alpha$ as a strip of uniform and absolute convergence. The intersection of the left half-plane of the first integral, the entire plane of the middle integral and the right halt-plane of the last integral gives the strip of uniform and absolute convergence $\sigma_1 < \alpha \leq \Re s \leq \beta < \sigma$. □

Remark. In the case of an original function $u(t)\, f(t)$, where $f(t)$ is assumed to be of bounded exponential growth as $t \to \infty$, the condition $\sigma_1 < \Re s < \infty$ is always fulfilled and the ordinary one-sided Laplace integral converges absolutely and uniformly for all s in the right half-plane $\sigma_1 < \alpha \leq \Re s$.

There are functions which are not of bounded exponential growth and for which the Laplace integral has still a non-void strip of convergence. There are other examples of functions which have only a strip of simple convergence. In other words the class of Laplace transformable functions (or originals) is wider than the class of functions (originals) of bounded exponential growth. But the task to prove properties, for example operational rules, for the widest class of Laplace transformable functions can be difficult and needs subtle and sophisticated mathematics. It is not the aim of this book to go in this direction. For a reader interested in accurate and sophisticated formulations of theorems and their proofs, we recommend the books of Doetsch, for example [Doetsch (1976)], [Doetsch and Nader (1974)]. For the proofs of the theorems in this book we will stay at a more elementary level that consists of assuming that the functions are of bounded exponential growth.

■ 1.2.2 Operational Properties

1.2.2.1 Linearity

It is clear that the two-sided Laplace transformation is a linear operator as the ordinary Laplace transformation. The two-sided image of a linear combination of functions is the linear combination of the individual images.

1.2.2.2 Integration Rules

The integration rule for the two-sided Laplace transformation splits into two parts.

Proposition. *If* $f(t) \circ\!\!-\!\!\bullet\, F(s)$ *for* $\sigma_1 < \Re s < \sigma_2$, *then*

$$\int_{-\infty}^{t} f(x)\, dx \circ\!\!-\!\!\bullet\, \frac{1}{s} F(s), \qquad \max\{\sigma_1, 0\} < \Re s < \sigma_2, \tag{1.55}$$

$$\int_{t}^{\infty} f(x)\, dx \circ\!\!-\!\!\bullet\, -\frac{1}{s} F(s), \qquad \sigma_1 < \Re s < \min\{\sigma_2, 0\} . \tag{1.56}$$

An outline of a *proof* is:

If s is restricted to $\max\{\sigma_1, 0\} < \Re s < \sigma_2$, then $\Re s > 0$ and we have

$$\int_{-\infty}^{t} f(x)\,dx \;\square\!\!-\!\!\blacksquare\; \int_{-\infty}^{\infty} e^{-st} \int_{-\infty}^{t} f(x)\,dx\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{t} e^{-st} f(x)\,dx\,dt$$

$$= \int_{-\infty}^{\infty}\int_{x}^{\infty} e^{-st} f(x)\,dt\,dx = \int_{-\infty}^{\infty} f(x) \int_{x}^{\infty} e^{-st}\,dt\,dx$$

where we interchanged the order of integration in the double integral. Then,

$$\int_{-\infty}^{\infty} f(x) \int_{x}^{\infty} e^{-st}\,dt\,dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{-st}}{-s} \right)_{t=x}^{t\to\infty} dx = \int_{-\infty}^{\infty} f(x)\,\frac{0 - e^{-sx}}{-s}\,dx$$

since $\Re s > 0$. Now

$$\int_{-\infty}^{\infty} f(x)\,\frac{0 - e^{-sx}}{-s}\,dx = \frac{1}{s}\int_{-\infty}^{\infty} e^{-sx} f(x)\,dx = \frac{1}{s} F(s).$$

The proof for the second rule is similar. □

1.2.2.3 The Differentiation Rule

In the following we assume that the function $f(t)$ is differentiable for $t < a$ and for $t > a$ for some a, and that the two-sided Laplace transform of $f'(t)$ exists. We allow $f(t)$ to have a discontinuity of the first kind at $t = a$.

Proposition. *Let* $f(t) \;\square\!\!-\!\!\blacksquare\; F(s)$ *for* $\sigma_1 < \Re s < \sigma_2$ *and* $f(t)$ *be continuous on the real axis, except at* $t = a$ *where* $f(t)$ *may have a discontinuity of the first kind. Then*

$$f'(t) \;\square\!\!-\!\!\blacksquare\; s\,F(s) - e^{-as}\{f(a+) - f(a-)\},\quad \sigma_1 < \Re s < \sigma_2. \tag{1.57}$$

For an outline of proof, we assume that $f(t)$ is of bounded exponential growth with the two growth indices σ_1 and σ_2, satisfying the condition $\sigma_1 < \sigma_2$.

For $\sigma_1 < \Re s < \sigma_2$ we have

$$\lim_{t\to-\infty} e^{-st} f(t) = 0 = \lim_{t\to\infty} e^{-st} f(t).$$

For, if α and β are such that $\sigma_1 < \alpha < \Re s < \beta < \sigma_2$, we have the estimates (1.53) and (1. 54)

$$|f(t)| \le M_1 e^{\alpha t} \quad \forall\, t > T_1 > 0,$$
$$|f(t)| \le M_2 e^{\beta t} \quad \forall\, t < T_2 < 0,$$

$$|e^{-st} f(t)| = e^{-\Re s\, t}\ |f(t)| \le M_2 e^{-(\Re s - \beta)t}$$
$$= M_2 e^{(\beta - \Re s)t} \to 0$$

for $t \to -\infty$, because $\beta - \Re s > 0$. Similarly for $t \to \infty$.

Now

$$
\begin{aligned}
f'(t) \circ\!\!-\!\!\bullet\, & \int_{-\infty}^{a} e^{-st} f'(t)\,dt + \int_{a}^{\infty} e^{-st} f'(t)\,dt \\
&= (e^{-st} f(t))\,|_{t\to-\infty}^{a} - \int_{-\infty}^{a} (-s)\, e^{-st} f(t)\,dt \\
&\quad + (e^{-st} f(t))\,|_{a}^{t\to\infty} - \int_{a}^{\infty} (-s)\, e^{-st} f(t)\,dt \\
&= e^{-sa} f(a-) + s \int_{-\infty}^{a} e^{-st} f(t)\,dt - e^{-sa} f(a+) + s \int_{a}^{\infty} e^{-st} f(t)\,dt \\
&= - e^{-sa} \{f(a+) - f(a-)\} + s \int_{-\infty}^{\infty} e^{-st} f(t)\,dt \\
&= s F(s) - e^{-sa} \{f(a+) - f(a-)\}. \quad \square
\end{aligned}
$$

If $f(t)$ has discontinuities of the first kind at $t = a_1, a_2, \ldots, a_n$, then the differentiation rule reads

$$
f'(t) \circ\!\!-\!\!\bullet\, s F(s) - \sum_{k=1}^{n} e^{-a_k s} \{f(a_k+) - f(a_k-)\}.
$$

Observe that in the special case of an original $u(t) f(t)$, where $f(t)$ is assumed to be continuous for all $t > 0$, the only discontinuity is at $t = 0$. Furthermore, we have $f(0-) = 0$ and the above differentiation rule becomes

$$
\begin{aligned}
& u(t) f'(t) \circ\!\!-\!\!\bullet\, s F(s) - e^{-0s} \{f(0+) - 0\} \\
&= s F(s) - f(0+),
\end{aligned}
$$

which is the differentiation rule for the ordinary Laplace transformation.

If the function $f(t)$ is differentiable for all t, and therefore continuous for all t, the differentiation rule becomes simply

$$
f(t) \circ\!\!-\!\!\bullet\, F(s) \Longrightarrow f'(t) \circ\!\!-\!\!\bullet\, s F(s), \qquad \sigma_1 < \Re s < \sigma_2 \tag{1.58}
$$

provided that $f'(t)$ is two-sided Laplace transformable.

Example.

$$
e^{-at^2} \circ\!\!-\!\!\bullet\, \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right), \quad -\infty < \Re s < \infty,\ a > 0
$$

implies

$$
-2at e^{-at^2} \circ\!\!-\!\!\bullet\, s \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right), \quad -\infty < \Re s < \infty
$$

and by linearity

$$t e^{-at^2} \circ\!\!-\!\!\bullet\; -\frac{s}{2a}\sqrt{\frac{\pi}{a}}\exp\left(\frac{s^2}{4a}\right), \quad -\infty < \Re s < \infty.$$

Remark. For functions with bounded exponential growth, the proof above shows that the strip of convergence remains the same when the differentiation rule is applied. For more general functions, this need no more be the case. The strip of convergence may become wider or narrower. In order to apply the differentiation rule

$$f(t) \circ\!\!-\!\!\bullet\; F(s) \Longrightarrow f^{(n)}(t) \circ\!\!-\!\!\bullet\; s^n F(s),$$

in a more general case, under the hypothesis that $f^{(n)}(t)$ is two-sided Laplace transformable, the following hypotheses should be satisfied:

$f(-\infty) = f'(-\infty) = \cdots = f^{(n-1)}(-\infty) = 0$ and s is restricted to $\Re s > 0$, respectively,

$f(\infty) = f'(\infty) = \cdots = f^{(n-1)}(\infty) = 0$, if s is restricted to $\Re s < 0$.

For a proof of this, see [Doetsch G. II (1972)].

1.2.2.4 The Image Shift Rule

This rule is similar to the one for the ordinary Laplace transformation, but the strip of convergence is translated.

Proposition. *If $F(s)$ is the two-sided image of $f(t)$ with the strip of convergence $\sigma_1 < \Re s < \sigma_2$, then*

$$e^{at} f(t) \circ\!\!-\!\!\bullet\; F(s-a), \quad \sigma_1 + \Re a < \Re s < \sigma_2 + \Re a, \tag{1.59}$$

Proof.

$$F(s-a) = \int_{-\infty}^{\infty} e^{-(s-a)t} f(t)\,dt, \quad \sigma_1 < \Re(s-a) < \sigma_2$$

which equals

$$\int_{-\infty}^{\infty} e^{-st} e^{at} f(t)\,dt \bullet\!\!-\!\!\circ e^{at} f(t), \; \sigma_1 + \Re a < \Re s < \sigma_2 + \Re a. \; \square$$

1.2.2.5 The t-Shift Rule

In contrast to the situation for ordinary Laplace transformation, right and left shifts of the function $f(t)$ need not be distinguished. The shift constant a may be positive or negative.

Proposition. *If $F(s)$ is the two-sided image of $f(t)$ with the strip of convergence $\sigma_1 < \Re s < \sigma_2$, then*

$$f(t-a) \circ\!\!-\!\!\bullet\; e^{-as} F(s), \quad \sigma_1 < \Re s < \sigma_2 \tag{1.60}$$

Proof.

$$f(t-a) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} e^{-st} f(t-a)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-s(t+a)} f(t)\,dt = e^{-as} \int_{-\infty}^{\infty} e^{-st} f(t)\,dt = e^{-as} F(s). \quad \square$$

1.2.2.6 The Similarity Rule

Again in contrast to the situation for ordinary Laplace transformation, the scale change factor a may be positive or negative. Observe the change of the strip of convergence.

Proposition. *If $F(s)$ is the two-sided image of $f(t)$ with the strip of convergence $\sigma_1 < \Re s < \sigma_2$, then*

$$f(at) \circ\!\!-\!\!\bullet \frac{1}{|a|} F\left(\frac{s}{a}\right), \qquad \begin{cases} a\sigma_1 < \Re s < a\sigma_2, & a > 0 \\ a\sigma_2 < \Re s < a\sigma_1, & a < 0 \end{cases} . \tag{1.61}$$

Proof for the case that $a < 0$:

$$f(at) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} e^{-st} f(at)\,dt = \int_{\infty}^{-\infty} e^{-sx/a} f(x)\,d(x/a);$$

by a change of variables $t = x/a$. Then

$$\int_{\infty}^{-\infty} e^{-sx/a} f(x)\,d(x/a) = -\frac{1}{a} \int_{-\infty}^{\infty} e^{-sx/a} f(x)\,dx$$

$$= \frac{1}{-a} \int_{-\infty}^{\infty} e^{-s/ax} f(x)\,dx = \frac{1}{|a|} \int_{-\infty}^{\infty} e^{-s/ax} f(x)\,dx$$

$$= \frac{1}{|a|} F\left(\frac{s}{a}\right), \quad \sigma_1 < \Re\left(\frac{s}{a}\right) < \sigma_2.$$

As $a < 0$, the last inequality becomes $a\sigma_1 > \Re s > a\sigma_2$. $\square$

Example. Two-sided Laplace transform of the Gaussian probability density. The normalized Gaussian probability density, corresponding to mean zero and to variance 1 is given by

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

By use of linearity and the correspondence

$$e^{-at^2} \circ\!\!-\!\!\bullet \sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right)$$

we find

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \circ\!\!-\!\!\bullet e^{-s^2/2}.$$

The Gaussian probability density corresponding to mean μ and variance σ^2 is given by

$$f(t) := \frac{1}{\sigma}\,\varphi\Big(\frac{t-\mu}{\sigma}\Big) \quad (\sigma > 0)\,.$$

By applying first the similarity rule, then the t-shift rule and finally linearity, we obtain

$$\varphi(t) := \frac{1}{\sigma}\,\varphi\Big(\frac{t-\mu}{\sigma}\Big) \circ\!-\!\bullet\; \Phi(s) = \frac{1}{\sigma}\,\sigma\, e^{-\mu s}\; e^{-(\sigma s)^2/2}$$
$$= \exp\Big(-\frac{\sigma^2}{2}\,s^2 - \mu s\Big).$$

The strip of convergence is the entire plane.

1.2.2.7 The t-Multiplication Rule

This rule is exactly the same as the one for the ordinary Laplace transformation. It reads as follows:

Proposition. *If $F(s)$ is the two-sided image of $f(t)$ with the strip of convergence $\sigma_1 < \Re s < \sigma_2$, then*

$$t^n f(t) \circ\!-\!\bullet\; (-1)^n F^{(n)}(s), \quad \sigma_1 < \Re s < \sigma_2. \tag{1.62}$$

One obtains this rule by differentiating n times under the integral sign of the Laplace integral.

1.2.2.8 The Convolution Rule

The *convolution of two functions* $f(t)$ and $g(t)$ defined on the entire real axis is defined as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t-x)\, g(x)\, \mathbf{d}x. \tag{1.63}$$

Observe that, if the two functions are originals, we obtain the defining expression of Section 1.1.2.9, i.e.,

$$(u(t)\, f(t) * u(t)\, f(t))(t) = \int_0^t f(t-x)\, g(x)\, \mathbf{d}x\,.$$

Again, one can prove that the convolution product is commutative, distributive and associative:

$$f * g = g * f\,, \quad f * (g + h) = f * g + f * h\,,$$
$$f * (g * h) = (f * g) * h = f * g * h,$$

where $f * g * h$ is

$$(f * g * h)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t - x_1 - x_2)\, g(x_1)\, h(x_2)\, \mathbf{d}x_1\, \mathbf{d}x_2.$$

The Convolution Theorem is analogous to the one for the one-sided Laplace transformation.

Convolution Theorem. *If we have $f(t) \circ\!-\!\bullet\; F(s)$, $\sigma_1 < \Re s < \sigma_2$ and $g(t) \circ\!-\!\bullet\; G(s)$, $\rho_1 < \Re s < \rho_2$, then*

$$(f * g)(t) \;\square\!\!-\!\!\blacksquare\; F(s)\,G(s), \quad \max\{\sigma_1,\ \rho_1\} < \Re s < \min\{\sigma_2, \rho_2\}. \tag{1.64}$$

We omit the proof, which again is based on a change of variables in a 2-dimensional integral.

Example. In the above integral for $f * g * h$, we take $f(t) = u(t)$ and obtain, by the convolution theorem with $g(t) \;\square\!\!-\!\!\blacksquare\; G(s)$, $\sigma_1 < \Re s < \sigma_2$ and $h(t) \;\square\!\!-\!\!\blacksquare\; H(s)$, $\rho_1 < \Re s < \rho_2$,

$$(u * g * h)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t - x_1 - x_2)\, g(x_1)\, h(x_2)\, dx_1\, dx_2$$

$$= \iint_{x_1 + x_2 \le t} g(x_1)\, h(x_2)\, dx_1\, dx_2 \;\square\!\!-\!\!\blacksquare\; \frac{1}{s}\, G(s)\, H(s)$$

$$\max\{0, \sigma_1,\ \rho_1\} < \Re s < \min\{\infty, \sigma_2, \rho_2\} = \min\{\sigma_2, \rho_2\}.$$

Example. Euler's Beta Function. Starting from the correspondences

$$\frac{1}{s^a} \;\blacksquare\!\!-\!\!\square\; u(t)\,\frac{t^{a-1}}{\Gamma(a)}, \quad \frac{1}{s^b} \;\blacksquare\!\!-\!\!\square\; u(t)\,\frac{t^{b-1}}{\Gamma(b)}, \quad \Re a > 0,\ \Re b > 0$$

we have, by the Convolution Theorem,

$$\frac{1}{s^a\, s^b} \;\blacksquare\!\!-\!\!\square\; \int_{-\infty}^{\infty} u(t - x)\,\frac{(t-x)^{a-1}}{\Gamma(a)}\, u(x)\,\frac{x^{b-1}}{\Gamma(b)}\, dx$$

$$= u(t)\int_0^t \frac{(t-x)^{a-1}}{\Gamma(a)}\,\frac{x^{b-1}}{\Gamma(b)}\, dx$$

$$= \frac{u(t)}{\Gamma(a)\,\Gamma(b)}\int_0^t (t-x)^{a-1}\, x^{b-1}\, dx\,.$$

On the other hand,

$$\frac{1}{s^a\, s^b} = \frac{1}{s^{a+b}} \;\blacksquare\!\!-\!\!\square\; u(t)\,\frac{t^{a+b-1}}{\Gamma(a+b)}.$$

Therefore,

$$u(t)\,\frac{t^{a+b-1}}{\Gamma(a+b)} = \frac{u(t)}{\Gamma(a)\,\Gamma(b)}\int_0^t (t-x)^{a-1}\, x^{b-1}\, dx.$$

Or, if we set $t = 1$, we obtain

$$\frac{1}{\Gamma(a+b)} = \frac{1}{\Gamma(a)\,\Gamma(b)}\int_0^1 (1-x)^{a-1}\, x^{b-1}\, dx.$$

The function

$$B(a, b) := \int_0^1 (1-x)^{a-1}\, x^{b-1}\, dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \quad \Re a > 0,\ \Re b > 0 \tag{1.65}$$

is called *Euler's Beta function.*

Example. The Reflection Formula for the Gamma Function. We start with the correspondence

$$\exp[-e^{-t}] \circ\!\!-\!\!\bullet\ \Gamma(s), \quad \Re s > 0$$

and apply the similarity rule with $a = -1$ to obtain

$$\exp[-e^{t}] \circ\!\!-\!\!\bullet\ \Gamma(-s), \quad \Re s < 0.$$

By the image shift rule,

$$e^{t}\exp[-e^{t}] \circ\!\!-\!\!\bullet\ \Gamma(-(s-1)) = \Gamma(-s+1) \quad \Re s < 1.$$

Now we use the convolution theorem.

$$\Gamma(s)\,\Gamma(-s+1) \bullet\!\!-\!\!\circ \int_{-\infty}^{\infty} \exp[-e^{-(t-x)}]\, e^{x} \exp[-e^{x}]\, dx$$

$$= \int_{-\infty}^{\infty} \exp[x - e^{-t}e^{x} - e^{x}]\, dx, \quad 0 < \Re s < 1.$$

A change of variables $u = e^{x}$ yields

$$\Gamma(s)\,\Gamma(-s+1) \bullet\!\!-\!\!\circ \int_{0}^{\infty} e^{-u[\exp(-t)+1]}\, du = \frac{1}{1+e^{-t}} \circ\!\!-\!\!\bullet \frac{\pi}{\sin \pi s}.$$

Therefore, we obtain the relation, called the *Reflection Formula for the Gamma Function,*

$$\Gamma(s)\,\Gamma(-s+1) = \frac{\pi}{\sin \pi s} \tag{1.66}$$

which we derived under the restriction $0 < \Re s < 1$. The relation can be extended to all s by analytic continuation.

Problem 1.26. Establish the correspondence

$$\exp[-(at^2 + bt + c)] \circ\!\!-\!\!\bullet\ \sqrt{\frac{\pi}{a}}\exp\left[\frac{(s+b)^2}{4a} - c\right], \quad -\infty < \Re s < \infty, \quad a > 0.$$

Problem 1.27. By use of the Mellin transform correspondence

$$e^{\pm it} \overset{M}{\rightarrow} e^{\pm i(\pi/2)z}\,\Gamma(z), \quad 0 < \Re z < 1,$$

establish the correspondences

$$\exp[\pm i e^{-t}] \circ\!\!-\!\!\bullet\ \exp\left[\pm \frac{\pi}{2} i s\right]\Gamma(s),$$

$$\cos(e^{-t}) \circ\!\!-\!\!\bullet\ \cos\left(\frac{\pi}{2} s\right)\Gamma(s),$$

$$\sin(e^{-t}) \circ\!\!-\!\!\bullet\ \sin\left(\frac{\pi}{2} s\right)\Gamma(s)$$

with $0 < \Re s < 1$. The last relation even holds for $-1 < \Re s < 1$.

Problem 1.28. Use the similarity rule to establish

$$\frac{1}{1+e^{t}} \circ\!\!-\!\!\bullet\ -\frac{\pi}{\sin \pi s} \quad (-1 < \Re s < 0).$$

Problem 1.29. Use the image shift rule, the Convolution Theorem, and appropriate changes of variables to establish the *Duplication Formula for the Gamma Function*

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right) = \frac{2\sqrt{\pi}}{4^s}\,\Gamma(2s). \tag{1.67}$$

■ 1.2.3 The Dirac Impulse and its Laplace Transform

1.2.3.1 A Heuristic Introduction of the Dirac Impulse and its Laplace Transform

We consider the sequence of functions

$$\delta_n(t-1) := \frac{n^{n+1}}{n!}\,u(t)\,t^n\,e^{-nt}.$$

It is easy to see that the peak (maximum value) of $\delta_n(t-1)$ is at $t=1$ for all n. Let us take the two-sided Laplace transform of each member of the sequence. By the image shift rule, we get

$$\delta_n(t-1) \circ\!\!-\!\!\bullet\; \Delta_n(s)$$
$$= \frac{n^{n+1}}{n!}\,\frac{n!}{(s+n)^{n+1}} = \left(\frac{n}{s+n}\right)^{n+1} = \frac{1}{(1+\frac{s}{n})^n}\left(\frac{n}{s+n}\right).$$

We can deduct two things. First,

$$\Delta_n(0) = \left(\int_{-\infty}^{\infty} e^{-st}\,\delta_n(t-1)\,dt\right)_{s=0} = \int_{-\infty}^{\infty}\delta_n(t-1)\,dt = 1$$

for all n. This means that for all n, we have unit area under the curve of each $\delta_n(t-1)$. Furthermore, by Stirling's Formula,

$$n! \approx \sqrt{\frac{2\pi}{n}}\,n^{n+1}\,e^{-n} \qquad (n\to\infty),$$

we have for the peak at $t=1$,

$$\delta_n(0) = \frac{n^{n+1}}{n!}\,e^{-n} = \sqrt{\frac{2\pi}{n}}\,n^{n+1}\,e^{-n}\,\sqrt{\frac{n}{2\pi}}\,\frac{1}{n!}$$
$$\approx n!\,\sqrt{\frac{n}{2\pi}}\,\frac{1}{n!} = \sqrt{\frac{n}{2\pi}}$$

as $n\to\infty$. This shows that as n increases, the graphs grow higher, but their widths become narrower due to the constant unit area under the curves. Fig.1.11 shows the graphs of $\delta_n(t-1)$ for increasing values of n.

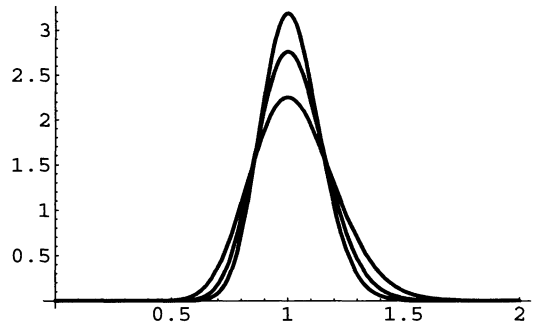


Fig.1.11: $\delta_n(t-1)$ for $n = 32$, 48, and 64.

Second,

$$\lim_{n\to\infty} \Delta_n(s) = \lim_{n\to\infty} \frac{1}{(1+\frac{s}{n})^n} \left(\frac{n}{s+n}\right) = \frac{1}{e^s} 1 = e^{-s} = : \Delta(s).$$

This leads to the following conclusion: The sequence $\delta_n(t-1)$ converges to something that is "infinitely high", "infinitely narrow" and still has "unit area under the curve". Moreover, this entity has $\Delta(s) = e^{-s}$ as its image function. This object, which cannot be considered as an ordinary function, as we will see below, is called the *Dirac Impulse located at t = 1* and is denoted by $\delta(t-1)$. It must be considered as some sort of a generalized function with the correspondence

$$\delta(t-1) \circ\!\!-\!\!\bullet\; e^{-s}.$$

It is seen that $\delta_n(t-1)$ is $n-1$ times continuously differentiable on the real axis and that $\phi_n(t) := \delta_n(t-1)$ satisfies $\phi_n^{(m)}(0+) = 0$, provided that $m < n$. Thus, by the differentiation rule we have, for sufficiently large n,

$$\delta_n^{(m)}(t-1) \circ\!\!-\!\!\bullet\; s^m \Delta_n(s).$$

Fig. 1.12 shows the first derivatives of $\delta_n(t-1)$ for increasing n.

As $n \to \infty$ one has the limit relation

$$\delta^{(m)}(t-1) \circ\!\!-\!\!\bullet\; s^m e^{-s}.$$

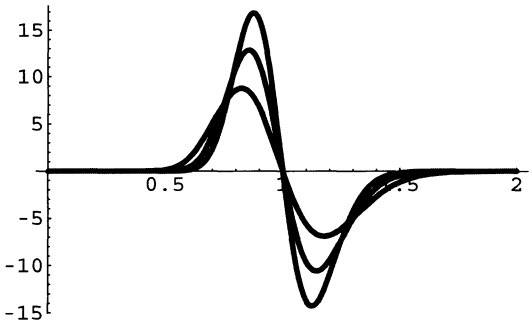


Fig.1.12: $\delta_n'(t-1)$ for $n = 32$, 48 and 64.

By the original shift rule with the shift $a-1$, we obtain

$$\delta_n(t-a+1-1) = \frac{n^{n+1}}{n!}\, u(t-a+1)\,(t-a+1)^n\, e^{-n(t-a+1)}$$

$$\square\square - \blacksquare\ e^{-(a-1)s}\,\Delta_n(s),$$

or $\delta_n(t-a) \square\square - \blacksquare\ e^{-(a-1)s}\,\Delta_n(s)$,

which as $n \to \infty$, and $\Delta_n(s) \to e^{-s}$ tends to

$$\delta(t-a) \square\square - \blacksquare\ e^{-as}. \tag{1.68}$$

Similarly, we find

$$\delta^{(m)}(t-a) \square\square - \blacksquare\ s^m\, e^{-as} \tag{1.69}$$

For $a = 0$ we have

$$\delta(t) \square\square - \blacksquare\ 1 \tag{1.70}$$

$$\delta^{(m)}(t) \square\square - \blacksquare\ s^m. \tag{1.71}$$

Let us recapitulate: We have introduced a new generalized function $\delta(t-a)$, called the Dirac Impulse at $t = a$ and its "derivatives" $\delta^{(m)}(t-a)$. This Dirac Impulse has a "peak" at $t = a$, infinitely high and infinitely narrow but with unit area under the curve:

$$\begin{aligned} &\delta(t-a) = 0, \quad t \neq a \\ &\delta(t-a) = \infty, \quad t = a \\ &\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1. \end{aligned} \tag{1.72}$$

Furthermore, we found the correspondences

$$\delta(t-a) \square\square - \blacksquare\ e^{-as}, \quad \delta^{(m)}(t-a) \square\square - \blacksquare\ s^m\, e^{-as}. \tag{1.73}$$

This gives an intuitive idea of what the Dirac Impulse and its derivatives are. Unfortunately, it cannot be put on a firm mathematical basis in this way. The three properties (1.72), for example, are contradictory. If $\delta(t-a)$ were an ordinary function satisfying the first two properties, its integral would be zero, for any definition of the integral.

You can turn it around either way, it will not be possible to establish a mathematically and logically correct theory of the Dirac Impulse in the manner sketched above. The problem is more profound and needs a whole new theory that generalizes the notion of functions. There are several approaches for doing so. The best known currently is Laurent Schwartz's Theory of Distribution [Schwartz (1965)], [Zemanian I (1987)], [Zemanian II (1987)]. Another interesting approach is the Theory of Sato's Hyperfunctions [Imai (1992].

We introduced the Dirac Impulse and its derivatives as the limit of the sequence of functions

$$\delta_n(t-1) := \frac{n^{n+1}}{n!}\, u(t)\, t^n\, e^{-nt}$$

and of their derivatives. By repeating the same steps, with the above sequence replaced by a sequence of Gaussian densities with standard deviation equal to $1/n$ and mean a,

$$\delta_n(t-a) := \frac{n}{\sqrt{2\pi}} \exp\left[-\frac{n}{2}(t-a)^2\right] \circ\!\!-\!\!\bullet \operatorname{Exp}\left[\frac{s^2}{2n^2} - as\right],$$

the same relations

$$\delta(t-a) \circ\!\!-\!\!\bullet e^{-as}, \quad \delta^{(m)}(t-a) \circ\!\!-\!\!\bullet s^m e^{-as}.$$

are obtained, as $n \to \infty$. There are many other sequences of functions that can be used in order to produce the above relations. See for example [Van der Pol and Bremmer (1987)]. But the fundamental objections are not removed in any of these classical approaches.

Remark. The approach of Section 1.2.3.1 with the sequence of original functions

$$\delta_n(t-1) := \frac{n^{n+1}}{n!} u(t)\, t^n e^{-nt}, \quad t > 0$$

can be used in the context of the one-sided Laplace transformation until the step where we invoke the original shift translation rule. But the shift has to be restricted to $a - 1 \geq 0$. If we used a left shift with $a < 1$, a part of the graph of $\delta_n(t-a)$ would be cut away and the area under the curve would become smaller than unity, since the left shifted original is in fact $u(t)\,\delta_n(t-a)$. Also $\varphi(0+) = \delta_n(-a)$ becomes different from zero with the effect that the differentiation rule is no more a simple multiplication by s. Besides the difficulty with the derivative, we would, in the case of $a = 0$, wind up with the limit relation $\delta(t) \circ\!\!-\!\!\bullet 1/2$. But this is not what is wanted. Some authors try to bypass these difficulties by defining the one–sided Laplace integral by

$$F(s) = \int_{0-}^{\infty} e^{-st} f(t)\, dt$$

in order to get hold of the entire Dirac Impulse at $t = 0$ rather than just half of it. But this ad hoc repair is not satisfactory and leaves other difficulties untouched. In order to avoid such difficulties the Dirac Impulse should be introduced in the context of the two-sided Laplace integral.

1.2.3.2 Further Relations involving the Dirac Impulse

By use of

$$\delta_n(t-1) := \frac{n^{n+1}}{n!} u(t)\, t^n e^{-nt} \circ\!\!-\!\!\bullet \left(\frac{n}{s+n}\right)^{n+1}$$

and the original shift rule we obtain

$$\delta_n(t-a) = \delta_n(t-(a-1)-1) \circ\!\!-\!\!\bullet e^{-(a-1)s}\left(\frac{n}{s+n}\right)^{n+1},$$

$$\delta_n(t-b) = \delta_n(t-(b-1)-1) \circ\!\!-\!\!\bullet e^{-(b-1)s}\left(\frac{n}{s+n}\right)^{n+1}.$$

The Convolution Theorem implies

$$\delta_n(t-a) * \delta_n(t-b) \circ\!\!-\!\!\bullet e^{-(a-1)s}\, e^{-(b-1)s}\left(\frac{n}{s+n}\right)^{n+1}\left(\frac{n}{s+n}\right)^{n+1},$$

and the expression on the right-hand side can be written as

$$e^{-(a-1)s}\, e^{-(b-1)s} \left(\frac{n}{s+n}\right)^{n+1} \left(\frac{n}{s+n}\right)^{n+1}$$
$$= e^{-(a+b-2)s} \left(\frac{n}{s+n}\right)^{2n+2}$$
$$= e^{-(a+b-2)s} \left(\frac{n}{s+n}\right)^{2} \left(\left[\frac{1}{(1+s/n)^n}\right]^n\right)^2 \longrightarrow e^{-(a+b-2)s}\, \frac{1}{e^{2s}}$$
$$= e^{-(a+b)s} \quad (n \to \infty).$$

The last expression is the image of $\delta(t-a-b)$. Hence

$$\delta(t-a) * \delta(t-b) = \delta(t-a-b). \tag{1.74}$$

Next, let $f(t) \circ\!\!-\!\!\bullet\, F(s)$ and

$$\delta_n(t) = \circ\!\!-\!\!\bullet\, e^s \left(\frac{n}{s+n}\right)^{n+1}.$$

By the convolution theorem we have

$$\delta_n(t) * f(t) \circ\!\!-\!\!\bullet\, e^s \left(\frac{n}{s+n}\right)^{n+1} F(s) = e^s \left(\frac{n}{s+n}\right) \frac{1}{(1+s/n)^n}\, F(s).$$

As $n \to \infty$ we obtain the limit

$$\delta(t) * f(t) \circ\!\!-\!\!\bullet\, e^s\, 1\, \frac{1}{e^s}\, F(s) = F(s) \bullet\!\!-\!\!\circ\, f(t),$$

thus

$$\delta(t) * f(t) = f(t). \tag{1.75}$$

This shows that the Dirac Impulse acts as a unit element with respect to the convolution product. The above relation is often written as

$$\int_{-\infty}^{\infty} \delta(t-x)\, f(x)\, dx = f(t).$$

Also, by the similarity rule with $a = -1$, we have

$$\delta_n(-t) = \circ\!\!-\!\!\bullet\, e^{-s} \left(\frac{n}{-s+n}\right)^{n+1} = e^{-s} \left(\frac{n}{-s+n}\right) \frac{1}{(1-s/n)^n}$$
$$\longrightarrow e^{-s}\, 1\, \frac{1}{e^{-s}} = 1.$$

Therefore the Dirac Impulse is an even generalized function:

$$\delta(-t) = \delta(t). \tag{1.76}$$

Thus, the previous integral can also be written as

$$\int_{-\infty}^{\infty} \delta(x-t)\, f(x)\, dx = f(t).$$

Again let $f(t) \circ\!\!-\!\!\bullet\, F(s)$ and

$$\delta_n(t) = \circ\!\!-\!\!\bullet\, e^s \left(\frac{n}{s+n}\right)^{n+1}.$$

By the differentiation rule

$$\delta_n{}^{(k)}(t) = \circ\!\!-\!\!\bullet\, s^k e^s \left(\frac{n}{s+n}\right)^{n+1},$$

and by the Convolution Theorem we obtain

$$\delta_n{}^{(k)}(t) * f(t) \circ\!\!-\!\!\bullet\, s^k e^s \left(\frac{n}{s+n}\right)^{n+1} F(s) = e^s \left(\frac{n}{s+n}\right) \frac{1}{(1+s/n)^n} \{s^k F(s)\}.$$

The limit $n \to \infty$ yields

$$\delta^{(k)}(t) * f(t) \circ\!\!-\!\!\bullet\, e^s \frac{1}{e^s} \{s^k F(s)\} = s^k F(s) \bullet\!\!-\!\!\circ\, f^{(k)}(t),$$

therefore

$$\delta^{(k)}(t) * f(t) = f^{(k)}(t). \tag{1.77}$$

Next, we consider the following sequence and use the multiplication rule.

$$t\,\delta_n(t) \circ\!\!-\!\!\bullet\, -\frac{d}{ds}\left\{e^s \left(\frac{n}{s+n}\right)^{n+1}\right\} = -e^s \left(\frac{n}{s+n}\right)^{n+1} \frac{s-1}{s+n}$$
$$= -e^s \left(\frac{n}{s+n}\right) \frac{1}{(1+s/n)^n} \frac{s-1}{s+n}.$$

The last expression tends to

$$-e^s \left(\frac{n}{s+n}\right) \frac{1}{(1+s/n)^n} \frac{s-1}{s+n} \longrightarrow -e^s\, 1\, \frac{1}{e^s}\, 0 = 0, \quad n \to \infty.$$

This yields the relation

$$t\,\delta(t) = 0. \tag{1.78}$$

1.3 Ordinary Linear Differential Equations

■ 1.3.1 Differential Equations with Constant Coefficients

1.3.1.1 Scalar Differential Equations of Order n

Let us introduce the *linear differential operator of order n*

$$L := a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{d}{dt} + a_0 , \qquad (a_n \neq 0).$$

The operator L acts on an original function $y(t)$ as follows:

$$L\,y(t) := a_n \frac{d^n\, y}{dt^n} + a_{n-1} \frac{d^{n-1}\, y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0\, y(t).$$

A linear scalar differential equation of order n and with constant coefficients can be written in the form

$$L\,y(t) = f(t). \tag{1.79}$$

The given function $f(t)$, is called the *excitation, solicitation, non-homogeneous term* or the *input* function. The solution $y(t)$ is called the *response, answer* or *output* function. As in system theory, we think of (1.79) as a mathematical model of a physical system represented by the block diagram (Fig.1.13):

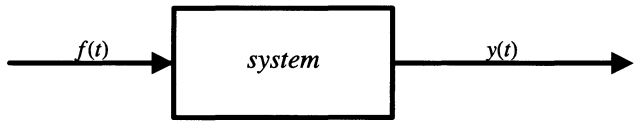


Fig.1.13: Block diagram of an input-output system.

A system is called *linear* if a linear combination of input functions produces an output that is the linear combination of the individual input functions (Fig.1.14):

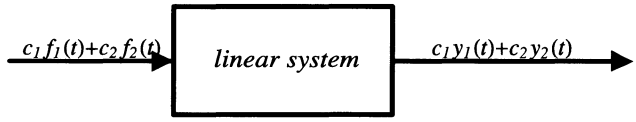


Fig.1.14: A linear system.

A system is called *time-invariant* or *fixed* if a delayed input $f(t-a)$ produces a corresponding delayed output $y(t-a)$ for any constant a.

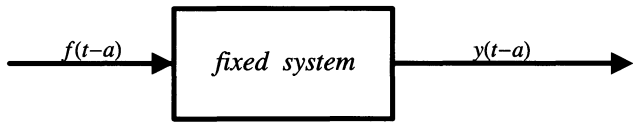


Fig.1.15: A fixed linear system.

Time-invariance means that the choice of the initial instant t_0 for a measurement or observation is irrelevant for the behavior of the system. No matter whether the system is observed today or tomorrow, identical inputs will always produce identical outputs. Thus for time-invariant systems, one in general chooses the initial instant $t_0 = 0$. If a system is modeled by a linear differential equation with constant coefficients, then it is a linear time-invariant system. The system modeled by (1.79) is a special case in terms of the input function. In applications, often the right-hand side $f(t)$ is replaced by

$$R\,f(t) := \left(b_0 + b_1 \frac{df}{dt} + \cdots + b_m \frac{d^m f}{dt^m}\right),$$

i.e. a linear differential operator R of order m acts on the input function. In this case the differential equation is

$$L\,y(t) = R\,f(t). \tag{1.80}$$

The solution of a scalar differential equation of order n is not unique. The general solution consists of a family of functions depending on n arbitrary parameters (integration constants). In order to obtain a unique solution, in the sense that a given input function $f(t)$ produces a uniquely determined output $y(t)$, we have to impose additional conditions on the solution. If the conditions are all related to the same initial time $t_0 = 0$, we call it an *initial value problem.* Thus an initial value problem consists of

$$\begin{aligned} &L\,y(t) = R\,f(t)\ , \\ &y(0+) = y_0\ , \\ &y'\,(0+) = y_0{}'\ , \\ &y''\,(0+) = y_0{}'' \ \ldots\ldots\ \ldots\ \ y^{(n-1)}(0+) = y_0{}^{(n-1)}, \end{aligned} \tag{1.81}$$

where the n initial values $y_0\,, \ldots\,, y_0{}^{(n-1)}$ are to be specified.

If the solution is restricted to a finite interval, for example to $[0, \ell]$, and if the additional conditions are related to the two endpoints of the interval, we deal with a *boundary value problem.* A boundary value problem, for example, is:

$$\begin{aligned} &\frac{d^2\,y}{dt^2} - y(t) = f(t)\,, \\ &y(0) = a,\ \ y(\ell) = b. \end{aligned}$$

The most effective method to handle problems with linear ordinary differential equations with constant coefficients is by means of Laplace transformation.

The left linear differential operator of order n

$$L := \frac{d^n}{dt^n} + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1\,\frac{d}{dt} + a_0\,, \qquad (a_n = 1) \tag{1.82}$$

is associated with the *characteristic polynomial* of degree n

$$P(s) := s^n + a_{n-1}\,s^{n-1} + \cdots + a_1\,s + a_0\,. \tag{1.83}$$

The leading coefficient of the left differential operator L is assumed to be 1. Similarly, the right differential operator

$$R := b_0 + b_1\,\frac{d}{dt} + \cdots + b_m\,\frac{d^m}{dt^m}$$

is associated with the polynomial

$$O(s) := b_0 + b_1\,s + \cdots + b_m\,s^m. \tag{1.84}$$

Now we assume that $y(t)$ and its derivatives as well as $f(t)$ have (one-sided) Laplace transforms: $y(t) \circ\!-\!\bullet\, Y(s)$, $f(t) \circ\!-\!\bullet\, F(s)$. By the differentiation rule and the linearity, the image of the solution of the initial value problem (1.81) can be arranged as follows:

$$
\begin{aligned}
Y(s) &= \frac{Q(s)}{P(s)} F(s) + y_0 \frac{s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_1}{P(s)} \\
&+ y_0' \frac{s^{n-2} + a_{n-1} s^{n-3} + \cdots + a_2}{P(s)} \\
&+ \cdots + y_0^{(n-2)} \frac{s + a_{n-1}}{P(s)} + y_0^{(n-1)} \frac{1}{P(s)} \\
&- f(0+) \frac{b_1 + \cdots + b_{m-1} s^{m-1}}{P(s)} - f'(0+) \frac{b_2 + \cdots + b_{m-1} s^{m-2}}{P(s)} \\
&- \cdots - f^{(m-2)}(0+) \frac{b_{m-1} + b_m s}{P(s)} - f^{(m-1)}(0+) \frac{b_m}{P(s)}.
\end{aligned}
$$

In the case of (1.79), where no differential operator is acting on the input function, the expression above reduces to

$$
\begin{aligned}
Y(s) &= \frac{1}{P(s)} F(s) + y_0 \frac{s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_1}{P(s)} \\
&+ y_0' \frac{s^{n-2} + a_{n-1} s^{n-3} + \cdots + a_2}{P(s)} \\
&\cdots \\
&+ y_0^{(n-2)} \frac{s + a_{n-1}}{P(s)} + y_0^{(n-1)} \frac{1}{P(s)}.
\end{aligned}
\tag{1.86}
$$

The original function $y(t)$ corresponding to (1.85) or (1.86) respectively is called the *global response* or *global answer* of the system. If all initial values of the response are zero, i.e.

$$
y(0+) = y'(0+) = \cdots = y^{(n-1)}(0+) = 0, \tag{1.87}
$$

then the response of the system consists of only

$$
\begin{aligned}
y_1(t) &= \mathcal{L}^{-1}\left[\frac{Q(s)}{P(s)} F(s)\right](t) \\
&- \mathcal{L}^{-1}\left[f(0+) \frac{b_1 + \cdots + b_{m-1} s^{m-1}}{P(s)} + f'(0+) \frac{b_2 + \cdots + b_{m-1} s^{m-2}}{P(s)}\right. \\
&\qquad \left. + \cdots + f^{(m-2)}(0+) \frac{b_{m-1} + b_m s}{P(s)} + f^{(m-1)}(0+) \frac{b_m}{P(s)}\right](t).
\end{aligned}
$$

If no differential operator acts on the input, the latter simplifies to

$$
y_1(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{P(s)}\right](t).
$$

The function $y_1(t)$ is called the *forced response* of the system. If (1.87) holds, we say that the system is in a *non-excited* or in a *quiet initial state.* If there is no input function, i.e. $f(t) = 0$, but the system is in an excited initial state (some or all of the initial values are non-zero), then the response of the system consists of

$$y_0(t) = y_0\,\mathcal{L}^{-1}\left[\frac{s^{n-1} + a_{n-1}\,s^{n-2} + \cdots + a_1}{P(s)}\right](t)$$
$$+\ y_0'\,\mathcal{L}^{-1}\left[\frac{s^{n-2} + a_{n-1}\,s^{n-3} + \cdots + a_2}{P(s)}\right]$$
$$+ \cdots + y_0^{(n-2)}\,\mathcal{L}^{-1}\left[\frac{s + a_{n-1}}{P(s)}\right](t) + y_0^{(n-1)}\,\mathcal{L}^{-1}\left[\frac{1}{P(s)}\right](t)$$

which is called the *free response of the system.* The global response of the system is the sum of the free response and the forced response. A system which is in a quiet initial state has no free response and its global response is identical with its forced response.

The general expression for the global response (1.85) clarifies that the system behaves like a linear system only if it is in a quiet, i.e. non-excited initial state. For non-zero and arbitrary initial values, the linearity condition, which requires that if the input $f_1(t)$ produces the output $y_1(t)$, and the input $f_2(t)$ produces the output $y_2(t)$, then the input $c_1\,f_1(t) + c_2\,f_2(t)$ produces the output $c_1\,y_1(t) + c_2\,y_2(t)$, can be violated.

1.3.1.2 Stable Systems and Transfer Functions

A linear system is called *asymptotically stable* if for arbitrary initial values the free response (the response corresponding to $f(t) = 0$) tends to zero as $t \to \infty$. This means that after a certain lapse of time the global response is essentially equal to the forced response because the influence of the initial values fades away. In other words, after a certain lapse of time, the output of the system is essentially determined by its input and not by the generally uncontrollable initial values. A real physical system should be an asymptotically stable system, otherwise it is of no reliable use.

From (1.85) it can be seen that a system modeled by (1.81) is asymptotically stable if and only if the sum of the rational image functions

$$y_0\,\frac{s^{n-1} + a_{n-1}\,s^{n-2} + \cdots + a_1}{P(s)} + y_0'\,\frac{s^{n-2} + a_{n-1}\,s^{n-3} + \cdots + a_2}{P(s)}$$
$$+ \cdots + y_0^{(n-2)}\,\frac{s + a_{n-1}}{P(s)} + y_0^{(n-1)}\,\frac{1}{P(s)},$$

which have the characteristic polynomial $P(s)$ as their denominator, give rise to originals that tend to zero as $t \to \infty$. These rational image functions all are proper rational functions. By applying the partial fraction decomposition, we obtain a sum of terms of the type

$$\sum_k \frac{c_k}{(s - \alpha_k)^{\ell_k}}$$

where the α_k's are the zeros of the characteristic polynomial $P(s)$, i.e. the poles of the rational function $Q(s)/P(s)$. In Section 1.1.4.1 we observed that a partial fraction term $1/(s - \alpha_k)^{\ell_k}$ has an original that tends to zero as $t \to \infty$ if and only if the pole α_k is in the left complex half-plane, i.e. if a_k has a strictly negative real part. Thus the following proposition holds.

Proposition. *The system modeled by (1.81) is asymptotically stable if and only if all poles of the rational function*

$$G(s) := \frac{Q(s)}{P(s)} \tag{1.88}$$

have strictly negative real parts, i.e. are located in the left complex half-plane.

$G(s)$ is called the *transfer function* of the system. It is obtained by merely inspecting the two differential operators.

If an asymptotically stable system is exposed to a non-zero input, the terms containing the initial values of $f(t)$ and its derivatives

$$-f(0+)\frac{b_1+\cdots+b_{m-1}s^{m-1}}{P(s)} - f'(0+)\frac{b_2+\cdots+b_{m-1}s^{m-2}}{P(s)}$$
$$-\cdots - f^{(m-2)}(0+)\frac{b_{m-1}+b_m s}{P(s)} - f^{(m-1)}(0+)\frac{b_m}{P(s)}$$

are improper rational functions if the order m of the right-hand side differential operator is greater than the order n of the left-hand side differential operator. Improper rational functions can be decomposed into a sum of a polynomial and a proper rational function, which in the above case, has again $P(s)$ as its denominator. The inverse Laplace transform of a polynomial in s is an expression of Dirac Impulses and their derivatives at $t = 0$. Since asymptotic stability describes the behavior of the system for large values of time t, the Dirac Impulses at $t = 0$ do not affect the behavior of the system for large t.

In an asymptotically stable system the output components which are produced by the non-zero initial values of the response and of the input will eventually fade away such that after a certain lapse of time only the part due to

$$G(s)F(s) = \frac{Q(s)}{P(s)}F(s)$$

prevails. First, this means that for sufficiently large times, the system behaves like a truly linear system. Secondly, that the system can be characterized, after an initial transitory phase, by its transfer function $G(s)$. This explains why an asymptotically stable system is represented by the block diagram (Fig.1.16):

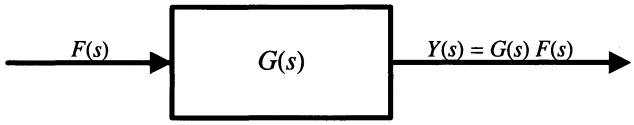


Fig.1.16: Representation of an asymptotically stable system.

The poles of the transfer function are the zeros of the characteristic polynomial $P(s)$. A polynomial is called a *stable polynomial* (or a Hurwitz polynomial), if all its roots have strictly negative real parts. Therefore, testing the stability of a system amounts to testing whether or not its characteristic polynomial is stable. The roots or zeros of a polynomial of degree higher than 2 are not readily available. We need a test to try out whether all roots are located in the left half-plane, by merely inspecting the coefficients of the polynomial. The best known stability test is the Hurwitz Test. We give it without proof. For a proof and other stability tests, see for example [Sarachik (1996)]. The Hurwitz Test uses the so-called $n \times n$ Hurwitz matrix

$$H = \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ 0 & a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where the first and second rows of H are completed by filling in zeros on the right to obtain n columns, i.e. $a_\nu = 0$, if $\nu < 0$. A new row $i\,(i > 2)$is formed by shifting row $i-2$ one position to the right, dropping the rightmost element and filling a zero in the first column. This process is carried out for $i = 3, 4, \dots, n$.

The Hurwitz Test. *The polynomial*

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \qquad (a_n \neq 0)$$

is stable if and only if all principal minors are positive.

The positive principal minors are:

$$\Delta_1 := a_{n-1} > 0\,,\ \Delta_2 := \det\begin{pmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{pmatrix} > 0\,,$$

$$\Delta_3 := \det\begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{pmatrix} > 0 \ \dots \Delta_n := \det H > 0.$$

The n conditions of the Hurwitz Test are redundant. Liénard–Chipart proved that it suffices to verify the conditions for Δ_i for all odd i's or for all even i's, provided that all $a_k > 0$.

Example. $P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$;

$$H = \begin{pmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{pmatrix}.$$

Provided that all five coefficients are positive, the polynomial is stable if and only if

$$\Delta_3 := \det\begin{pmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{pmatrix} > 0.$$

The condition that all coefficients of the polynomial must have the same sign (all positive or all negative, but none zero) is a necessary condition for the stability of the polynomial. The condition is not sufficient, except for the case $n = 2$. A polynomial of second degree is stable, if and only if all three coefficients are all negative or all positive. If one coefficient of a polynomial of degree n is zero, i.e. if some powers of s are absent, the polynomial is necessarily unstable.

Example. Consider

$$y'' + 2\delta y' + \omega^2 y(t) = f(t)\,,$$
$$y(0+) = A\,,$$
$$y'(0+) = B\,,$$

$$P(s) = s^2 + 2\delta s + \omega^2 = (s+\delta)^2 + \Omega^2, \quad \Omega := \sqrt{\omega^2 - \delta^2},$$
$$Q(s) = 1.$$

The system is asymptotically stable if $\delta > 0$, unstable if $\delta \leq 0$. We assume $\delta > 0$ and $\omega^2 - \delta^2 > 0$. The Laplace transform of the solution is:

$$Y(s) = \frac{1}{(s+\delta)^2 + \Omega^2} F(s) + A \frac{s + 2\delta}{(s+\delta)^2 + \Omega^2} + B \frac{1}{(s+\delta)^2 + \Omega^2}.$$

The free response is given by

$$y_0(t) = \mathcal{L}^{-1}\left[A \frac{(s + \delta)}{(s+\delta)^2 + \Omega^2} + \frac{B + A\delta}{\Omega} \frac{\Omega}{(s+\delta)^2 + \Omega^2}\right](t)$$
$$= u(t)\, e^{-\delta t}\left\{A \cos \Omega t + \frac{B + A\delta}{\Omega} \sin \Omega t\right\}.$$

For the unit-step response (the forced response of the unit-step function as an input) we have:

$$F(s) = 1/s,$$
$$y_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s[(s+\delta)^2 + \Omega^2]}\right](t)$$
$$= \frac{1}{\omega^2} u(t)\left\{1 - e^{-\delta t}\left[\cos \Omega t + \frac{\delta}{\Omega} \sin \Omega t\right]\right\}.$$

The unit-ramp response (forced response of $u(t)\, t$ as an input) is:

$$F(s) = 1/s^2,$$
$$y_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2[(s+\delta)^2 + \Omega^2]}\right](t)$$
$$= u(t)\left\{-\frac{2\delta}{\omega^4} + \frac{1}{\omega^2} t\right\} + u(t) \frac{e^{-\delta t}}{\omega^4}\left\{2\delta \cos \Omega t - \frac{\Omega^2 - \delta^2}{\Omega} \sin \Omega t\right\}.$$

Due to the notation of originals, the factor $u(t)$ can also be omitted. For the computation of the last two inverse Laplace transforms a table [Oberhettinger F. and Badii L. (1970)] was used.

Example. Consider the following initial value problem:

$$y^{(3)} + 3y'' + 3y' + y(t) = e^t,$$
$$y(0+) = 0,$$
$$y'(0+) = 0,$$
$$y''(0+) = 0,$$

$$P(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3, \quad Q(s) = 1, \quad F(s) = 1/(s+1).$$

The system is asymptotically stable. The solution is

$$Y(s) = \frac{F(s)}{P(s)} = \frac{1}{(s+1)^4} \blacksquare\!-\!\square\; y(t) = \frac{t^3}{6} e^{-t}.$$

Example. Consider the system modeled by

$$y^{(3)} + 3y'' + 3y' + y(t) = f(t) + 2f'(t),$$
$$y(0+) = 1,$$
$$y'(0+) = -1,$$
$$y''(0+) = 2,$$

$$P(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3, \quad Q(s) = 1 + 2s.$$

The transfer function of this asymptotically stable system is

$$G(s) = \frac{1 + 2s}{(s+1)^3}.$$

The free response is given by

$$Y_0(s) = \frac{s^2 + 3s + 3}{(s+1)^3} - \frac{s+3}{(s+1)^3} + 2\frac{1}{(s+1)^3}$$
$$= \frac{s^2 + 2s + 2}{(s+1)^3} = \frac{(s+1)^2 + 1}{(s+1)^3}$$
$$= \frac{1}{(s+1)} + \frac{1}{(s+1)^3} \blacksquare - \square \; y_0(t) = e^{-t} + \frac{t^2}{2}e^{-t}.$$

The forced response to the input $f(t) = t^2$ is, observing that $f(0+) = 0$,

$$Y_1(s) = \frac{1 + 2s}{(s+1)^3}\frac{2}{s^3} = \frac{4s + 2}{s^3(s+1)^3}$$
$$= \frac{a_{11}}{s^3} + \frac{a_{12}}{s^2} + \frac{a_{13}}{s} + \frac{a_{21}}{(s+1)^3} + \frac{a_{22}}{(s+1)^2} + \frac{a_{23}}{s+1},$$

$$a_{11} = \left(\frac{4s + 2}{(s+1)^3}\right)_{s=0} = 2, \; a_{12} = \left(\frac{d}{ds}\left(\frac{4s + 2}{(s+1)^3}\right)\right)_{s=0} = -2,$$

$$a_{13} = \left(\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{4s + 2}{(s+1)^3}\right)\right)_{s=0} = 0, \; a_{21} = \left(\frac{4s + 2}{s^3}\right)_{s=-1} = 2,$$

$$a_{22} = \left(\frac{d}{ds}\left(\frac{4s + 2}{s^3}\right)\right)_{s=-1} = 2, \; a_{23} = \left(\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{4s + 2}{s^3}\right)\right)_{s=-1} = 0.$$

Therefore $y_1(t) = t^2 - 2t + t^2 e^{-t} + 2te^{-t}$.

1.3.1.3 Special System Responses

Suppose that the input-output system is asymptotically stable and that it is described by its transfer function $G(s)$. Thus all poles of the transfer function are located in the complex left half-plane, i.e. they all have strictly negative real parts. Furthermore, suppose that the system is real, implying $G(\bar{s}) = \overline{G(s)}$.

The *unit-impulse response* $g(t)$ is the response of the system to an applied Dirac Impulse $\delta(t)$, where all initial values (of the input and the output) are assumed to be zero.

Because $f(t) = \delta(t) \square - \blacksquare F(s) = 1$ and (1.85) we see at once that the Laplace transform of the unit-impulse response is

$$Y(s) = G(s)F(s) = G(s)\,1 = G(s),$$

i.e. the unit-impulse response is equal to the inverse Laplace transform of the transfer function of the system:

$$g(t) = \mathcal{L}^{-1}[G(s)]\,(t). \tag{1.89}$$

The *unit-step response* is defined as

$$y_{\mathrm{st}}(t) = \mathcal{L}^{-1}\Big[\frac{G(s)}{s}\Big](t). \tag{1.90}$$

From (1.85) we see that up to a transient the unit-step response is equal to the response of the system if the input is the unit-step function. A *transient* is a function of time that tends to zero as $t \to \infty$. The terms corresponding to the initial values of both the input and the output in (1.85) yield transients when transformed back. In the case where there is no differential operator acting on the input function, the unit-step response is the response to a unit-step, where all initial values of the response are assumed to be zero, meaning that the system is in a non-excited initial state. Let us now compute the unit-step response. Because the system is assumed to be asymptotically stable, the transfer function $G(s)$ is holomorphic on a right half-plane that contains the imaginary axis in its interior (the right most singularity still has a strictly negative real part). The Laplace transform of the unit-step response is given by $G(s)/s$. The integration rule gives

$$\begin{aligned}
\mathcal{L}^{-1}\Big[\frac{G(s)}{s}\Big](t) &= u(t)\int_0^t g(t)\,dt = u(t)\Big[\int_0^\infty g(t)\,dt - \int_t^\infty g(t)\,dt\Big] \\
&= \Big(u(t)\int_0^\infty e^{-st}\,g(t)\,dt\Big)_{s=0} - u(t)\int_t^\infty g(t)\,dt \\
&= u(t)\,G(0) - u(t)\int_t^\infty g(t)\,dt\,.
\end{aligned}$$

The last term on the right is a transient which tends to zero as $t \to \infty$. We have proved:

Proposition. *The unit-step response of an asymptotically stable system described by its transfer function $G(s)$ is, up to a transient, given by $G(0)\,u(t)$. We also say that the steady-state part of the unit-step response is given by $G(0)\,u(t)$.*

The *frequency response* of the system is defined as

$$\mathcal{L}^{-1}\Big[G(s)\,\frac{1}{s - i\omega}\Big](t). \tag{1.91}$$

As above the frequency response is, up to a transient, the response of the system to an applied sinusoidal input of the form $\exp(i\omega t)$. If there is no differential operator acting on the input function the frequency response is the response to $\exp(i\omega t)$, provided that the system is in a non-excited initial state (all initial values of the response are zero).

We obtain the frequency response by using the Convolution Theorem

$$\mathcal{L}^{-1}\left[\frac{1}{s-i\omega}\,G(s)\right](t) = \int_0^t e^{i\omega(t-x)}\,g(x)\,dx$$

$$= e^{i\omega t}\int_0^t e^{-i\omega x}\,g(x)\,dx$$

$$= e^{i\omega t}\left[\int_0^\infty e^{-i\omega x}\,g(x)\,dx - \int_t^\infty e^{-i\omega x}\,g(x)\,dx\right]$$

$$= e^{i\omega t}\,G(i\omega) - e^{i\omega t}\int_t^\infty e^{-i\omega x}\,g(x)\,dx.$$

Because the imaginary axis is inside the half-plane of convergence of $G(s)$, the last term on the right-hand side is a transient which tends to zero as $t \to \infty$. We proved:

Proposition. *The frequency response to $f(t) = \exp(i\omega t)$ of an asymptotically stable system, described by its transfer function $G(s)$, is, up to a transient, given by $y(t) = G(i\omega)\exp(i\omega t)$. If the complex-valued function $G(i\omega)$ of the real variable ω is written in polar coordinates:*

$$G(i\omega) = a(\omega)\,e^{i\varphi(\omega)}, \quad a(\omega) = |G(i\omega)|, \quad \varphi(\omega) = \arg G(i\omega),$$

then, up to a transient, the frequency response is

$$y(t) = a(\omega)\,e^{i[\omega t + \varphi(\omega)]}. \tag{1.92}$$

By superposition, and assuming that the system is real, we also conclude that up to a transient the frequency response to $\sin\omega t$ and $\cos\omega t$ is

$$y(t) = a(\omega)\,\sin[\omega t + \varphi(\omega)], \quad y(t) = a(\omega)\,\cos[\omega t + \varphi(\omega)],$$

respectively. This means that up to transients an asymptotically stable linear system responds to a sinusoidal input of angular frequency ω and amplitude a by multiplying the amplitude a by the amplification factor $a(\omega) = |G(i\omega)|$ and by producing a phase shift of $\varphi(\omega) = \arg G(i\omega)$ in the response. Both amplification factor and phase shift are functions of the frequency. The response has the same frequency as the input. The frequency response and the unit-step response are related to each other. One can be computed from the other. If the unit-step response $y_{st}(t)$ is given, the frequency response $u_f(t)$ is obtained as follows:

$$Y_{st}(s) = \frac{G(s)}{s} \Longrightarrow y_{st}(t) = \int_0^t g(\tau)\,d\tau \Longrightarrow y_{st}'(t) = g(t)$$

$$\Longrightarrow G(s) = \mathcal{L}[y_{st}'(t)](s) \Longrightarrow G(i\omega) = \mathcal{L}[y_{st}'(t)](i\omega).$$

Hence we obtain the frequency response from the unit-step response as

$$y_f(t) = \mathcal{L}\left[\frac{dy_{st}}{dt}\right](i\omega)\;e^{i\omega t}.$$

More interesting is determining the unit-step response from the frequency response. Let us introduce two notions from electrical engineering.

The real part of $G(i\omega)$ is called the *conductance* of the system and is denoted by $U(\omega)$. The imaginary part of $G(i\omega)$ is called the *susceptance* of the system and is denoted by $V(\omega)$, thus $G(i\omega) = U(\omega) + iV(\omega)$.

Conductance and susceptance are measurable quantities. It was shown, see for example [Doetsch (1976)], that the unit-step response can be computed from the conductance or from the susceptance alone. The result is

$$\begin{aligned} y_{st}(t) &= u(t)\,\frac{2}{\pi}\int_0^\infty \frac{U(\omega)}{\omega}\sin\omega t\,d\omega \\ &= u(t)\,\frac{2}{\pi}\int_0^\infty \frac{V(\omega)}{\omega}\sin\omega t\,d\omega + U(0)\,. \end{aligned} \tag{1.93}$$

The advantage of these formulas is that conductance or susceptance can often be measured more easily than the step-response itself.

Example. Suppose that a system is modeled by the differential equation

$$y'' + 2y' + 5y(t) = 3f(t) - f'(t)\,.$$

This is an asymptotically stable system since the characteristic polynomial is of degree 2 with all three coefficients positive.

$$P(s) = s^2 + 2s + 5, \quad Q(s) = 3 - s, \quad G(s) = \frac{-s+3}{s^2+2s+5}\,.$$

The steady-state part of the unit-step response is $u(t)\,G(0) = (3/5)\,u(t)$. The total unit-step response is

$$\mathcal{L}^{-1}\Big[\frac{-s+3}{s(s^2+2s+5)}\Big](t) = u(t)\Big\{\frac{3}{5} + 2e^{-t}\Big(-\frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t)\Big)\Big\}.$$

The forced response to a unit-step in the general case where a differential operator acts on the input function is not the same as the unit-step response defined above! For this example, the forced response to the unit-step is

$$\mathcal{L}^{-1}\Big[\frac{3}{s(s^2+2s+5)}\Big](t) = u(t)\Big\{\frac{3}{5} + 2e^{-t}\Big(-\frac{3}{10}\cos(2t) - \frac{3}{20}\sin(2t)\Big)\Big\}.$$

In order to compute the frequency response to $\cos\omega t$, for example, we first write

$$G(i\omega) = \frac{-i\omega+3}{-\omega^2+2i\omega+5} = \frac{15-5\omega^2}{4\omega^2+(5-\omega^2)^2} + i\,\frac{\omega(\omega^2-11)}{4\omega^2+(5-\omega^2)^2}\,.$$

Therefore the conductance and the susceptance are

$$U(\omega) = \frac{15-5\omega^2}{4\omega^2+(5-\omega^2)^2}, \qquad V(\omega) = \frac{\omega(\omega^2-11)}{4\omega^2+(5-\omega^2)^2}\,.$$

The amplification factor and the phase shift are

$$|G(i\omega)| = \frac{|-i\omega+3|}{|-\omega^2+2i\omega+5|} = \sqrt{\frac{\omega^2+9}{4\omega^2+(5-\omega^2)^2}}\,,$$

$$\begin{aligned} \varphi(\omega) &= \arg\Big[\frac{15-5\omega^2}{4\omega^2+(5-\omega^2)^2} + i\,\frac{\omega(\omega^2-11)}{4\omega^2+(5-\omega^2)^2}\Big] \\ &= \arg[15-5\omega^2+i\omega(\omega^2-11)] \end{aligned}$$

giving finally the frequency response

$$y(t) = \sqrt{\frac{\omega^2 + 9}{4\omega^2 + (5 - \omega^2)^2}} \cos[\omega t + \varphi(\omega)].$$

Problem 1.30. Show that, in the general case of (1.81), where the differential equation also has a right-hand differential operator acting on the input function, the forced response to the unit-step function is given by

$$\mathcal{L}^{-1}\left[\frac{b_0}{s\,P(s)}\right](t),$$

which is not the same as the unit-step response defined as

$$\mathcal{L}^{-1}\left[\frac{Q(s)}{s\,P(s)}\right](t) \qquad (Q(s) = b_0 + b_1 s + \cdots + b_m s^m).$$

Problem 1.31. Verify the formulas (1. 93) for $G(s) = 1/(s+1)$.

1.3.1.4 Systems of Ordinary Differential Equations

The mathematical modeling of a physical system often results in a system of differential equations rather than in a single scalar differential equation. The method of Laplace transformation is well adapted for solving linear systems of differential equations with constant coefficients. It is not necessary that the system be a system of first order differential equations, but the reader should be aware of the occurrence of particular effects concerning initial value problems in connection with a subclass of systems of higher order differential equations, called anomalous systems. Anomalous systems are widespread in applications and we shall treat them in a later chapter. The basic method for solving systems of differential equations is straightforward and is best explained by way of an example.

Example. Solve the following initial value problem:

$$\begin{aligned} &x'' - y(t) = t, \\ &y' + x(t) = 0, \\ &x(0+) = 1,\ x'(0+) = 0,\ y(0+) = -1. \end{aligned}$$

Let us assume that the two variables $x(t)$, $y(t)$ and their derivatives are Laplace transformable. By use of the differentiation rule, the linearity and the specified initial values, the image system becomes

$$\begin{aligned} &s^2 X(s) - s - 0 - Y(s) = \frac{1}{s^2}, \\ &s\,Y(s) - (-1) + X(s) = 0. \end{aligned}$$

This is a linear system for the two unknown image functions $X(s)$ and $Y(s)$. In matrix notation, the system is written as

$$\begin{pmatrix} s^2 & -1 \\ 1 & s \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} (s^3+1)/s^2 \\ -1 \end{pmatrix}.$$

By Cramer's rule,

$$X(s) = \frac{s^3 - s + 1}{s(s^3 + 1)} = \frac{s^3 - s + 1}{s(s + 1)(s^2 - s + 1)},$$

$$Y(s) = -\frac{s^4 + s^3 + 1}{s^2(s^3 + 1)} = -\frac{s^4 + s^3 + 1}{s^2(s + 1)(s^2 - s + 1)}.$$

By applying the inverse Laplace transform to these image functions we obtain the solution. This is done for $X(s)$:

$$\begin{aligned} X(s) &= \frac{s^3 - s + 1}{s(s + 1)(s^2 - s + 1)} \\ &= \frac{s^3 - s + 1}{s(s + 1)\left[s - \left(1/2 + \sqrt{3}/2\,i\right)\right]\left[s - \left(1/2 - \sqrt{3}/2\,i\right)\right]} \\ &= \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s - \left(1/2 + \sqrt{3}/2\,i\right)} + \frac{\overline{c_3}}{s - \left(1/2 - \sqrt{3}/2\,i\right)}. \end{aligned}$$

The coefficients of the partial fractions are determined as follows:

$$c_1 = \left(\frac{s^3 - s + 1}{(s + 1)(s^2 - s + 1)}\right)_{s=0} = 1, \quad c_2 = \left(\frac{s^3 - s + 1}{s(s^2 - s + 1)}\right)_{s=-1} = -1/3,$$

$$c_3 = \left(\frac{s^3 - s + 1}{s(s + 1)\left[s - \left(1/2 - \sqrt{3}/2\,i\right)\right]}\right)_{s = 1/2 + \sqrt{3}/2\,i} = \frac{1 + \sqrt{3}\,i}{6}.$$

Hence,

$$x(t) = 1 - \frac{1}{3}e^{-t} + e^{t/2}\left[\frac{1}{3}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right].$$

Similarly,

$$y(t) = -t - \frac{1}{3}e^{-t} - \frac{2}{3}e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right).$$

A special, but important class of systems of ordinary differential equations are the systems of first order equations. A system of n linear first order differential equations with constant coefficients is best written in matrix notation.

$$\mathbf{y}'(t) - \mathbf{A}\mathbf{y}(t) = \mathbf{f}(t) \tag{1.94}$$

with the constant $n \times n$-matrix $\mathbf{A}$ and the two n-vectors $\mathbf{y} = \mathbf{y}(t)$, $\mathbf{f} = \mathbf{f}(t)$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \dots \\ y_n(t) \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \dots \\ f_n(t) \end{pmatrix}.$$

We also need n initial conditions

$$\mathbf{y}(0+) = \begin{pmatrix} y_1(0+) \\ \dots \\ y_n(0+) \end{pmatrix} = \begin{pmatrix} y_{10} \\ \dots \\ y_{n0} \end{pmatrix} =: \mathbf{y}_0.$$

Introducing the following matrix-Laplace transforms

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \dots \\ y_n(t) \end{pmatrix} \circ\!-\!\bullet\, \mathbf{Y}(s) = \begin{pmatrix} Y_1(s) \\ \dots \\ Y_n(s) \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \dots \\ f_n(t) \end{pmatrix} \circ\!-\!\bullet\, \mathbf{F}(s) = \begin{pmatrix} F_1(s) \\ \dots \\ F_n(s) \end{pmatrix}$$

we obtain, by the differentiation rule applied to each component,

$$\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ \dots \\ y_n'(t) \end{pmatrix} \circ\!-\!\bullet \begin{pmatrix} s\,Y_1(s) - y_{10} \\ \dots \\ s\,Y_n(s) - y_{n0} \end{pmatrix} = s\,\mathbf{Y}(s) - \mathbf{y}_0\,,$$

the image system

$$s\,\mathbf{Y}(s) - \mathbf{y}_0 - \mathbf{A}\mathbf{Y}(s) = \mathbf{F}(s).$$

With the $n \times n$-identity matrix $\mathbf{I}$ the system can be written as

$$(s\,\mathbf{I} - \mathbf{A})\,\mathbf{Y}(s) = \mathbf{F}(s) + \mathbf{y}_0.$$

Using the resolvent of $\mathbf{A}$,

$$\mathbf{R}(s, \mathbf{A}) := (s\,\mathbf{I} - \mathbf{A})^{-1} \tag{1.95}$$

we can write the image vector of the solution as

$$\mathbf{Y}(s) = (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{F}(s) + (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{y}_0 = \mathbf{R}(s, \mathbf{A})\,\mathbf{F}(s) + \mathbf{R}(s, \mathbf{A})\,\mathbf{y}_0.$$

The resolvent of $\mathbf{A}$ is an $n \times n$-matrix with elements that are rational functions of s. The resolvent exists for all values of s where the determinant

$$P(s) := \det(s\,\mathbf{I} - \mathbf{A}) \neq 0.$$

This determinant $P(s)$ of the matrix $s\,\mathbf{I} - \mathbf{A}$ is a polynomial in s of degree n. $P(s)$ is also the common denominator polynomial of all entries of the matrix $\mathbf{R}(s, \mathbf{A})$. This fact together with the expression for $\mathbf{Y}(s)$ show that the system is asymptotically stable, if and only if all zeros of $P(s)$ have strictly negative real parts (they are situated in the left complex half-plane), or which is the same, if and only if all poles of the resolvent are in the left half-plane. In this case all components of $\mathbf{R}(s, \mathbf{A})\,\mathbf{y}_0$ will be transients and the response of the system, after a certain lapse of time, is essentially determined by the forced response

$$\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{R}(s, \mathbf{A})\,\mathbf{F}(s)]\,(t),$$

and thus independent of the initial values. We see that for a first order system the determinant of the matrix $s\,\mathbf{I} - \mathbf{A}$ plays the same part concerning the stability behavior as does the characteristic polynomial of a scalar differential equation of order n.

Problem 1.32. Establish the matrix correspondence

$$\mathbf{R}(s, \mathbf{A}) := (s\,\mathbf{I} - \mathbf{A})^{-1} \bullet\!-\!\circ\, e^{t\mathbf{A}}\,, \tag{1.96}$$

where the matrix exponential function $\mathbf{Y}(t) = e^{t\mathbf{A}}$ is defined as the solution of the matrix differential equation

$$\frac{d}{dt}\mathbf{Y} = \mathbf{A}\,\mathbf{Y}(t), \quad \mathbf{Y}(0) = \mathbf{I}.$$

A is a constant square matrix. and $\mathbf{Y}(t)$ is a square matrix whose elements are functions of t.

Chapter 2
z-Transformation

2.1 z-Transforms and Inverse z-Transforms

The *z-transformation* emerged, at least under this curious name, in the middle of the 20th century [Jury (1977)]. The basic idea is old and is known by the name of "generating function method" among the mathematicians. Indeed, the z-transform of a sequence is the generating function of this sequence, where the independent variable z is replaced by its reciprocal $1/z$. The z-transformation or z-transform today is applied to model sample-data control systems or other discrete-data systems. Its role for discrete time systems is similar to the method of Laplace transformation for continuous time systems. From a mathematical point of view, the method of z-transformation is an operational calculus for solving difference equations or systems of such equations, similar to the method of Laplace transformation in connection with differential equations. The name z-transformation or z-transform is nonsense but, unfortunately, is in common use today (it is as if the Laplace transformation would be called the s-transform or p-transform). A reasonable name for this method would perhaps be "Laurent transform or Laurent transformation" because the defining series is a Laurent series. But it is too late for that. The z-transformation is intimately connected with the discrete Laplace transformation and with other discrete transformation methods. As is the case with the Laplace transformation, there is an ordinary or one-sided z-transform and a two-sided one. In this book we study only the one-sided z-transformation. However, the one-sided z-transformation has a generalization which is called the *advanced or modified z-transformation.* If the latter is applied to a sequence, there is no reason for a new name. It is just the ordinary z-transform applied to another sequence depending on a supplementary parameter. But if the domain of the originals is considered to be a set of continuous time functions rather than discrete time sequences, a new name is justified. The Laplace transformation maps a continuous function of the real variable t to a function of the complex variable s. Similarly, the z-transformation maps a discrete sequence to a function of the complex variable z. Both methods benefit from the theory of functions of a complex variable.

■ 2.1.1 Discrete Transformations

We first introduce the discrete Laplace transform.

Definition. The *discrete Laplace transform* of the sequence $f(kT)$ evaluated at kT with $k = 0, 1, 2, \ldots$ and $T > 0$, is the function of a complex variable s, defined as

$$F^*(s) := \sum_{k=0}^{\infty} f(kT)\, e^{-kTs} =: \mathcal{L}_T[f(kT)](s), \tag{2.1}$$

provided that the series converges for some value of the variable s.

Observe that $|e^{-k\,T\,s}| = \exp[-k\,T\,\Re s]$. This implies that if the discrete Laplace transform converges absolutely for some real x_0, it will converge absolutely for all complex s with $\Re s > x_0$. Thus, the domain of convergence of a discrete Laplace transform is a right half-plane. The discrete Laplace transform (with $T = 1$) is a discrete version of the ordinary Laplace transform

$$\mathcal{L}[f](s) = \int_0^\infty f(t)\, e^{-t\,s}\, \mathbf{d}t\,.$$

Indeed, the discrete Laplace transform of the sequence $f(k\,T)$ is the two-sided Laplace transform of the Dirac row

$$f(t) \sum_{k=0}^{\infty} \delta(t - k\,T)$$

$$= \sum_{k=0}^{\infty} f(t)\, \delta(t - k\,T) = \sum_{k=0}^{\infty} f(k\,T)\, \delta(t - k\,T)\,,$$

i.e. $\mathcal{L}_T[f(k\,T)](s) = \mathcal{L}_2[f(t) \sum_{k=0}^{\infty} \delta(t - k\,T)](s)$, because

$$\int_{-\infty}^{\infty} e^{-t\,s}\, f(t) \sum_{k=0}^{\infty} \delta(t - k\,T)\, \mathbf{d}t$$

$$= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-t\,s}\, f(t)\, \delta(t - k\,T)\, \mathbf{d}t = \sum_{k=0}^{\infty} e^{-k\,T\,s}\, f(k\,T)\,.$$

Definition (2.1) has the advantage that it is "close to" the definition of the Laplace transform, but the disadvantage of being unnecessarily complicated. When computing some discrete Laplace image functions, the sub-expression $e^{T\,s}$ will appear in the resulting image functions, and one is tempted to substitute z for $e^{T\,s}$. By doing so one obtains the z-transform.

Definition. The *z-transform* of the sequence $f(k\,T)$ $k = 0, 1, 2, \ldots$ is the function of the complex variable z, defined as

$$F(z) := \sum_{k=0}^{\infty} f(k\,T)\, z^{-k}\,, \tag{2.2}$$

provided that the series converges for some value of z.

The z-transform and the discrete Laplace transform of a sequence $f(k\,T)$ are related by

$$F^*(s) = \mathcal{L}_T[f(k\,T)](s) = F(e^{T\,s}). \tag{2.3}$$

In the following, for simplicity, we assume $T = 1$ and consider the z-transform of the sequence $f(k)$, $k = 0, 1, 2, \ldots$, i.e.,

$$F(z) := \sum_{k=0}^{\infty} f(k)\, z^{-k}\,.$$

A convergent power series

$$F\left(\frac{1}{z}\right) := \sum_{k=0}^{\infty} f(k)\, z^k ,$$

always has a disk $|z| < 1/\varrho$ as its domain of convergence. The mapping $z \mapsto 1/z$ maps the interior of the circle of radius $1/\varrho$ onto the exterior of the circle $|z| > \varrho$. Thus the domain of convergence of a z-transform is always the exterior of a circle centered at the origin. Since a power series always converges uniformly and absolutely on its disk of convergence, the z-transform converges uniformly and absolutely on the outside of its circle of convergence. Furthermore, a z-transform is a holomorphic or an analytic function on the outside of its circle of convergence.

The function

$$G(z) := F\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} f(k)\, z^k$$

is said to be the *generating function* of the sequence $f(k)$. Thus the z-transform and the generating function of a given sequence are related by a change of the independent variable from z to $1/z$ and vice versa. The Cauchy-Hadamard formula gives the radius of convergence $1/\rho$ of the power series $G(z)$, see for example [Henrici (1974)], i.e. there is convergence if

$$|z| < 1/\rho = \left\{ \lim_{k\to\infty} \sup \sqrt[k]{|f(k)|} \right\}^{-1} .$$

Hence we conclude that the *radius of convergence of the z-transform* $F(z)$ is given by

$$\rho = \lim_{k\to\infty} \sup \sqrt[k]{|f(k)|}$$

and the z-transform $F(z)$ is an analytic function on the exterior of the circle $|z| = \varrho$.

Clearly, if the sequence $\sqrt[k]{|f(k)|}$ is convergent, we simply have

$$\rho = \lim_{k\to\infty} \sqrt[k]{|f(k)|} .$$

Furthermore, if $f(k) \neq 0$ for all sufficiently large k, the radius of convergence is also given by

$$\rho = \lim_{k\to\infty} \left| \frac{f(k+1)}{f(k)} \right| ,$$

provided the limit on the right-hand side exists (see for example [Henrici (1974)]).

If for all sufficiently large k the sequence $f(k)$ satisfies the relation

$$|f(k)| \leq K r^k ,$$

for some $K > 0$ and some $r > 0$, it is readily seen that the sequence will have a z-transform with a radius of convergence $\rho \leq r$. On the other hand, a sequence like $f(k) = 2^{k^2}$ has no z-transform because the defining series does not converge for any finite z. We also have the fact that any function $F(z)$, holomorphic outside some circle with radius ρ and centered at the origin, can be considered as the z-transform of a sequence $f(k)$. The sequence consists of the coefficients of the Laurent series of $F(z)$ and is given by

$$f(k) = \frac{1}{2\pi i}\int_C F(z)\, z^{k-1}\, dz \qquad (k = 0, 1, 2, \ldots) \tag{2.4}$$

where the path of integration C is a circle of radius $r > \rho$ traversed in the positive (anti-clockwise) sense. This is a convenient formula for the *inverse z-transform.* Another formula for the inverse z-transform follows from Taylor's theorem since a z-transform is a power series in $s = 1/z$. Hence

$$f(k) = \frac{1}{k!}\left(\frac{d^k}{dz^k} F\left(\frac{1}{z}\right)\right)_{z=0} \quad (k = 0, 1, 2, \ldots).$$

Notation. The correspondence between an original sequence $f(k)$ and its z-image function $F(z)$ is denoted by

$$f(k) \circ\!-\!\diamond\, F(z),$$
$$F(z) = Z[f(k)](z), \quad F = Z[f], \quad f(k) = Z^{-1}[F(z)](k), \quad f = Z^{-1}[F].$$

In applications, the sequence $f(k)$ is often generated by the sampling values of a continuous original function $y(t)$ sampled at kT with a period or step $T > 0$, i.e.,

$$f(k) := y(kT), \quad (k = 0, 1, 2, \ldots).$$

The continuous time signal $y(t)$ generates a discrete signal $f(k) := y(kT)$ by a sampling process. The z-transform of the discrete signal $y(kT)$ then is

$$Y(z) := \sum_{k=0}^{\infty} y(kT)\, z^{-k}.$$

Instead of sampling the given time function at instants kT, one can sample at instants $kT + \tau$ where $0 \le \tau < T$. In this case the sequence is $f(k) = y(kT + \tau)$ and the z-transform of the sequence becomes

$$Y(z) := \sum_{k=0}^{\infty} y(kT + \tau)\, z^{-k}.$$

Clearly, as long as τ is regarded as a constant parameter there is no need to introduce a new name. $Y(z)$ is just the ordinary z-transform of the sequence $f(k) = y(kT + \tau)$. But things change whenever τ is considered as a second independent variable. In this case we write

$$Y(z, \tau) := \sum_{k=0}^{\infty} y(kT + \tau)\, z^{-k}, \quad (k = 0, 1, 2, \ldots;\ 0 \le \tau < T) \tag{2.5}$$

and $Y(z, \tau)$ is called the *advanced z-transform of the continuous time signal* $y(t)$. Clearly, we have $Y(z, 0) = Y(z)$, the ordinary z-transform.

Following the same line of thought allows us to generalize the discrete Laplace transform and define the *generalized discrete Laplace transform of* $f(t)$, evaluated at $kT + \tau$ with step of length $T > 0$ and with $0 \le \tau < T$, by

$$F^*(s, \tau) := \sum_{k=0}^{\infty} f(kT + \tau)\, e^{-kTs} \\ = \mathcal{L}_T[f(kT + \tau)](s). \tag{2.6}$$

Let us introduce yet another discrete transform of a sequence $f(k)$:

$$\hat{F}(z) = \sum_{k=0}^{\infty} \frac{f(k)}{k!} z^k . \tag{2.7}$$

We call it the *MacLaurin transform* but this is not a recognized name.

As for continuous originals in the framework of Laplace transforms (see Section 1.2.1), the domain of the given sequence $f(k)$ may be extended by including the negative integers and defining $f(k) = 0$ for $k < 0$. A typical original sequence, or a discrete signal, is then represented in Fig.2.1.

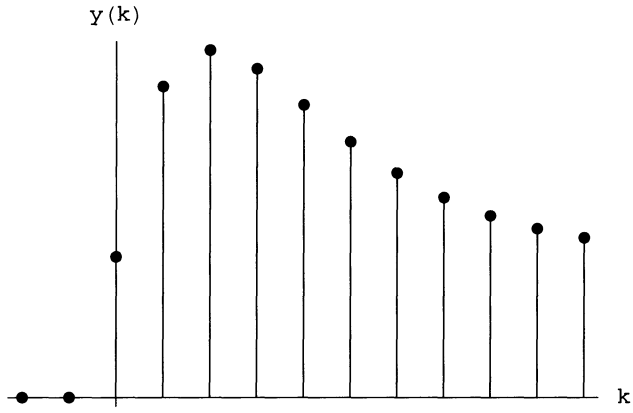


Fig.2.1: A typical original sequence

The *discrete unit-step* is the sequence

$$u(k) := \begin{cases} 0, & k = -1, -2, \ \ldots \\ 1, & k = 1, 2, \ \ldots \end{cases} .$$

Its z-transform is

$$\sum_{k=0}^{\infty} 1\, z^{-k} = \frac{1}{1 - 1/z} = \frac{z}{z-1} , \quad |z| > 1 .$$

Hence, we obtain the correspondence

$$u(k) \circ\!-\!\diamond \frac{z}{z-1}.$$

Thus, the z-transform of the discrete unit-step is a holomorphic function in the exterior of the unit-circle. The MacLaurin transform of the discrete unit-step is $\hat{F}(z) = e^z$. The z-transform of the sequence $f(k) = a^k$ is given by

$$\sum_{k=0}^{\infty} a^k z^{-k} = \frac{1}{1-a/z} = \frac{z}{z-a}, \quad \left|\frac{a}{z}\right| < 1 \Longleftrightarrow |z| > \frac{1}{|a|}$$

and we obtain the correspondence

$$a^k \circ\!-\!\diamond \frac{z}{z-a}$$

which is a holomorphic function outside of the circle of radius $1/|a|$. The MacLaurin transform of $f(k) = a^k$ is $\hat{F}(z) = e^{az}$. An immediate consequence is

$$e^{ak} \circ\!-\!\diamond \frac{z}{z-e^a},$$

whence

$$e^{i\omega k} \circ\!-\!\diamond \frac{z}{z-e^{i\omega}}, \quad e^{-i\omega k} \circ\!-\!\diamond \frac{z}{z-e^{-i\omega}}$$

and, by linearity (see below),

$$\sin \omega k = \frac{1}{2i}(e^{i\omega k} - e^{-i\omega k}),$$
$$\circ\!-\!\diamond \frac{1}{2i}\left\{\frac{z}{z-e^{i\omega}} - \frac{z}{z-e^{-i\omega}}\right\}$$
$$= \frac{z \sin \omega}{z^2 - 2z\cos\omega + 1}, \quad |z| > 1.$$

Therefore

$$\sin \omega k \circ\!-\!\diamond \frac{z \sin \omega}{z^2 - 2z\cos\omega + 1}.$$

Similarly,

$$\cos \omega k \circ\!-\!\diamond \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1}.$$

We shall show below how the MacLaurin transform of the above sequences can be found by using the inverse Laplace transformation.

The z-transform of the sequence $f(k) = 1/k!$ is

$$\frac{1}{k!} \circ\!-\!\diamond e^{1/z},$$

while its MacLaurin transform is

$$\hat{F}(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+1)} z^k = \sum_{k=0}^{\infty} \frac{\left(\frac{2\sqrt{z}}{2}\right)^{2k}}{k!\,\Gamma(k+1)} = I_0\left(2\sqrt{z}\right)$$

where I_0 is the modified Bessel function of order 0.

We noted that a z-transform is a holomorphic or analytic function on the exterior of its circle of convergence. On the circle itself there must be at least one singularity. In most cases the

analytic function element formed by the z-transform and the exterior of the circle of convergence can be extended analytically into the interior of the circle of convergence. The singularities of the resulting global analytic function located on and perhaps inside the circle of convergence are isolated singularities (essential singularities or poles) or branch points. Let us look at the following three simple examples.

Example. $f(k) := 1/k!$ has the z-transform $F(z) = \exp(1/z)$ which is holomorphic for all $|z| > 0$. In this case the circle of convergence shrinks to the point $z = 0$ and the z-transform has an isolated essential singularity at the origin.

Example. $f(k) := 1$ constant. The z-transform is $F(z) = z/(z-1)$ and the circle of convergence is $|z| = 1$. The resulting global analytic function has the only singularity $z = 1$ which is a pole located on the circle of convergence.

Example. $f(0) = 0$, $f(k) = 1/k$, $k \geq 1$. The z-transform is

$$F(z) = \ln \frac{z}{z-1}, \quad |z| > 1.$$

The circle of convergence is the unit-circle. If ln denotes the principal branch of the logarithm (the one taking on real values for real z) then $z = 1$ and $z = 0$ are two branch points and the global analytic function is defined on $\mathbb{C} \setminus [0, 1]$, i.e. on the finite complex plane with an incision from 0 to 1.

2.1.1.1 Laplace Transforms and Discrete Laplace Transforms

In this section we shall establish a theorem that will be needed in Chapter 5 dealing with sampled-data control systems. First of all, we observe that the generalized discrete Laplace transform of the sequence $y(kT + \tau)$,

$$Y^*(s, \tau) = \sum_{k=0}^{\infty} y(kT + \tau)\, e^{-kTs},$$

is a periodic function of s with the imaginary period $2\pi i/T$:

$$Y^*\left(s + i\frac{2\pi}{T}, \tau\right) = Y^*(s, \tau).$$

Proof.

$$Y^*\left(s + i\frac{2\pi}{T}, \tau\right) = \sum_{k=0}^{\infty} y(kT + \tau)\, e^{-kT(s + i2\pi/T)}$$

$$= \sum_{k=0}^{\infty} y(kT + \tau)\, e^{-kTs} \exp\left(-kT\frac{i2\pi}{T}\right)$$

$$= \sum_{k=0}^{\infty} y(kT + \tau)\, e^{-kTs} = Y^*(s, \tau). \quad \square$$

For a given original function $y(t)$ with its Laplace image function $Y(s)$ and the discrete Laplace transform $Y^*(s, \tau)$, we shall now examine in detail the mapping $Y(s) \mapsto Y^*(s, \tau)$ and find a formula that expresses $Y^*(s, \tau)$ in terms of $Y(s)$.

Theorem. *Let the original function $y(t)$ be continuous for $t > 0$ with a finite $y(0+)$. If $y(t) \circ\!-\!\bullet\, Y(s)$ and if $y(kT+\tau) \xrightarrow{\mathcal{L}_T} Y^*(s, \tau)$ (discrete Laplace transform), then for $\tau = 0$ we have*

$$Y^*(s, 0) = \frac{y(0+)}{2} + \text{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} Y\Big(s + i\,\frac{2\pi}{T}\,n\Big)\,. \tag{2.8}$$

For $\tau > 0$ we have

$$Y^*(s, \tau) = \text{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} Y\Big(s + i\,\frac{2\pi}{T}\,n\Big)\exp\Big[\Big(s + i\,\frac{2\pi}{T}\,n\Big)\tau\Big]\,. \tag{2.9}$$

The two formulas show that if $y(0+) \neq 0$ the discrete Laplace transform $Y^*(s, \tau)$ has a discontinuity at $\tau = 0$ with respect to the parameter τ. The symbol pv (principal value) means that

$$\text{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} \ldots := \frac{1}{T}\lim_{m\to\infty}\sum_{n=-m}^{m} \ldots$$

These formulas can be proved, for example, by making use of *Poisson's summation formula.* It says

$$\sum_{k=-\infty}^{\infty} f(kT+\tau) = \text{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} \hat{f}\Big(\frac{2\pi}{T}\,n\Big)\,e^{i\,n\,(2\pi/T)\,\tau} \tag{2.10}$$

where

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-ixy}\,f(x)\,dx$$

is the Fourier transform of $f(x)$. The formula holds if

1. $\int_{-\infty}^{\infty} |\,f(x)\,|\,dx < \infty$,
2. $f(x)$ is locally of bounded variation, and at points of discontinuity f is normalized such that

$$f(x) := \frac{f(x-) + f(x+)}{2}.$$

For a proof of Poisson's summation formula, see any textbook on Fourier transforms, for example [Chandrasekharan (1989)] or [Gasquet and Witomski (1998)].

Proof of the theorem. We apply Poisson's summation formula to the original $f(t) := u(t)\,e^{-st}\,y(t)$ where we have explicitly written the unit-step function $u(t)$, and where the parameter s has a sufficiently large real part in order to satisfy the condition 1 above. The Fourier transform of $f(t) = u(t)\,e^{-st}\,y(t)$ becomes

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\, u(t)\, e^{-st}\, y(t)\, dt$$
$$= \int_{0}^{\infty} e^{-i\omega t}\, e^{-st}\, y(t)\, dt = \int_{0}^{\infty} e^{-(s+i\omega)t}\, y(t)\, dt$$
$$= \mathcal{L}[y(t)]\,(s + i\omega) = Y(s + i\omega).$$

Because we assumed that $y(t)$ is continuous for $t > 0$, the normalization of $f(t) = u(t)\, e^{-st}\, y(t)$ is $y(0+)/2$ at $t = 0$ and equals itself everywhere else. We apply Poisson's formula, first for $\tau > 0$. The left-hand side becomes

$$\sum_{k=-\infty}^{\infty} f(kT + \tau) = \sum_{k=-\infty}^{\infty} u(kT + \tau)\, e^{-s(kT+\tau)}\, y(kT + \tau)$$
$$= \sum_{k=0}^{\infty} e^{-s(kT+\tau)}\, y(kT + \tau) = e^{-s\tau} \sum_{k=0}^{\infty} e^{-skT}\, y(kT + \tau).$$

The right-hand side is

$$\mathrm{pv}\, \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2\pi}{T} n\right) e^{in(2\pi/T)\tau}$$
$$= \mathrm{pv}\, \frac{1}{T} \sum_{n=-\infty}^{\infty} Y\left(s + i\, \frac{2\pi}{T} n\right) e^{in(2\pi/T)\tau}.$$

Finally, we obtain for $\tau > 0$

$$\sum_{k=0}^{\infty} e^{-skT}\, y(kT + \tau)$$
$$= \mathrm{pv}\, \frac{1}{T} \sum_{n=-\infty}^{\infty} Y\left(s + i\, \frac{2\pi}{T} n\right) e^{i\left(s + i\frac{2\pi}{T} n\right)\tau}.$$

In the case $\tau = 0$ the left-hand side of Poisson's formula is

$$\sum_{k=-\infty}^{\infty} f(kT) = \sum_{k=-\infty}^{\infty} u(kT)\, e^{-skT}\, y(kT)$$
$$= \frac{y(0+)}{2} + \sum_{k=1}^{\infty} e^{-skT}\, y(kT)$$
$$= -\frac{y(0+)}{2} + \sum_{k=0}^{\infty} e^{-skT}\, y(kT)$$

therefore,

$$\sum_{k=0}^{\infty} e^{-skT}\, y(kT) = \frac{y(0+)}{2} + \mathrm{pv}\, \frac{1}{T} \sum_{n=-\infty}^{\infty} Y\left(s + i\, \frac{2\pi}{T} n\right). \quad \square$$

Let us apply the above formulas. Formula (2.8) can be written as

$$Y^{\dagger}(e^{Ts}) = \frac{y(0+)}{2} + \operatorname{pv} \frac{1}{T} \sum_{n=-\infty}^{\infty} Y\left(s + i\frac{2\pi}{T} n\right)$$

where $y(t) \circ\!-\!\blacksquare\ Y(s)$ and $y(kT) \circ\!-\!\diamond\ Y^{\dagger}(z)$. If we take for $y(t)$ the unit-step function we obtain

$$\frac{e^{Ts}}{e^{Ts}-1} = \frac{1}{2} + \frac{1}{T}\left\{\frac{1}{s} + \sum_{n=1}^{\infty}\left[\frac{1}{s + i\frac{2\pi}{T} n} + \frac{1}{s - i\frac{2\pi}{T} n}\right]\right\}$$

$$= \frac{1}{2} + \frac{1}{T}\left\{\frac{1}{s} + \sum_{n=1}^{\infty} \frac{2s}{s^2 + \left(\frac{2\pi}{T} n\right)^2}\right\}$$

hence

$$\frac{1}{1-e^{-Ts}} = \frac{e^{Ts}}{e^{Ts}-1}$$

$$= \frac{1}{2} + \frac{1}{T}\frac{1}{s} + \frac{2}{T}\sum_{n=1}^{\infty} \frac{s}{s^2 + \left(\frac{2\pi}{T} n\right)^2},$$

or, if we subtract 1 from both sides,

$$\frac{1}{e^{Ts}-1} = -\frac{1}{2} + \frac{1}{T}\frac{1}{s} + \frac{2}{T}\sum_{n=1}^{\infty} \frac{s}{s^2 + \left(\frac{2\pi}{T} n\right)^2}.$$

The first of the last two formulas has an interesting interpretation if we formally take the inverse Laplace transform on both sides. We obtain a relation in the distributional sense:

$$\sum_{n=0}^{\infty} \delta(t - nT) = \frac{1}{2}\delta(t) + \frac{1}{T}u(t) + \frac{2}{T}\sum_{n=1}^{\infty} u(t)\cos\frac{2\pi n}{T} t,$$

or

$$\frac{1}{2}\delta(t) + \sum_{n=1}^{\infty} \delta(t - nT) = u(t)\left\{\frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty} \cos\frac{2\pi n}{T} t\right\}.$$

Replacing t by $-t$ and adding the two equations gives

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty} \cos\frac{2\pi n}{T} t = \frac{1}{T}\sum_{n=-\infty}^{\infty} \exp\left[i\frac{2\pi n}{T} t\right].$$

2.1.1.2 z-Transforms and MacLaurin Transforms

Let $f(k) \circ\!-\!\diamond\ F(z)$, $|z| > \rho$ and assume that $F(s)/s$ is a Laplace image function with the half-plane of convergence $\Re s > x_0$. Then

$$\frac{F(s)}{s} = \sum_{k=0}^{\infty} \frac{f(k)}{s^{k+1}}$$

by the definition of the z-transform. The range of s is restricted to the right half-plane $\Re s > \max\{\rho, x_0\}$. Now

$$\sum_{k=0}^{\infty} \frac{f(k)}{s^{k+1}} = \sum_{k=0}^{\infty} f(k)\, \mathcal{L}\left[\frac{t^k}{k!}\right](s) = \sum_{k=0}^{\infty} f(k) \int_0^{\infty} e^{-st}\, \frac{t^k}{k!}\, dt$$

$$= \int_0^{\infty} e^{-st} \sum_{k=0}^{\infty} f(k)\, \frac{t^k}{k!}\, dt = \int_0^{\infty} e^{-st}\, \hat{F}(t)\, dt = \mathcal{L}\left[\hat{F}(t)\right](s).$$

We obtain the following proposition.

Proposition. *Let $F(z)$ be the z-transform and $\hat{F}(t)$ the MacLaurin transform of the sequence $f(k)$. Then*

$$\hat{F}(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right](t), \quad F(z) = z\, \mathcal{L}\left[\hat{F}(t)\right](z), \tag{2.11}$$

provided $F(s)/s$ is a Laplace image function.

This proposition can be used to compute MacLaurin transforms. For example:

$$\frac{1}{k!} \circ\!-\!\diamond\; e^{1/z} = F(z) \Longrightarrow \frac{F(s)}{s} = \frac{e^{1/s}}{s} \blacksquare\!-\!\square\; I_0\left[2\sqrt{t}\right].$$

Again we find that the MacLaurin transform of $1/k!$ is given by $I_0\left[2\sqrt{z}\right]$. As a second example let us compute the MacLaurin transform of $\sin k\,\omega$. We found earlier that

$$\sin \omega k \circ\!-\!\diamond\; \frac{z \sin \omega}{z^2 - 2z\cos\omega + 1}$$

$$= F(z) \Longrightarrow \frac{F(s)}{s} = \frac{\sin\omega}{s^2 - 2s\cos\omega + 1}$$

$$= \frac{\sin\omega}{(s - \cos\omega)^2 + 1 - \cos^2\omega}$$

$$= \frac{\sin\omega}{(s - \cos\omega)^2 + \sin^2\omega} \blacksquare\!-\!\square\; e^{t\cos\omega} \sin[(\sin\omega)\, t].$$

Therefore the MacLaurin transform of the sequence $f(k) = \sin\omega k$ is given by $\hat{F}(z) = e^{z\cos\omega} \sin[(\sin\omega)\, z]$.

The right formula of (2.11) can be used to find z–correspondences. But be careful! It only works if $f(k)$ has a z–transform $F(z)$. Consider the following example. Let us start with the generating function for the Hermite polynomials which is known to be

$$\hat{F}(t) = \sum_{k=0}^{\infty} \frac{H_k(x)}{k!}\, t^k = e^{2tx - t^2}.$$

By use of the correspondence

$$\exp[-a t^2] \circ\!-\!\bullet \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{s^2}{4a}\right)\operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right)$$

and the image-shift property we have

$$\hat{F}(t) = e^{2tx-t^2} \circ\!-\!\bullet \frac{1}{2}\sqrt{\pi}\exp\left(\frac{(s-2x)^2}{4}\right)\operatorname{erfc}\left(\frac{s-2x}{2}\right).$$

Thus the "z-transform of the sequence of Hermite polynomials" should be

$$H_k(x) \circ\!-\!\diamond \sqrt{\pi}\,\frac{z}{2}\exp\left[\left(\frac{z}{2}-x\right)^2\right]\operatorname{erfc}\left(\frac{z}{2}-x\right).$$

But this is wrong! The sequence of Hermite polynomials is not z–transformable. The above image function is not holomorphic at ∞.

Problem 2.1. The z-transform of the Legendre polynomials $P_k(x)$ is given by

$$P_k(\cos\vartheta) \circ\!-\!\diamond \frac{z}{\sqrt{z^2-2z\cos\vartheta+1}}$$

where ϑ is a parameter. Establish that the MacLaurin transform of $P_k(x)$ is

$$\hat{F}(z) = \sum_{k=0}^{\infty}\frac{P_k(\cos\vartheta)}{k!}z^k = e^{z\cos\vartheta}J_0(z\sin\vartheta)$$

where $J_0(x)$ is the Bessel function of order 0.

Problem 2.2. A generalization of the MacLaurin transform of a sequence $f(k, \alpha)$ depending on a parameter α is defined by

$$\hat{F}(z, \alpha) := \sum_{k=0}^{\infty}\frac{f(k, \alpha)}{\Gamma(k+\alpha+1)}z^k .$$

Show that if $f(k, \alpha)$ has the z-transform $F(z, \alpha)$, then

$$\hat{F}(z, \alpha) = z^{-\alpha}\,\mathcal{L}^{-1}\left[\frac{F(s, \alpha)}{s^{\alpha+1}}\right](z)$$

provided that $F(s, \alpha)/s^{\alpha+1}$ is a Laplace image function.

Problem 2.3. The z-transform of the generalized Laguerre polynomials $L_k^{(\alpha)}(x)$ is given by

$$L_k^{(\alpha)}(x) \circ\!-\!\diamond \left(\frac{z}{z-1}\right)^{\alpha+1}\exp\left(-\frac{x}{z-1}\right)$$

(for $\alpha = 0$ this is the ordinary Laguerre polynomial). Use Problem 2.2 to establish the generalized MacLaurin transform of $L_k^{(\alpha)}(x)$, i.e.,

$$\hat{F}(z, \alpha) := \sum_{k=0}^{\infty}\frac{L_k^{(\alpha)}(x)}{\Gamma(k+\alpha+1)}z^k = (xz)^{-\alpha/2}e^z J_\alpha\left(2\sqrt{xz}\right)$$

where $J_\alpha(x)$ is the Bessel function of order α. Also see the example in Section 2.1.2.5.

■ 2.1.2 Operational Properties

2.1.2.1 Linearity

The defining series of a z-transform implies that a z-transformation is a linear operator. The z-image of a linear combination of original sequences is the linear combination of the corresponding images:

$$Z[c_1 f_1(k) + c_2 f_2(k)](k) = c_1 Z[f_1(k)](k) + c_2 Z[f_2(k)](k). \tag{2.12}$$

2.1.2.2 Original Shift Properties

There are two shift properties relative to the independent discrete variable, similar to those of the Laplace transformation of Section 1.1.2.5. The right shift property is:

Proposition. *Let m be a positive fixed integer and let $f(k) \circ\!-\!\diamond F(z)$. For the original sequence shifted m units to the right, i.e. $u(k-m)\,f(k-m)$, we then have*

$$u(k-m)\,f(k-m) \circ\!-\!\diamond z^{-m} F(z). \tag{2.13}$$

Proof.

$$u(k-m)\,f(k-m) \circ\!-\!\diamond \sum_{k=m}^{\infty} f(k-m)\,z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k)\,z^{-(k+m)} = z^{-m} \sum_{k=0}^{\infty} f(k)\,z^{-k} = z^{-m} F(z)\ . \ \square$$

The left shift property is more important. If an original sequence is shifted m units to the left, the values over the negative discrete axis are replaced by zero. Thus the new original is $u(k)\,f(k+m)$. As for the Laplace transformation, see Section 1.1.1, the discrete unit-step $u(k)$ is omitted. The rule for a fixed positive integer m is:

Proposition. *Let m be a positive fixed integer and let $f(k) \circ\!-\!\diamond F(z)$. For the original sequence shifted m units to the left, we then have*

$$f(k+m) \circ\!-\!\diamond z^m F(f) - \sum_{k=0}^{m-1} f(k)\,z^{m-k}, \tag{2.14}$$

$$f(k+1) \circ\!-\!\diamond z\,F(f) - f(0)\,z,$$
$$f(k+2) \circ\!-\!\diamond z^2 F(f) - f(0)\,z^2 - f(1)\,z,$$
$$f(k+3) \circ\!-\!\diamond z^3 F(f) - f(0)\,z^3 - f(1)\,z^2 - f(2)\,z,$$
$$\ldots$$

This rule is similar to the differentiation rule of the Laplace transformation and it is used to solve forward difference equations.

Proof.

$$f(k+m) \circ\!-\!\diamond \sum_{k=0}^{\infty} f(k+m)\, z^{-k} = \sum_{k=m}^{\infty} f(k)\, z^{-(k-m)} = z^m \sum_{k=m}^{\infty} f(k)\, z^{-k}$$

$$= z^m \left\{ \sum_{k=0}^{\infty} f(k)\, z^{-k} - \sum_{k=0}^{m-1} f(k)\, z^{-k} \right\} = z^m F(f) - \sum_{k=0}^{m-1} f(k)\, z^{m-k} \,. \quad \square$$

2.1.2.3 The Image Scale Change or Damping Rule

Proposition. *Let $a \neq 0$ be any complex number, then*

$$f(k) \circ\!-\!\diamond F(z) \implies a^{-k} f(k) \circ\!-\!\diamond F(a\,z)\,. \tag{2.15}$$

Proof.

$$a^{-k} f(k) \circ\!-\!\diamond \sum_{k=0}^{\infty} a^{-k} f(k)\, z^{-k} = \sum_{k=0}^{\infty} f(k)\, (\mathrm{az})^{-k} = F(a\,z)\,. \quad \square$$

This rule implies

$$e^{-k\lambda} f(k) \circ\!-\!\diamond F(e^{\lambda} z)\,,$$
$$e^{-ki\omega} f(k) \circ\!-\!\diamond F(e^{i\omega} z)\,, \quad e^{ki\omega} f(k) \circ\!-\!\diamond F(e^{-i\omega} z)\,,$$
$$\cos k\omega\ f(k) \circ\!-\!\diamond \frac{1}{2} \{ F(e^{-i\omega} z) + F(e^{i\omega} z) \}\,,$$
$$\sin k\omega\ f(k) \circ\!-\!\diamond \frac{1}{2i} \{ F(e^{-i\omega} z) - F(e^{i\omega} z) \}\,.$$

2.1.2.4 Multiplication and Image Differentiation Rules

Proposition. *If $f(k) \circ\!-\!\diamond F(z)$, then*

$$k\, f(k) \circ\!-\!\diamond -z \frac{dF}{dz}\,. \tag{2.16}$$

Proof.

$$k\, f(k) \circ\!-\!\diamond \sum_{k=0}^{\infty} k\, f(k)\, z^{-k} = -z \sum_{k=0}^{\infty} f(k)\, (-k)\, z^{-k-1}$$

$$= -z \sum_{k=0}^{\infty} f(k) \frac{d}{dz} z^{-k} = -z \frac{d}{dz} \sum_{k=0}^{\infty} f(k)\, z^{-k} = -z \frac{dF}{dz}\,. \quad \square$$

Applying this rule repeatedly, we obtain for a fixed positive integer m,

$$f(k) \circ\!-\!\diamond F(z) \implies k^m f(k) \circ\!-\!\diamond \left(-z \frac{d}{dz}\right)^m F(z)\,.$$

For example,

$$k^2 f(k) \circ\!-\!\diamond \left(-z\frac{d}{dz}\right)^2 F(z)$$
$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)F(z) = \left(-z\frac{d}{dz}\right)\{-zF'(z)\}$$
$$= z\{F'(z) + zF''(z)\} = zF'(z) + z^2F''(z).$$

Starting from the definition of the z-transform, we obtain by successive differentiation,

$$F'(z) = \sum_{k=0}^{\infty} f(k)(-k)z^{-k-1}$$
$$= \sum_{k=1}^{\infty}(-1)f(k)k\,z^{-(k+1)} = \sum_{k=2}^{\infty}(-1)f(k-1)(k-1)z^{-k},$$
$$F''(z) = \sum_{k=2}^{\infty}(-1)f(k-1)(k-1)(-k)z^{-k-1}$$
$$= \sum_{k=2}^{\infty}(-1)^2 f(k-1)(k-1)kz^{-(k+1)}$$
$$= \sum_{k=3}^{\infty}(-1)^2 f(k-2)(k-2)(k-1)z^{-k}$$
$$= \sum_{k=0}^{\infty}(-1)^2 u(k-3)f(k-2)(k-2)(k-1)z^{-k}$$
$$\diamond\!-\!\circ\ (-1)^2 u(k-3)f(k-2)(k-2)(k-1).$$

This yields the image differentiation rule.

Proposition. *If m is a positive integer and $F(z) \diamond\!-\!\circ f(k)$, then*

$$F^{(m)}(z) \diamond\!-\!\circ (-1)^m u(k-1-m)(k-1)(k-2)\cdots(k-m)f(k-m). \tag{2.17}$$

Examples.

$$k \circ\!-\!\diamond -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2},$$
$$k^2 \circ\!-\!\diamond -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3},$$
$$k^3 \circ\!-\!\diamond -z\frac{d}{dz}\frac{z(z+1)}{(z-1)^3} = \frac{z(z^2+4z+1)}{(z-1)^4}.$$

The first original shift rule applied to $z/(z-a) \diamond\!-\!\circ a^k$ yields

$$F(z) = \frac{1}{(z-a)} \diamond\!-\!\circ u(k-1)\,a^{k-1}.$$

Differentiating m times and using the above image differentiation rule induces

$$F^{(m)}(z) = \frac{(-1)^m\, m\,!}{(z-a)^{m+1}}$$
$$\diamond\!-\!\circ\ (-1)^m\,(k-1)\,(k-2)\,\ldots\,(k-m)\,u(k-1-m)\,a^{k-1-m}.$$

The second original shift rule gives

$$\frac{z\, m\,!}{(z-a)^{m+1}} \diamond\!-\!\circ\ k(k-1)\cdots(k-m+1)\,u(k-m)\,a^{k-m}$$

and finally, by linearity

$$\frac{z}{(z-a)^{m+1}} \diamond\!-\!\circ\ \frac{k(k-1)\cdots(k-m+1)}{m!}\,u(k-m)\,a^{k-m}$$
$$= u(k-m)\binom{k}{m}a^{k-m}.$$

Replacing m by $m-1$ yields the important correspondence, valid for $m = 1, 2, \ldots,$

$$\frac{z}{(z-a)^m} \diamond\!-\!\circ\ u(k-m+1)\binom{k}{m-1}a^{k-m+1} = \binom{k}{m-1}a^{k-m+1} \tag{2.18}$$

which is the inverse z-transform of a partial fraction.

The *Pochhammer Symbol* is defined by

$$(a)_m := \begin{cases} 1, & m = 0 \\ a(a+1)(a+2)\ldots(a+m-1), & m \in \mathbb{N}. \end{cases}$$

Let us find the z-transform of $(k)_m$, where m is a nonnegative fixed integer. We have

$$(k)_0 = 1 \circ\!-\!\diamond\ \frac{z}{z-1},$$
$$(k)_1 = k \circ\!-\!\diamond\ \frac{z}{(z-1)^2},$$
$$(k)_2 = k(k+1) = k^2 + k \circ\!-\!\diamond\ \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} = \frac{2z^2}{(z-1)^3};$$

similarly,

$$(k)_3 \circ\!-\!\diamond\ \frac{6z^3}{(z-1)^4}.$$

Obviously, we have for $m = 1, 2, 3, \ldots$

$$(k)_m \circ\!-\!\diamond\ \frac{m!\,z^m}{(z-1)^{m+1}}.$$

Let us prove this by an induction on m. The step from m to $m+1$ is

$$(k)_{m+1} = (k)_m\,(k+m) = k\,(k)_m + m\,(k)_m$$
$$\circ\!-\!\diamond\; -z\,\frac{d}{dz}\left[\frac{m!\,z^m}{(z-1)^{m+1}}\right] + m\,\frac{m!\,z^m}{(z-1)^{m+1}}$$
$$= \frac{(m+1)!\,z^{m+1}}{(z-1)^{m+2}} = \frac{(m+1)!\,z^{m+1}}{(z-1)^{(m+1)+1}}\,.\quad \square$$

We can include the case $m = 0$ by writing

$$u(k-1)\,(k)_m \;\circ\!-\!\diamond\; \frac{m!\,z^m}{(z-1)^{m+1}} \qquad (m = 0,\,1,\,2,\,\ldots), \tag{2.19}$$

$$u(k-1)\,\frac{(k)_m}{m!} \;\circ\!-\!\diamond\; \frac{z^m}{(z-1)^{m+1}} \qquad (m = 0,\,1,\,2,\,\ldots). \tag{2.20}$$

By replacing k by $k+1$ and using the second original shift property, we obtain

$$\frac{(k+1)_m}{m!} \;\circ\!-\!\diamond\; \left(\frac{z}{z-1}\right)^{m+1} \qquad (m = 0,\,1,\,2,\,\ldots). \tag{2.21}$$

The image scale change property yields

$$a^k\,\frac{(k+1)_m}{m!} \;\circ\!-\!\diamond\; \left(\frac{z}{z-a}\right)^{m+1} \qquad (m = 0,\,1,\,2,\,\ldots). \tag{2.22}$$

2.1.2.5 The Image Shift Property

If $F(z) \diamond\!-\!\circ\; f(k)$, $|z| > \rho$, what can be said about $F(z-a) \diamond\!-\!\circ\; g(k)$? First of all, the circle of convergence of $F(z-a)$ is given by $|z| = |a| + \rho$, i.e. $F(z-a)$ will be a holomorphic function outside the circle of radius $|a| + \rho$. The reason is that

$$|z| > |a| + \rho \Longrightarrow |z| - |a| > \rho\,,$$

but $|z| - |a| \le \big||z| - |a|\big| \le |z-a|$ and $|z| > |a| + \rho$ implies $|z-a| > \rho$. Now

$$F(z-a) = \sum_{k=0}^{\infty} f(k)\,(z-a)^{-k}\,,$$

$$(z-a)^{-k} = \begin{cases} 1, & k = 0 \\ \sum_{n=0}^{\infty} \frac{(k)_n}{n!}\,a^n\,z^{-k-n}, & k \in \mathbb{N}\,, \end{cases}$$

therefore,

$$F(z-a) = f(0) + \sum_{k=1}^{\infty} f(k) \sum_{n=0}^{\infty} \frac{(k)_n}{n!} a^n z^{-(k+n)}$$

$$= f(0) + \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell} f(k) \frac{(k)_{(\ell-k)}}{(\ell-k)!} a^{(\ell-k)} z^{-\ell}$$

$$= f(0) + \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell} f(k) \binom{\ell-1}{\ell-k} a^{\ell-k} z^{-\ell}.$$

We have proved the following rule:

Proposition. *If* $F(z) \diamond\!-\!\circ f(k)$ *for* $|z| > \rho$, *then*

$$F(z-a) \diamond\!-\!\circ g(k) = f(0)\,\delta(k) + \sum_{i=1}^{k} f(i) \binom{k-1}{i-1} a^{k-i}\,, \quad k > 0 \tag{2.23}$$

for $|z| > |a| + \rho$. *The symbol* $\delta(k-m) := \delta_{km}$ *is the Kronecker symbol.*

Example. Assume that the z-transform of the Laguerre polynomial $L_k(x)$ is

$$\left(\frac{z}{z-1}\right) \exp\left(-\frac{x}{z-1}\right).$$

Let us find $L_k(x)$. The inverse z-transform of $\exp(-x/z)$ is the sequence $(-x)^k/k!$. The first original shift property implies

$$\frac{\exp(-x/z)}{z} \diamond\!-\!\circ u(k-1) \frac{(-x)^{k-1}}{(k-1)!} = f(k).$$

The above image shift property yields

$$\frac{\exp(-x/(z-1))}{z-1} \diamond\!-\!\circ g(k)$$

$$= u(k-1) \sum_{i=1}^{k} u(i-1) \frac{(-x)^{i-1}}{(i-1)!} \binom{k-1}{i-1}$$

$$= u(k-1) \sum_{i=0}^{k-1} \frac{(-x)^i}{i!} \binom{k-1}{i}.$$

The second original shift property implies

$$\left(\frac{z}{z-1}\right) \exp\left(-\frac{x}{z-1}\right) \diamond\!-\!\circ \sum_{i=0}^{k} \frac{(-x)^i}{i!} \binom{k}{i} = L_k(x).$$

2.1.2.6 The Division Rule

Let $f(k)$ be a sequence with $f(0) = 0$ and with z-transform $F(z)$. What is the z-transform of $f(k)/k$? For any z-image $Y(z)$ we can write

$$\int_z^\infty Y'(\xi)\,d\xi = Y(\infty) - Y(z) = y(0) - Y(z),$$

provided z is outside the circle of convergence of $Y(z)$. Let $y(k) := f(k)/k$ for $k > 0$ and $y(0) := 0$, then $f(k) = k\,y(k)$. Hence

$$f(k) \circ\!-\!\diamond\; -z\,Y'(z) = F(z) \Longrightarrow Y'(z) = -\frac{F(z)}{z}.$$

By the reasoning above

$$Y(z) = y(0) - \int_z^\infty Y'(\xi)\,d\xi = \int_z^\infty \frac{F(\xi)}{\xi}\,d\xi.$$

We have proved the following rule:

Proposition. *If* $f(k) \circ\!-\!\diamond\; F(z)$ *and* $f(0) = 0$, *then*

$$\frac{f(k)}{k} \circ\!-\!\diamond \int_z^\infty \frac{F(\xi)}{\xi}\,d\xi. \tag{2.24}$$

Example. $f(k) = u(k-1)\,a^k$. We have

$$f(k) = u(k-1)\,a^k \circ\!-\!\diamond \frac{a}{z-a},$$

$$\int_z^\infty \frac{a}{\xi(\xi-a)}\,d\xi = \int_z^\infty \left(\frac{1}{\xi-a} - \frac{1}{\xi}\right)d\xi$$

$$= \left(\ln\left(\frac{\xi-a}{\xi}\right)\right)_z^\infty = 0 - \ln\left(\frac{z-a}{z}\right),$$

finally,

$$u(k-1)\,\frac{a^k}{k} \circ\!-\!\diamond \ln\left(\frac{z}{z-a}\right), \quad u(k-1)\,\frac{1}{k} \circ\!-\!\diamond \ln\left(\frac{z}{z-1}\right).$$

Problem 2.4. Apply again the above proposition and find

$$u(k-1)\,\frac{1}{k^2} \circ\!-\!\diamond \int_z^\infty \frac{1}{\xi} \ln\left(\frac{\xi}{\xi-1}\right)d\xi,$$

...

$$u(k-1)\,\frac{1}{k^n} \circ\!-\!\diamond \frac{1}{(n-2)!} \int_z^\infty \frac{1}{\xi}\left(\ln\left(\frac{\xi}{2}\right)\right)^{n-2} \ln\left(\frac{\xi}{\xi-1}\right)d\xi \quad (n \in \mathbb{N},\ n \geq 2).$$

2.1.2.7 Convolution of Sequences

Let $f(k)$ and $g(k)$ be two original sequences.

Definition. The *convolution* $(f * g)(k)$ *of the two sequences* is the original sequence defined by

$$(f * g)(k) := \sum_{i=0}^{k} f(k-i)\, g(i). \tag{2.25}$$

Remark. For $(f * g)(k)$ we also write $f(k) * g(k)$.

Because of

$$\sum_{i=0}^{k} f(k-i)\, g(i) = \sum_{i=0}^{k} f(i)\, g(k-i) = \sum_{i=0}^{k} g(k-i)\, f(i)$$

the convolution product is commutative, i.e $f * g = g * f$. The convolution is also distributive, i.e. $f * (g + h) = f * g + f * h$. This follows immediately from the definition. The convolution of three originals is defined by

$$(f * g * h)(k) := \sum_{0 \le i+j \le k} f(k-i-j)\, g(i)\, h(j).$$

The convolution product is associative, i.e. $f * (g * h) = (f * g) * h$, because

$$(f * g * h)(k) := \sum_{0 \le i+j \le k} f(k-i-j)\, g(i)\, h(j)$$

$$= \sum_{\ell=0}^{k} \sum_{i+j=\ell} f(k-i-j)\, g(i)\, h(j) = \sum_{\ell=0}^{k} \sum_{j=0}^{\ell} f(k-\ell)\, g(\ell-j)\, h(j)$$

$$= \sum_{\ell=0}^{k} f(k-\ell) \sum_{j=0}^{\ell} g(\ell-j)\, h(j) = \sum_{\ell=0}^{k} f(k-\ell)\, (g * h)(\ell) = (f * (g * h))(k).$$

This shows that $f * g * h = f * (g * h)$. Similarly, $f * g * h = (f * g) * h$.

We now prove the convolution theorem which is similar to the convolution theorem of Laplace transformation. We start with the product of two z-images:

$$F(z)\, G(z) = \left(\sum_{k=0}^{\infty} f(k)\, z^{-k} \right) \left(\sum_{\ell=0}^{\infty} g(\ell)\, z^{-\ell} \right)$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} f(k)\, g(\ell)\, z^{-(k+\ell)} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} f(n-\ell)\, g(\ell)\, z^{-n} = \sum_{n=0}^{\infty} (f * g)(n)\, z^{-n}.$$

We have established the

Convolution Theorem. *If* $f(k) \circ\!-\!\diamond\ F(z)$ *and* $g(k) \circ\!-\!\diamond\ G(z)$, *then*

$$(f * g)(k) \circ\!-\!\diamond F(z)\,G(z). \tag{2.26}$$

The ordinary product of two z-image functions corresponds to the convolution product of two original sequences.

By taking $f(k) = u(k) = 1$, we obtain

$$(u * g)(k) = \sum_{i=0}^{k} 1\, g(i) \circ\!-\!\diamond \frac{z}{z-1}\, G(z).$$

This is the **Summation Rule.** *If* $f(k) \circ\!-\!\diamond F(z)$ *then*

$$s(k) := \sum_{i=0}^{k} f(i) \circ\!-\!\diamond S(z) = \frac{z}{z-1}\, F(z). \tag{2.27}$$

Example.

$$a^k * a^k = \sum_{i=0}^{k} a^{k-i}\, a^i = a^k \sum_{i=0}^{k} 1 = (k+1)\, a^k \circ\!-\!\diamond \left(\frac{z}{z-a}\right)^2$$

$$(k+1)\, a^k * a^k$$

$$= \sum_{i=0}^{k} a^{k-i}(i+1)\, a^i = a^k \sum_{i=0}^{k} (i+1) = a^k\, \frac{(k+1)(k+2)}{2}.$$

Therefore

$$a^k\, \frac{(k+1)(k+2)}{2} \circ\!-\!\diamond \left(\frac{z}{z-a}\right)^3.$$

By induction, we again find the correspondence

$$a^k\, \frac{(k+1)(k+2)\ldots(k+m)}{m!} = a^k\, \frac{(k+1)_m}{m!} = a^k \binom{k+m}{m} \circ\!-\!\diamond \left(\frac{z}{z-a}\right)^{m+1}.$$

Example. From the example at the end of Section 2.1.2.6 we immediately obtain the z-transform of the partial sums of the harmonic series:

$$f(k) := \sum_{\nu=1}^{k} \frac{1}{\nu} \circ\!-\!\diamond \frac{z}{z-1} \ln\left(\frac{z}{z-1}\right).$$

2.1.2.8 The Product Rule and Parseval's Relation

Let $f(k) \circ\!-\!\diamond F(z)$ for $|z| > r_1$ and $g(k) \circ\!-\!\diamond G(z)$ for $|z| > r_2$. We form the product

$$F(p)\, G\left(\frac{z}{p}\right) \frac{1}{p} = \left(\sum_{k=0}^{\infty} f(k)\, p^{-k}\right) \left(\sum_{\ell=0}^{\infty} g(\ell) \left(\frac{z}{p}\right)^{-\ell}\right) \frac{1}{p},$$

where p is assumed to be on the circle $|p| = r$ with $r_1 < r < |z|/r_2$. This implies that $|z| > r_1 r_2$ and $|z/p| > r r_2 / r = r_2$, hence

$$F(p)\,G\Big(\frac{z}{p}\Big)\,\frac{1}{p} = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} f(k)\,g(\ell)\,z^{-\ell}\,\frac{1}{p^{-\ell+1+k}}.$$

Now we take the line integral along the circle $|p| = r$ in the positive sense

$$\frac{1}{2\pi i}\int_C F(p)\,G\Big(\frac{z}{p}\Big)\,\frac{1}{p}\,dp = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} f(k)\,g(\ell)\,z^{-\ell}\;\frac{1}{2\pi i}\int_C \frac{1}{p^{-\ell+1+k}}\,dp,$$

where we interchanged integration and summation. For the integral on the right-hand side we have

$$\frac{1}{2\pi i}\int_C \frac{1}{p^{-\ell+1+k}}\,dp = \begin{cases} 1, & \ell = k \\ 0, & \ell \neq k \end{cases},$$

which implies

$$\frac{1}{2\pi i}\int_C F(p)\,G\Big(\frac{z}{p}\Big)\,\frac{1}{p}\,dp = \sum_{k=0}^{\infty} f(k)\,g(k)\,z^{-k}.$$

We have proved the following *product rule:*

Proposition. *Let* $f(k) \circ\!-\!\diamond\; F(z)$ *for* $|z| > r_1$ *and* $g(k) \circ\!-\!\diamond\; G(z)$ *for* $|z| > r_2$, *then*

$$f(k)\,g(k) \circ\!-\!\diamond\; H(z) = \frac{1}{2\pi i}\int_C \frac{F(p)}{p}\,G\Big(\frac{z}{p}\Big)\,dp, \tag{2.28}$$

where the image function $H(z)$ *is defined for* $|z| > r_1 r_2$, *and the line integral is taken along the circle* $|p| = r$ *in the positive sense with* $r_1 < r < |z|/r_2$.

Example. From

$$u(k-1)\,\frac{1}{k} \circ\!-\!\diamond\; \ln\Big(\frac{z}{z-1}\Big),$$

$$u(k-1)\,\frac{1}{k^2} \circ\!-\!\diamond\; \frac{1}{2\pi i}\int_C \frac{1}{p}\ln\Big(\frac{p}{p-1}\Big)\,\ln\Big(\frac{z/p}{z/p-1}\Big)\,dp,$$

where C is the circle centered at the origin and with radius r, $1 < r < |z|$. Finally

$$u(k-1)\,\frac{1}{k^2} \circ\!-\!\diamond\; \frac{1}{2\pi i}\int_C \frac{1}{p}\ln\Big(\frac{p}{p-1}\Big)\,\ln\Big(\frac{z}{z-p}\Big)\,dp$$

which gives the image function as a line integral.

Now assume that all singularities of $F(z)$ are strictly inside the unit circle. Taking $r_1 = 1$ in (2.28) and $|z| > 1$, we obtain

$$[f(k)]^2 \circ\!-\!\diamond\; \frac{1}{2\pi i}\int_C \frac{F(p)}{p}\,F\Big(\frac{z}{p}\Big)\,dp \tag{2.29}$$

which is called **Parseval's Theorem.**

If in the relation

$$\sum_{k=0}^{\infty} [f(k)]^2 z^{-k} = \frac{1}{2\pi i} \int_C \frac{F(p)}{p} F\left(\frac{z}{p}\right) dp,$$

the limit for $z \to 1$ taken from the outside of the unit-circle exists, we obtain

$$\sum_{k=0}^{\infty} [f(k)]^2 = \frac{1}{2\pi i} \int_C \frac{F(p)}{p} F\left(\frac{1}{p}\right) dp \tag{2.30}$$

which may be used to compute the energy of a discrete signal.

From the example above we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2\pi i} \int_C \frac{1}{p} \ln\left(\frac{p}{p-1}\right) \ln\left(\frac{1}{1-p}\right) dp.$$

Indeed the line integral can be computed along the contour in Fig.2.2.

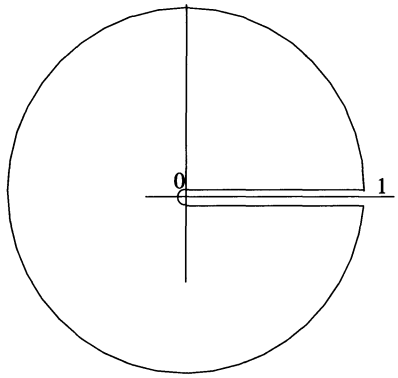


Fig.2.2: Integration contour

By Cauchy's integral theorem the line integral along this contour is zero since the integrand is holomorphic inside the contour. On the small semi-circle around the origin the line integral tends to zero if the radius shrinks to zero. On the upper side of the segment [0, 1] the integrand is

$$\frac{1}{x}\left\{\ln\left(\frac{x}{1-x}\right) - \pi i\right\} \ln\left(\frac{1}{1-x}\right),$$

and on the lower side it is

$$\frac{1}{x}\left\{\ln\left(\frac{x}{1-x}\right) + \pi i\right\} \ln\left(\frac{1}{1-x}\right).$$

Therefore we have

$$\frac{1}{2\pi i} \int_C \frac{1}{p} \ln\left(\frac{p}{p-1}\right) \ln\left(\frac{1}{1-p}\right) dp$$
$$= \int_0^1 \frac{1}{x} \ln\left(\frac{1}{1-x}\right) dx = \int_1^{\infty} \frac{\ln u}{u(u-1)} du = \frac{\pi^2}{6}.$$

2.1.2.9 The z-Transform of a Periodic Sequence

Definition. An original sequence $f(k)$ is periodic with the integer period N, if $f(k+N) = f(k)$ for all nonnegative integer k.

For the z-transform of $f(k)$ we then have

$$f(k) = f(k+N) \circ\!-\!\diamond \sum_{k=0}^{\infty} f(k)\, z^{-k} = \sum_{\ell=0}^{\infty}\sum_{k=0}^{N-1} f(k+\ell N)\, z^{-(k+\ell N)}$$

$$= \sum_{\ell=0}^{\infty}\sum_{k=0}^{N-1} f(k)\, z^{-(k+\ell N)} = \sum_{\ell=0}^{\infty} z^{-\ell N} \sum_{k=0}^{N-1} f(k)\, z^{-k} = \frac{1}{1-z^{-N}} \sum_{k=0}^{N-1} f(k)\, z^{-k}.$$

We have proved:

Proposition. *The z-transform of the periodic sequence $f(k)$ with period N is given by*

$$f(k) = f(k+N) \circ\!-\!\diamond \frac{z^N}{z^N-1} \sum_{k=0}^{N-1} f(k)\, z^{-k}. \tag{2.31}$$

The right-hand member can be written in the form

$$\frac{z\,P(z)}{z^N-1}$$

where $P(z)$ is a polynomial of degree $N-1$. This shows that the z-image of a periodic sequence is a rational function with at most N distinct simple poles which are N-th unit-roots of 1.

2.1.2.10 The Initial and Final Value Theorems

Initial Value Theorem. *If $f(k) \circ\!-\!\diamond F(z)$*, then $f(0) = \lim_{z\to\infty} F(z)$.

This follows immediately from the definition of the z-transform. Assume now that $\lim_{n\to\infty} f(n) = : a$ exists. We can write for any finite n,

$$f(n+1) - f(0) = \sum_{k=0}^{n} f(k+1) - f(k) = \lim_{z\to 1} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}.$$

By letting $n \to \infty$ and by the assumed fact that the limit on the left-hand side exists, we obtain

$$a - f(0) = \lim_{z\to 1} \sum_{k=0}^{\infty} [f(k+1) - f(k)]\, z^{-k}.$$

On the right-hand side is the z-transform of the sequence $f(k+1) - f(k)$ which is

$$z\,F(z) - z\,f(0) - F(z) = (z-1)\,F(z) - f(0).$$

Hence, we obtained the

Final Value Theorem. *If* $\lim\limits_{n\to\infty} f(n)$ *exists it can be computed by* $\lim_{z\to 1}(z-1)F(z)$.

Example. Let us test some established rules. We start with

$$u(k-1)\,\frac{1}{k} \circ\!-\!\diamond\, \ln\!\left(\frac{z}{z-1}\right)$$
$$\Longrightarrow \frac{1}{k+1} \circ\!-\!\diamond\, z\,\ln\!\left(\frac{z}{z-1}\right) - z\,0\,,$$

due to the second original shift rule. By the k-multiplication rule

$$k\,\frac{1}{k+1} \circ\!-\!\diamond\, -z\,\frac{d}{dz}\left\{z\,\ln\!\left(\frac{z}{z-1}\right)\right\}$$
$$= -z\left\{\ln\!\left(\frac{z}{z-1}\right) + z\,\frac{z-1}{z}\,\frac{-1}{(z-1)^2}\right\}$$
$$= -z\,\ln\!\left(\frac{z}{z-1}\right) + \frac{z}{(z-1)}\,.$$

Thus,

$$\lim_{z\to 1}(z-1)\left\{-z\ln\!\left(\frac{z}{z-1}\right) + \frac{z}{(z-1)}\right\}$$
$$= -\lim_{z\to 1} z\,(z-1)\ln\!\left(\frac{z}{z-1}\right) + \lim_{z\to 1} z \;=\; 0+1=1$$

which equals $\lim\limits_{n\to\infty} n/(n+1)$.

Problem 2.5. Establish the following correspondences

$$a^k\,\frac{(\gamma)_k}{k!} \circ\!-\!\diamond\, \left(1-\frac{a}{z}\right)^{-\gamma} = \left(\frac{z}{z-a}\right)^{\gamma}, \quad |z| > |a|,$$
$$a^k\binom{\gamma}{k} \circ\!-\!\diamond\, \left(1+\frac{a}{z}\right)^{\gamma} = \left(\frac{z+a}{z}\right)^{\gamma}, \quad |z| > |a|\,.$$

Problem 2.6. Use Problem 2.3 to establish the following relation between the Laguerre polynomials $L_k(x)$ and the generalized Laguerre polynomials $L_k^{(\alpha)}(x)$:

$$L_k^{(\alpha)}(x) = \sum_{\ell=0}^{k} \frac{(\alpha)_\ell}{\ell!}\,L_{k-\ell}(x)\,.$$

■ 2.1.3 The Inverse z-Transformation

2.1.3.1 Inversion Formulas

In Section 2.1.1 we introduced the following

Inversion formula. *If* $F(z)$ *for* $|z| > \rho$ *is a given z-image function, the inverse z-transform of* $F(z)$ *is obtained by*

$$f(k) = \frac{1}{2\pi i}\int_C F(z)\, z^{k-1}\, dz \quad (k = 0, 1, 2, \ldots), \tag{2.32}$$

where the line integral is taken in the positive sense along a simple closed path C, which encircles the origin and lies completely inside the domain $|z| > \rho$, *where* $F(z)$ *is a holomorphic function.*

If the singularities on or inside the circle of convergence $|z| = \rho$ are all isolated, i.e. there are no branch points, the line integral can be evaluated by the computation of residues.

Example.

$$F(z) = \frac{z(z-c)}{(z-a)(z-b)},$$

$$f(k) = \frac{1}{2\pi i}\int_C \frac{z^k(z-c)}{(z-a)(z-b)}\, dz = \operatorname*{Res}_{a,b}\left\{\frac{z^k(z-c)}{(z-a)(z-b)}\right\}$$

$$= \frac{a^k(a-c)}{a-b} + \frac{b^k(b-c)}{b-a} = \frac{1}{a-b}[(a-c)a^k - (b-c)b^k].$$

Another inversion formula, which stems from Taylor's theorem of power series, is

$$f(k) = \frac{1}{k!}\left(\frac{d^k}{dz^k}F\left(\frac{1}{z}\right)\right)_{z=0} \quad (k = 0, 1, 2, \ldots). \tag{2.33}$$

Example.

$$F(z) = \arctan\frac{1}{z},$$

$$F\left(\frac{1}{z}\right) = \arctan z = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - + \cdots$$

thus

$$f(k) = \frac{1-(-1)^k}{2}\,\frac{(-1)^{(k-1)/2}}{k} \quad (k = 0, 1, 2, \ldots).$$

2.1.3.2 The Inverse of Rational Image Functions

If the z-image $F(z)$ of a sequence $f(k)$ with $f(0) \neq 0$ is a rational function, the degree of the numerator polynomial equals the degree of the denominator polynomial. Only if $f(0) = 0$ is the image a proper rational function. Thus, if $F(z)$ is a rational z-image, we have two cases:

1. $F(z)$ is a proper rational function.
2. $F(z)/z$ is a proper rational function.

Furthermore, most rational image functions encountered in applications are real, i.e. a quotient of two polynomials with real coefficients. Such an image satisfies $F(\bar{z}) = \overline{F(z)}$. As we did for Laplace rational image functions, we assume in the following that the z-image functions are real rational functions. We now refer to Section 1.1.4.1. If we assume that the denominator polynomial $P(s)$ is completely factored out, the partial fraction decomposition of $F(z)$ in the first case is given by

$$F(z) = \frac{Q(z)}{P(z)}$$

$$= Q(z)/(z-\xi_1)^{\nu_1} \ldots (z-\xi_r)^{\nu_r}\ [(z-\alpha_1)(z-\overline{\alpha_1})]^{\mu_1} \ldots\ [(z-\alpha_c)(z-\overline{\alpha_c})]^{\mu_c}$$

$$= \sum_{k=1}^{r}\left[\frac{b_{k1}}{(z-\xi_k)^{\nu_k}} + \frac{b_{k2}}{(z-\xi_k)^{\nu_k-1}} + \cdots + \frac{b_{k\nu_1}}{(z-\xi_k)}\right]$$

$$+ \sum_{k=1}^{c}\left[\left\{\frac{a_{k1}}{(z-\alpha_k)^{\nu_k}} + \frac{\overline{a_{k1}}}{(z-\overline{\alpha_k})^{\nu_k}}\right\} + \cdots + \left\{\frac{a_{k\nu_1}}{(z-\alpha_k)} + \frac{\overline{a_{k\nu_1}}}{z-\overline{\alpha_k}}\right\}\right]$$

where ξ_ν are all distinct real zeros and $(\alpha_\mu, \overline{\alpha_\mu})$ are the pairs of all distinct nonreal zeros of $P(z)$. In the second case the left-hand side must be replaced by $F(z)/z$. We see that the inverse z-transform of a rational image function can be computed if we know the inverse z-transforms of the following partial fractions:

1.

$$\frac{1}{(z-\xi)^\nu}, \quad \frac{a}{(z-\alpha)^\nu} + \frac{\overline{a}}{(z-\overline{\alpha})^\nu}$$

for the first case, and

2.

$$\frac{z}{(z-\xi)^\nu}, \quad \frac{az}{(z-\alpha)^\nu} + \frac{\overline{a}z}{(z-\overline{\alpha})^\nu}$$

for the second case. From (2.18) we have for a real or a nonreal zero a

$$\frac{z}{(z-a)^\nu} \diamond\!-\!\circ \binom{k}{\nu-1} a^{k-\nu+1}, \quad (\nu = 1, 2 \ldots), \tag{2.34}$$

or explicitly,

$$\frac{z}{z-a} \diamond\!-\!\circ a^k, \quad \frac{z}{(z-a)^2} \diamond\!-\!\circ k\,a^{k-1},$$

$$\frac{z}{(z-a)^3} \diamond\!-\!\circ \frac{k(k-1)}{2} a^{k-2}, \quad \frac{z}{(z-a)^4} \diamond\!-\!\circ \frac{k(k-1)(k-2)}{3!} a^{k-3}$$

etc.

The first original shift property then implies

$$\frac{1}{(z-a)^\nu} \diamond\!-\!\circ u(k-1)\binom{k-1}{\nu-1} a^{k-\nu}, \quad (\nu = 1, 2 \ldots), \tag{2.35}$$

or explicitly

$$\frac{1}{z-a} \diamond\!-\!\circ\; u(k-1)\,a^{k-1},$$

$$\frac{1}{(z-a)^2} \diamond\!-\!\circ\; u(k-1)\,(k-1)\,a^{k-2},$$

$$\frac{1}{(z-a)^3} \diamond\!-\!\circ\; u(k-1)\,\frac{(k-1)\,(k-2)}{2}\,a^{k-3},$$

$$\frac{1}{(z-a)^4} \diamond\!-\!\circ\; u(k-1)\,\frac{(k-1)\,(k-2)\,(k-3)}{3!}\,a^{k-4}$$

etc. Thus, if $a = \xi$ is real, we have what we need. For the pair

$$\frac{a\,z}{(z-\alpha)^{\nu}} + \frac{\overline{a}\,z}{(z-\overline{\alpha})^{\nu}}$$

with a nonreal $\alpha = |\alpha|\,e^{i\gamma}$ and $a = |a|\,e^{i\varphi}$ we obtain

$$\begin{aligned}
&\frac{a\,z}{(z-\alpha)^{\nu_k}} + \frac{\overline{a}\,z}{(z-\overline{\alpha})^{\nu}} \\
&\diamond\!-\!\circ\; |a|\,e^{i\varphi}\binom{k}{\nu-1}(|\alpha|\,e^{i\gamma})^{k-\nu+1} \\
&\qquad + |a|\,e^{-i\varphi}\binom{k}{\nu-1}(|\alpha|\,e^{-i\gamma})^{k-\nu+1} \\
&= |a|\,(|\alpha|)^{k-\nu+1}\binom{k}{\nu-1}\{e^{I[\gamma(k-\nu+1)+\varphi]} + e^{-I[\gamma(k-\nu+1)+\varphi]}\} \\
&= 2\,|a|\,(|\alpha|)^{k-\nu+1}\binom{k}{\nu-1}\cos[\gamma(k-\nu+1)+\varphi]\,.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{a\,z}{(z-\alpha)^{\nu_k}} + \frac{\overline{a}\,z}{(z-\overline{\alpha})^{\nu}} \\
&\diamond\!-\!\circ\; 2\,|a|\,(|\alpha|)^{k-\nu+1}\binom{k}{\nu-1}\cos[\gamma(k-\nu+1)+\varphi],
\end{aligned} \tag{2.36}$$

and similarly,

$$\begin{aligned}
&\frac{a}{(z-\alpha)^{\nu_k}} + \frac{\overline{a}}{(z-\overline{\alpha})^{\nu}} \\
&\diamond\!-\!\circ\; 2\,u(k-1)\,|a|\,(|\alpha|)^{k-\nu}\binom{k-1}{\nu-1}\cos[\gamma(k-\nu)+\varphi].
\end{aligned} \tag{2.37}$$

Example.

$$F(z) = \frac{z(z+1)\,(z^2+2)}{(z-2)^2\,(z^2+2z+2)}.$$

The partial fraction decomposition yields

$$\frac{F(z)}{z} = \frac{(z+1)(z^2+2)}{(z-2)^2(z^2+2z+2)}$$

$$= \frac{9}{5}\frac{1}{(z-2)^2} + \frac{18}{25}\frac{1}{z-2} + \left(\frac{7}{50} - \frac{i}{50}\right)\frac{1}{z-(-1+i)} + \left(\frac{7}{50} + \frac{i}{50}\right)\frac{1}{z-(-1-i)},$$

therefore

$$F(z) =$$

$$\frac{9}{5}\frac{z}{(z-2)^2} + \frac{18}{25}\frac{z}{z-2} + \left(\frac{7}{50} - \frac{i}{50}\right)\frac{z}{z-(-1+i)} + \left(\frac{7}{50} + \frac{i}{50}\right)\frac{z}{z-(-1-i)},$$

$$\frac{7}{50} + \frac{i}{50} = \frac{1}{\sqrt{50}}\, e^{i\arctan(1/7)},\quad -1+i = \sqrt{2}\, e^{i3\pi/4},$$

$$f(k) = \frac{9}{5}k\,2^{k-1} + \frac{18}{25}2^k + \frac{2}{\sqrt{50}}\left(\sqrt{2}\right)^k \cos\left[\frac{3\pi}{4}k + \arctan\frac{1}{7}\right].$$

Problem 2.7. Show that we also can write, with $\alpha = |\alpha|\, e^{i\gamma}$ and $a = A + Bi$:

$$\frac{az}{(z-\alpha)^{\nu_k}} + \frac{\overline{a}z}{(z-\overline{\alpha})^{\nu}}$$

$$\diamond\!-\!\circ\; 2\,(|\alpha|)^{k-\nu+1}\binom{k}{\nu-1}\{A\cos[\gamma(k-\nu+1)] - B\sin[\gamma(k-\nu+1)]\},$$

$$\frac{a}{(z-\alpha)^{\nu_k}} + \frac{\overline{a}}{(z-\overline{\alpha})^{\nu}}$$

$$\diamond\!-\!\circ\; 2\,u(k-1)\,(|\alpha|)^{k-\nu}\binom{k-1}{\nu-1}\{A\cos[\gamma(k-\nu)] - B\sin[\gamma(k-\nu)]\}\,.$$

2.2 Difference Equations

■ 2.2.1 Difference Equations of Order n

A linear forward difference equation of order n with constant coefficients is a relation of the type

$$y(k+n) + c_{n-1}\,y(k+n-1) + \cdots + c_1\,y(k+1) + c_0\,y(k) = f(k). \tag{2.38}$$

If the first n initial values are given, i.e.,

$$y(0) = y_0,\ y(1) = y_1,\ \ldots,\ y(n-1) = y_{n-1}, \tag{2.39}$$

then (2.38) with $k = 0$ enables one to compute $y(n)$. By applying again (2.38), now with $k = 1$, $y(n+1)$ is computed and so on. The two relations (2.38) and (2.39) constitute an initial value problem and we just saw that it has a unique solution. It is easy to compute recursively the sequence $y(k)$ or the first few k's, but our aim is to obtain a closed expression for the unknown

sequence $y(k)$. In order to achieve this, we assume that the unknown sequence has a z-transform, i.e $y(k) \circ\!-\!\diamond\, Y(z)$. By the second original shift rule we have

$$
\begin{aligned}
&y(k+1) \circ\!-\!\diamond\, z\,Y(z) - z\,y_0\,,\\
&y(k+2) \circ\!-\!\diamond\, z^2\,Y(z) - z^2\,y_0 - z\,y_1\,,\\
&\cdots\\
&y(k+n) \circ\!-\!\diamond\, z^n\,Y(z) - z^n\,y_0 - z^{n-1}\,y_1 - \cdots - z^2\,y_{n-2} - z\,y_{n-1}\,.
\end{aligned}
$$

If we also assume that $f(k) \circ\!-\!\diamond\, F(z)$, we obtain, by use of the linearity rule, the image equation

$$
\begin{aligned}
&(z^n + c_{n-1}\,z^{n-1} + \cdots + c_1\,z + c_0)\,Y(z)\\
&- y_0(z^n + c_{n-1}\,z^{n-1} + \cdots + c_1\,z) - y_1(z^{n-1} + c_{n-1}\,z^{n-2} + \cdots + c_2\,z)\\
&- \cdots - y_{n-2}(z^2 + c_{n-1}\,z) - y_{n-1}\,z = F(z)\,.
\end{aligned}
$$

To (2.38) we associate the *characteristic polynomial*

$$
P(z) = z^n + c_{n-1}\,z^{n-1} + \cdots + c_1\,z + c_0. \tag{2.40}
$$

The z-transform of the global solution of the initial value problem can then be expressed as

$$
\begin{aligned}
Y(z) = {} & \frac{F(z)}{P(z)} + y_0\,\frac{z^n + c_{n-1}\,z^{n-1} + \cdots + c_1\,z}{P(z)}\\
&+ y_1\,\frac{z^{n-1} + c_{n-1}\,z^{n-2} + \cdots + c_2\,z}{P(s)}\\
&+ \cdots + y_{n-2}\,\frac{z^2 + c_{n-1}\,z}{P(z)} + y_{n-1}\,\frac{z}{P(z)}\,.
\end{aligned} \tag{2.41}
$$

Again we call

$$
y_1(k) := Z^{-1}\left[\frac{F(z)}{P(z)}\right](k) \tag{2.42}
$$

the *forced response* of the discrete system, modeled by the difference equation, and

$$
\begin{aligned}
&y_0(k) :=\\
&\quad y_0\,Z^{-1}\left[\frac{z^n + c_{n-1}\,z^{n-1} + \cdots + c_1\,z}{P(z)}\right](k)\\
&+ y_1\,Z^{-1}\left[\frac{z^{n-1} + c_{n-1}\,z^{n-2} + \cdots + c_2\,z}{P(z)}\right](k)\\
&+ \ldots\\
&+ y_{n-2}\,Z^{-1}\left[\frac{z^2 + c_{n-1}\,z}{P(z)}\right](k)\\
&+ y_{n-1}\;Z^{-1}\left[\frac{z}{P(z)}\right](k)
\end{aligned} \tag{2.43}
$$

is said to be its *free response.*

In the difference equation (2.38) we often have another difference operator that is acting on the input sequence. The difference equation is then

$$y(k+n) + c_{n-1}\, y(k+n-1) + \cdots + c_1\, y(k+1) + c_0\, y(k) = b_0\, f(k) + b_1\, f(k+1) + \cdots + b_m\, f(k+m). \tag{2.44}$$

By proceeding as above, we can now write

$$y_1(k) := Z^{-1}[G(z)]\,(k) \tag{2.45}$$

for the forced response, where

$$G(s) = \frac{b_0 + b_1\, z + \cdots + b_m\, z^m}{P(z)} \tag{2.46}$$

is called the *transfer function* of the discrete system. The expression (2.43) for the free response has to be replaced in this more general case by

$$y_0(k) := y_0\, Z^{-1}\left[\frac{z^n + c_{n-1}\, z^{n-1} + \cdots + c_1\, z}{P(z)}\right](k)$$

$$+ y_1\, Z^{-1}\left[\frac{z^{n-1} + c_{n-1}\, z^{n-2} + \cdots + c_2\, z}{P(z)}\right](k)$$

$$+ \ldots$$

$$+ y_{n-2}\, Z^{-1}\left[\frac{z^2 + c_{n-1}\, z}{P(z)}\right](k) + y_{n-1}\, Z^{-1}\left[\frac{z}{P(z)}\right](k)$$

$$+ y_{n-1}\, Z^{-1}\left[\frac{z}{P(z)}\right](k)$$

$$- f(0)\, Z^{-1}\left[\frac{b_m\, z^m + b_{m-1}\, z^{m-1} + \cdots + b_1\, z}{P(z)}\right](k)$$

$$- f(1)\, Z^{-1}\left[\frac{b_m\, z^{m-1} + b_{m-1}\, z^{m-2} + \cdots + b_2\, z}{P(z)}\right](k)$$

$$- \cdots$$

$$- f(m-2)\, Z^{-1}\left[\frac{b_m\, z^2 + b_{m-1}\, z}{P(z)}\right](k)$$

$$- f(m-1)\, Z^{-1}\left[\frac{b_m\, z}{P(z)}\right](k).$$

Example. $y(k+2) - 7\,y(k+1) + 10\,y(k) = 16\,k$; $y(0) = 6$, $y(1) = 2$,

$$P(z) = z^2 - 7z + 10 = (z-5)(z-2),$$
$$F(z) = 16z/(z-1)^2,$$

$$Y(z) = \frac{16z}{(z-1)^2(z-5)(z-2)} + 6\,\frac{z^2 - 7z}{(z-5)(z-2)} + 2\,\frac{z}{(z-5)(z-2)}$$
$$= \frac{2z(3z^3 - 26z^2 + 43z - 12)}{(z-1)^2(z-5)(z-2)}.$$

The decomposition in partial fractions yields

$$\frac{Y(z)}{z} = 4\,\frac{1}{(z-1)^2} + 5\,\frac{1}{z-1} - 3\,\frac{1}{z-5} + 4\,\frac{1}{z-2}$$

thus

$$Y(z) = 4\,\frac{z}{(z-1)^2} + 5\,\frac{z}{z-1} - 3\,\frac{z}{z-5} + 4\,\frac{z}{z-2}\,.$$

By applying the inverse z-transform, we finally obtain

$$y(k) = 4k + 5 - 3\,(5^k) + 4\,(2^k)\,.$$

Example. $y(k+2) - 2\,y(k+1) + 2\,y(k) = 0;\ y(0) = 0,\ y(1) = 1,$

$$P(z) = z^2 - 2z + 2 = (z-1)^2 + 1 = [z-(1+i)]\,[z-(1-i)],$$

$$Y(z) = \frac{z}{[z-(1+i)]\,[z-(1-i)]}.$$

Since this image function is already a proper rational function, the partial fraction decomposition can directly be applied. Nevertheless it is in general advisable to apply the partial fraction decomposition to $Y(z)/z$ in the case where the numerator polynomial contains the factor z.

$$\frac{Y(z)}{z} = \frac{c}{[z-(1+i)]} + \frac{\overline{c}}{[z-(1-i)]}$$

$$= -\frac{i}{2}\,\frac{1}{[z-(1+i)]} + \frac{i}{2}\,\frac{1}{[z-(1-i)]}\,,$$

$$Y(z) = -\frac{i}{2}\,\frac{z}{[z-(1+i)]} + \frac{i}{2}\,\frac{z}{[z-(1-i)]}$$

$$\left(A = 0,\ B = -1/2,\ |\,\alpha\,| = \sqrt{2},\ \gamma = \pi/4\right),$$

$$y(k) = 2\left(\sqrt{2}\right)^k \left\{0 + \frac{1}{2}\sin\left(k\,\frac{\pi}{4}\right)\right\} = \left(\sqrt{2}\right)^k \sin\left(k\,\frac{\pi}{4}\right).$$

The two last examples were forward difference equations. The next one is a backward difference equation.

Example. $3\,y(k) - 4\,y(k-1) + y(k-2) = k.$

The initial conditions are given by the fact that the sequence $y(k)$ is supposed to be an original sequence, i.e. $y(-1) = y(-2) = \cdots = 0$. We apply the first original shift rule:

$$y(k) \circ\!-\!\diamond\ Y(z),\quad y(k-1) \circ\!-\!\diamond\ \frac{Y(z)}{z},\quad y(k-2) \circ\!-\!\diamond\ \frac{Y(z)}{z^2},$$

$$\left(3 - 4\,\frac{1}{z} + \frac{1}{z^2}\right) Y(z) = \frac{z}{(z-1)^2}\,,$$

$$Y(z) = \frac{1}{3}\left(\frac{z}{z-1}\right)^3 \frac{1}{z-1/3} = \frac{1}{3}\,\frac{z^3}{(z-1)^3\,(z-1/3)}.$$

The first expression of the right-hand side calls for an application of the convolution theorem by making use of the correspondences

$$a^k \frac{(k+1)_m}{m!} \circ\!-\!\diamond \left(\frac{z}{z-a}\right)^{m+1}, \quad \frac{1}{z-1/3} \diamond\!-\!\circ\, u(k-1)\left(\frac{1}{3}\right)^{k-1},$$

$$y(k) = \frac{1}{3}\sum_{\nu=0}^{k} u(k-\nu-1)\left(\frac{1}{3}\right)^{k-\nu-1} \frac{(\nu+1)_2}{2!},$$

$$= \frac{u(k-1)}{2}\sum_{\nu=0}^{k-1} \frac{(\nu+1)(\nu+2)}{3^{k-\nu}}.$$

By applying the partial fraction decomposition to the second expression, a closed expression for the solution is obtained

$$\frac{1}{3}\frac{z^3}{(z-1)^3(z-1/3)}$$

$$= \frac{1}{2(-1+z)^3} + \frac{3}{4(-1+z)^2} + \frac{3}{8(-1+z)} - \frac{1}{24(-\frac{1}{3}+z)}$$

$$y(k) = u(k-1)\left\{\frac{(k-1)(k-2)}{4} + \frac{3}{4}(k-1) + \frac{3}{8} - \frac{1}{24}\frac{1}{3^{k-1}}\right\}.$$

2.2.1.1 Stability of a Discrete System

A discrete system modeled by (2.44) is said to be *asymptotically stable* if any free response to the system is a transient, i.e. goes to zero for k→∞. The behavior of a discrete asymptotically stable system is up to a transient determined by its transfer function, and can therefore be represented by the block diagram in Fig.2.3.

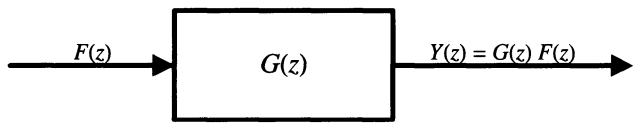


Fig.2.3: Block diagram of a discrete system

An inspection of the original sequences corresponding to the partial fractions (Section 2.1.3.2) shows immediately that the free response is a transient, if and only if all zeros of the characteristic polynomial $P(s)$ are in absolute value strictly smaller than 1, i.e. are located inside the unit-circle of the complex z-plane. These zeros are the poles of the transfer function.

Proposition. *The discrete system described by the transfer function G(z) is asymptotically stable, if all poles of the transfer function are located strictly inside the unit-circle.*

A stability test for a discrete system tests whether or not all zeros of a given polynomial $P(z)$ are inside the unit-circle. There are several possibilities to conceive such a test. One can use the Hurwitz test which permits us to know whether or not a polynomial has all roots located in the left complex half-plane (see Section 1.3.1.1). The following Moebius transformation maps the left half-plane of the complex s-plane onto the interior of the unit-circle of the z-plane:

$$s \mapsto z = w(s) := \frac{1+s}{1-s}.$$

The imaginary s-axis is mapped on the unit-circle, the point $s = -1$ is mapped on $z = 0$. By writing

$$\begin{aligned} P(z) &= z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \\ &= \left(\frac{1+s}{1-s}\right)^n c_{n-1}\left(\frac{1+s}{1-s}\right)^{n-1} + \cdots + c_1 \frac{1+s}{1-s} + c_0 \\ &= \frac{1}{(1-s)^n}\left((1+s)^n + c_{n-1}(1+s)^{n-1}(1-s)\right. \\ &\qquad \left. + \cdots + c_1 (1+s)(1-s)^{n-1} + c_0(1-s)^n\right) \\ &=: \frac{h(s)}{(1-s)^n}. \end{aligned}$$

we have

$$h(s) = 0 \;\&\; \Re s \le 0 \Longleftrightarrow P(z) = 0 \;\&\; |z| \le 1.$$

By applying the Hurwitz test to the polynomial

$$\begin{aligned} h(s) := (1+s)^n &+ c_{n-1}(1+s)^{n-1}(1-s) + \cdots \\ &+ c_1 (1+s)(1-s)^{n-1} + c_0(1-s)^n \end{aligned}$$

we obtain a desired test which informs us whether or not the zeros of $P(z)$ are situated in the interior of the unit-circle. The first three polynomials $h(s)$ are:

$$\begin{aligned} n = 1: \;& h(s) = (c_0 + 1) + s(-c_0 + 1), \\ n = 2: \;& h(s) = (c_0 + c_1 + 1) + s(-2c_0 + 2) + s^2(c_0 - c_1 + 1), \\ n = 3: \;& h(s) = (c_0 + c_1 + c_2 + 1) + s(-3c_0 - c_1 + c_2 + 3) \\ & \quad + s^2(3c_0 - c_1 - c_2 + 3) + s^3(-c_0 + c_1 - c_2 + 1). \end{aligned}$$

Example. The polynomial $P(z) = z^2 + c_1 z + c_0$ has all its roots inside the unit-circle if and only if

$$c_0 + c_1 + 1 > 0 \;\&\; -2c_0 + 2 > 0 \;\&\; c_0 - c_1 + 1 > 0.$$

This means that the point (c_1, c_0) has to be inside the triangle A(–1,0), B(1,–1), C(1,1) of the (c_1, c_0)-plane.

Jury (see, for example, [Jury (1977)], [Sarachik (1996)]) has developed a test which works directly with the coefficients of the polynomial $P(z)$, without passing through the Moebius transformation.

Another test is based on the Principle of the Argument from complex analysis (see for example [Chabat, Lavrentiev (1972)]). This theorem says: if $P(z)$ is a polynomial of degree n which is non-vanishing on the unit-circle of the complex z-plane, then the number of zeros inside the unit-circle is equal to the number of complete revolutions around the origin of the image point $w = P(z)$, when z traverses once the unit-circle.

Test. Assume that the contour $w = P(z)$ does not pass through the origin $w = 0$ if z traverses the unit-circle. All zeros of the polynomial of degree n are strictly inside the unit-circle, if the point $w = P(z)$ encircles exactly n times the origin $w = 0$ when z traverses once the unit-circle. n is the degree of $P(s)$.

Example (Fig.2.4). The contour of the polynomial $P(z) = 4z^3 - 6z^2 + 4z - 1$ encircles the origin exactly three times which implies that all zeros are inside the unit-circle.

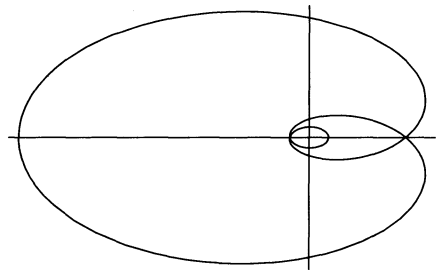


Fig.2.4: Contour in the z-plane.

Example (Fig.2.5). The contour of the polynomial $P(z) = 2z^3 - 6z^2 + 5z - 2$ encircles the origin only twice which implies that there is at least one zero outside the unit-circle.

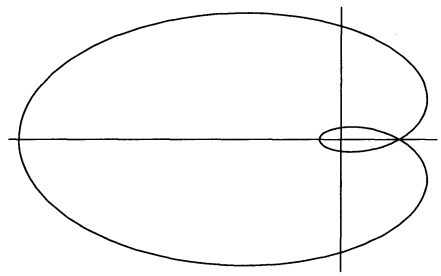


Fig.2.5: Contour in the z-plane.

■ 2.2.2 Systems of Difference Equations

Systems of difference equations are treated straightforwardly by the means of z-transformation. A first example is given by the problem to find a closed formula for the k^{th} power of a square matrix. The advantage of z-transformation over the methods of linear algebra is that no complicated theory of normal forms of matrices is needed. We explain the method for a general 2×2 -matrix. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad A^k := \begin{pmatrix} y_1(k) & y_2(k) \\ y_3(k) & y_4(k) \end{pmatrix}; \quad A^0 := \begin{pmatrix} y_1(0) & y_2(0) \\ y_3(0) & y_4(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix product yields then

$$\begin{aligned} A^{k+1} &:= \begin{pmatrix} y_1(k+1) & y_2(k+1) \\ y_3(k+1) & y_4(k+1) \end{pmatrix} \\ &= A\,A^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1(k) & y_2(k) \\ y_3(k) & y_4(k) \end{pmatrix} = \begin{pmatrix} a\,y_1(k) + b\,y_3(k) & a\,y_2(k) + b\,y_4(k) \\ c\,y_1(k) + d\,y_3(k) & c\,y_2(k) + d\,y_4(k) \end{pmatrix}, \end{aligned}$$

i.e. a system of four coupled difference equations

$$\begin{aligned} y_1(k+1) &= a\,y_1(k) + b\,y_3(k); \quad y_2(k+1) = a\,y_2(k) + b\,y_4(k); \\ y_3(k+1) &= c\,y_1(k) + d\,y_3(k); \quad y_4(k+1) = c\,y_2(k) + d\,y_4(k). \end{aligned}$$

By assuming $y_1(k) \circ\!-\!\diamond Y_1(z)$, $y_2(k) \circ\!-\!\diamond Y_2(z)$, $y_3(k) \circ\!-\!\diamond Y_3(z)$, and finally $y_4(k) \circ\!-\!\diamond Y_4(z)$, we can write the image system

$$z\,Y_1(z) - z = a\,Y_1(z) + b\,Y_3(z);\; z\,Y_2(z) - 0 = a\,Y_2(z) + b\,Y_4(z);$$
$$z\,Y_3(z) - 0 = c\,Y_1(z) + d\,Y_3(z);\; z\,Y_4(z) - z = c\,Y_2(z) + d\,Y_4(z),$$

or, in matrix form,

$$\begin{pmatrix} z-a & -b \\ -c & z-d \end{pmatrix}\begin{pmatrix} Y_1(z) \\ Y_3(z) \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}, \quad \begin{pmatrix} z-a & -b \\ -c & z-d \end{pmatrix}\begin{pmatrix} Y_2(z) \\ Y_4(z) \end{pmatrix} = \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

With the inverse of the coefficient matrix we obtain

$$\begin{pmatrix} Y_1(z) \\ Y_3(z) \end{pmatrix} = \frac{1}{P(z)}\begin{pmatrix} -d+z & b \\ c & -a+z \end{pmatrix}\begin{pmatrix} z \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} Y_2(z) \\ Y_4(z) \end{pmatrix} = \frac{1}{P(z)}\begin{pmatrix} -d+z & b \\ c & -a+z \end{pmatrix}\begin{pmatrix} 0 \\ z \end{pmatrix},$$

where

$$P(z) = z^2 - (a+d)\,z + (a\,d - b\,c) = z^2 - \operatorname{tr}(A)\,z + \det A.$$

Therefore

$$Y_1(z) = \frac{z(z-d)}{P(z)}, \quad Y_2(z) = \frac{b\,z}{P(z)}, \quad Y_3(z) = \frac{c\,z}{P(z)}, \quad Y_4(z) = \frac{z(z-a)}{P(z)}.$$

A) If the characteristic equation

$$P(z) = z^2 - (a+d)\,z + (a\,d - b\,c) = z^2 - \operatorname{tr}(A)\,z + \det A = 0$$

has two distinct roots, i.e. $P(z) = (z-\alpha_1)\,(z-\alpha_2)$, then

$$Y_1(z) = \frac{z(z-d)}{(z-\alpha_1)\,(z-\alpha_2)}$$
$$= \frac{1}{\alpha_1-\alpha_2}\left\{(\alpha_1-d)\,\frac{z}{(z-\alpha_1)} - (\alpha_2-d)\,\frac{z}{(z-\alpha_2)}\right\},$$
$$Y_2(z) = \frac{b\,z}{(z-\alpha_1)\,(z-\alpha_2)} = \frac{b}{\alpha_1-\alpha_2}\left\{\frac{z}{(z-\alpha_1)} - \frac{z}{(z-\alpha_2)}\right\},$$
$$Y_3(z) = \frac{c\,z}{(z-\alpha_1)\,(z-\alpha_2)} = \frac{c}{\alpha_1-\alpha_2}\left\{\frac{z}{(z-\alpha_1)} - \frac{z}{(z-\alpha_2)}\right\},$$
$$Y_4(z) = \frac{z(z-a)}{(z-\alpha_1)\,(z-\alpha_2)}$$
$$= \frac{1}{\alpha_1-\alpha_2}\left\{(\alpha_1-a)\,\frac{z}{(z-\alpha_1)} - (\alpha_2-a)\,\frac{z}{(z-\alpha_2)}\right\},$$

which gives

$$y_1(k) = \frac{1}{\alpha_1-\alpha_2}\left\{(\alpha_1-d)\,\alpha_1^k - (\alpha_2-d)\,\alpha_2^k\right\},$$
$$y_2(k) = \frac{b}{\alpha_1-\alpha_2}\left\{\alpha_1^k - \alpha_2^k\right\},$$

$$y_3(k) = \frac{c}{\alpha_1 - \alpha_2}\{\alpha_1^k - \alpha_2^k\},$$

$$y_4(k) = \frac{1}{\alpha_1 - \alpha_2}\{(\alpha_1 - a)\alpha_1^k - (\alpha_2 - a)\alpha_2^k\}.$$

If the roots are nonreal, i.e. $\alpha_1 = |\alpha| e^{i\varphi}$ and $\alpha_2 = |\alpha| e^{-i\varphi}$ $(\varphi \neq 0, \pi)$,

$$y_1(k) = \frac{|\alpha|^{k-1}}{\sin\varphi}\{|\alpha| \sin(k+1)\varphi - d \sin k\varphi\},$$

$$y_2(k) = b\frac{|\alpha|^{k-1}}{\sin\varphi}\sin k\varphi,$$

$$y_3(k) = c\frac{|\alpha|^{k-1}}{\sin\varphi}\sin k\varphi,$$

$$y_4(k) = \frac{|\alpha|^{k-1}}{\sin\varphi}\{|\alpha| \sin(k+1)\varphi - a \sin k\varphi\}.$$

B) If the characteristic equation

$$P(z) = z^2 - (a+d)z + (ad - bc) = z^2 - \mathrm{tr}(A)z + \det A = 0$$

has one root of multiplicity 2, i.e. $P(z) = (z-\alpha)^2$, then

$$Y_1(z) = \frac{z(z-d)}{(z-\alpha)^2} = (\alpha - d)\frac{z}{(z-\alpha)^2} + \frac{z}{z-a} \diamond\!-\!\circ (\alpha - d)k\alpha^{k-1} + \alpha^k,$$

$$Y_2(z) = \frac{bz}{(z-\alpha)^2} \diamond\!-\!\circ bk\alpha^{k-1},$$

$$Y_3(z) = \frac{cz}{(z-\alpha)^2} \diamond\!-\!\circ ck\alpha^{k-1},$$

$$Y_4(z) = \frac{z(z-a)}{(z-\alpha)^2}$$

$$= (\alpha - a)\frac{z}{(z-\alpha)^2} + \frac{z}{z-a} \diamond\!-\!\circ (\alpha - a)k\alpha^{k-1} + \alpha^k.$$

Problem 2.8. Introduce, parallel to the matrix-Laplace transform of Section 1.3.1.4, the matrix-z-transform. Show thereon that the inverse z-transform of

$$z\mathbf{R}(z, \mathbf{A}) := z(z\mathbf{I} - \mathbf{A})^{-1},$$

where $\mathbf{A}$ is a constant square matrix, yields the matrix sequence $\mathbf{A}^k$, i.e., we have the correspondence

$$z\mathbf{R}(z, \mathbf{A}) := (z\mathbf{I} - \mathbf{A})^{-1} \diamond\!-\!\circ \mathbf{A}^k.$$

Chapter 3
Laplace Transforms with the Package

This chapter introduces the *Mathematica* package **LaplaceAndzTransforms V.5.0** . It can be studied in parallel to the first chapter. Readers familiar with Laplace transformation can start with this chapter. It is assumed that the reader has some level of expertise with *Mathematica,* see [Wolfram S. (1999)].

3.1 Basics

The *Mathematica* package **LaplaceAndzTransforms** (subsequently called the Package) was developed by the author during the last ten years. The Package offers a substantial extension to the built-in *Mathematica* functions LaplaceTransform, InverseLaplaceTransform, ZTransform and Inverse–ZTransform. In order to do the computations presented in this book the package has first to be laoded. Once the package is loaded, the scope of the above mentioned commands is extended and over a hundred new commands will be added. The Package is on the CD in the back cover of this book and runs on the computer platforms LINUX and Windows. The Package continues to be developed. In order to obtain future versions, consult the homepage of the author. In order to run the Package, *Mathematica* V4.0 or later is needed.

■ 3.1.1 Installation

The folder **LaplaceAndzTransforms** on the CD has to be copied in the folder `Applications` of the folder `AddOns` of the folder `mathematica`. See the Readmefirst on the CD.

■ 3.1.2 Loading the Package

The Package is loaded by

```
<< LaplaceAndzTransforms`LZSession`;
InitLZPackage[]
```

You get the prompt `InitOK!` which indicates that the Package is ready. By executing any command of the Package the needed files will be automatically loaded. For example, let us compute the inverse Laplace transform of the image function

```
F = (s + a) / (s^2 + 2 s + 5)^2;

InverseLaplaceTransform[F, s, t]
```

$$e^{-t}\left(2\,t\left(\frac{1}{16}\,(1-a)\,\text{Cos}[2\,t]+\frac{1}{8}\,\text{Sin}[2\,t]\right)-\frac{1}{16}\,(1-a)\,\text{Sin}[2\,t]\right)$$

By taking the Laplace transform of the result the given image function is returned.

```
LaplaceTransform[%, t, s] // Together
```

$$\frac{a+s}{(5+2s+s^2)^2}$$

■ 3.1.3 Basic Syntax and a Survey

The command `LaplaceTransform` has three arguments. The second argument is the independent variable of the original function, usually t or x. The first argument is the expression for the original, i.e. a function $f(t)$ or $f(x)$ of the variable which is the second argument. The third argument is the independent variable of the image function which is to be computed, usually s or p. For example:

```
F = LaplaceTransform[x^3 Cos[ω x], x, p]
```

$$-p\left(-\frac{48p^3}{(p^2+\omega^2)^4}+\frac{24p}{(p^2+\omega^2)^3}\right)-3\left(\frac{8p^2}{(p^2+\omega^2)^3}-\frac{2}{(p^2+\omega^2)^2}\right)$$

The command `InverseLaplaceTransform` is similar. The first argument is the expression of the given image function, the second is its independent variable, and the third argument is the independent variable of the original function which is to be computed. For example (by making use of the above computed image function):

```
InverseLaplaceTransform[F, p, t] // Together
```

```
t^3 Cos[t ω]
```

The second and the third argument of both commands must be symbols and no expressions! Most users will probably mainly encounter rational image functions. These are the Laplace transforms of elementary functions such as polynomials, trigonometric and hypergeometric functions. Also the image functions coming up in linear control theory are rational as solutions of ordinary linear differential equations with constant coefficients. However people working with partial differential equations (heat and mass transfer, vibrations in solids, transmission lines in electricity etc.) will encounter higher transcendental functions, the so-called special functions of mathematical physics. Most of these special functions are Laplace transformable, i.e. have a Laplace transform. For example, the Bessel function $J_n(t)$ has the Laplace transform ($n = 3$)

```
LaplaceTransform[BesselJ[3, t], t, s]
```

$$\frac{\left(-s+\sqrt{1+s^2}\right)^3}{\sqrt{1+s^2}}$$

which is no more a rational function. There are image functions whose corresponding originals come up as an infinite series, for example:

```
InverseLaplaceTransform[Cosh[x s] / (s^3 Cosh[a s]), s, t]
```

$$\frac{1}{2}\left(-a^2+t^2+x^2\right)+\text{Summation}\left[\frac{16(-1)^{k1}a^2\cos\left[\frac{\left(\frac{1}{2}+k1\right)\pi t}{a}\right]\cos\left[\frac{\left(\frac{1}{2}+k1\right)\pi x}{a}\right]}{(1+2k1)^3\pi^3},\{k1,0,\infty\}\right]$$

If the `Summation` of the output is replace by `Sum`, *Mathematica* will try to evaluate the infinite series.

```
% /. Summation → Sum // Short
```

$$\frac{1}{2}\left(-a^2+t^2+x^2\right)+\frac{4\,a^2\ (\ll 1\gg)}{\pi^3}$$

(In order to save space the output of large expressions is given in a short form; omit the `//Short` to obtain the full output). A main application of Laplace transformation is to obtain the solution of linear ordinary differential equations with constant coefficients and of systems of such equations. The method is well suited for initial value problems. For example, take the following differential equation

```
deqn = y''[t] + 2 y'[t] + 17 y[t] == t Exp[-2 t];
```

with the specified initial conditions

```
initcd = {y[0] → 2, y'[0] → -3};
```

`LinearIDSolve` computes the solution by means of Laplace transformation.

```
LinearIDSolve[deqn, initcd, y[t], t]
```

$$\left\{y[t]\to\frac{2\,e^{-2t}}{289}+\frac{1}{17}e^{-2t}\,t+2\,e^{-t}\left(\frac{288}{289}\text{Cos}[4\,t]-\frac{38}{289}\text{Sin}[4\,t]\right)\right\}$$

3.1.3.1 Defining the Type of Parameters

Sometimes you need to define the type or range of parameters in a given expression. The reason is that certain Laplace or inverse Laplace transforms are only defined under the assumption that a certain parameter is positive or is an integer, for example. Also certain simplifications may only be possible if it is known that a parameter or a variable is an integer, for example. In general the concept that a type or range of a parameter has to be defined is on purpose only applied if a dangerous mistake could be the consequence when omitted, otherwise the use of the Package would become unhandy.

```
LaplaceTransform[1 / (t^2 + c^2), t, s]
```

$$\frac{1}{\sqrt{c^2}}\left(\text{CosIntegral}\left[\sqrt{c^2}\ s\right]\text{Sin}\left[\sqrt{c^2}\ s\right]-\text{Cos}\left[\sqrt{c^2}\ s\right]\text{sineIntegralsi}\left[\sqrt{c^2}\ s\right]\right)$$

In order that the usual simplification works define c positive.

```
DefPositive[c];

LaplaceTransform[1 / (t^2 + c^2), t, s]
```

$$\frac{\text{CosIntegral}[c\,s]\,\text{Sin}[c\,s]-\text{Cos}[c\,s]\,\text{sineIntegralsi}[c\,s]}{c}$$

The following definitions are available:

```
DefReal[x], DefInteger[n], DefEven[n], DefOdd[n],
DefPositive[x], DefNegative[x], DefNonNegative[x],
```

`DefLess[a,b]` (defines $a < b$), `DefLessEqual[a,b]` (defines $a \leq b$),

`DefGreater[a,b]` (defines $a > b$), `DefGreaterEqual[a,b]` (defines $a \geq b$).

For all commands the usual short information is obtained by use of the ?operator, for example:

```
? DefLessEqual

   DefLessEqual[a,b] defines a smaller or equal
   than b.
```

3.2 The Use of Transformation Rules

■ 3.2.1 Linearity

The linearity property for the Laplace and inverse Laplace transforms (see Section 1.2.2.1) is automatically applied.

```
LaplaceTransform[A Cos[ω t] + B Sin[ω t], t, s]
```

$$\frac{A\,s}{s^2+\omega^2}+\frac{B\,\omega}{s^2+\omega^2}$$

```
InverseLaplaceTransform[%, s, t]
```

```
A Cos[t ω] + B Sin[t ω]
```

■ 3.2.2 The Integration Rule

If you want to apply the integration rule of Section 1.2.2.2, use `IntegrationTo[t]`, for example:

```
LaplaceTransform[t Sin[t] // IntegrationTo[t], t, s]
```

$$\frac{2}{(1+s^2)^2}$$

This is the same result as if you would first integrate $x\sin(x)$ from 0 to t and then take the Laplace transform:

```
LaplaceTransform[Integrate[x Sin[x], {x, 0, t}], t, s] // Together
```

$$\frac{2}{(1+s^2)^2}$$

■ 3.2.3 The Image Shift Rule

The image shift rule of Section 1.2.2.4 is applied by use of the command `ImageShift[c,s]`. By inserting the expression `F[s]//ImageShift[c,s]` the inverse Laplace transform of $F(s-c)$ is computed.

```
InverseLaplaceTransform[(1 / Sqrt [1 + s^2]) // ImageShift[c, s], s, t]
```

$$e^{c\,t}\ \text{BesselJ}[0,\ t]$$

■ 3.2.4 The Original Shift Rule

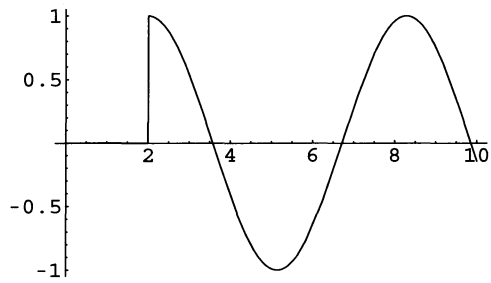
There are two rules for an original shift (see Section 1.2.2.5): for a right shift of the original function, and for a left shift. By use of the rule `f[t]//RightShift[c,t]` the Laplace transform of the $u(t-c)\,f(t-c)$ is computed.

```
LaplaceTransform[Cos[t] // RightShift[2, t], t, s]
```

$$\frac{e^{-2\,s}\,s}{1+s^2}$$

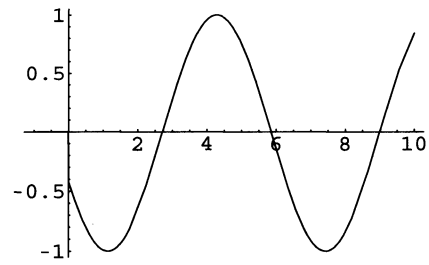
By the use of `Evaluate[N[f[t]//RightShift[c,t]]]` the shifted original function is plotted.

```
Plot[Cos[t] // RightShift[2, t] // N // Evaluate, {t, 0, 10}];
```



If you want to compute the Laplace transform of an original which is shifted to the left, use `f[t]//LeftShift[c,t]`. Let us first see what an original which is shifted 2 units to the left looks like.

```
Plot[Cos[t] // LeftShift[2, t] // N // Evaluate, {t, -1, 10}];
```



The Laplace transform of the shifted original is:

```
LaplaceTransform[Cos[t] // LeftShift[2, t], t, s] // Simplify
```

$$\frac{s\,\text{Cos}[2]-\text{Sin}[2]}{1+s^2}$$

The Laplace transform of a right shift can also be obtained by making use of the unit-step function.

```
LaplaceTransform[UnitStep[t - 2] Cos[t - 2], t, s]
```

$$\frac{e^{-2s}\,s}{1+s^2}$$

■ 3.2.5 The Similarity or Scale Change Rule

If `f[t]//ScaleChange[c,t]` is used, the Laplace transform of $f(c\,t)$ is computed. This is the implementation of the rule of Section 1.1.2.6. It is assumed that $c > 0$.

```
LaplaceTransform[BesselJ[3, t] // ScaleChange[c, t], t, s]
```

$$\frac{\left(-\frac{s}{c}+\sqrt{1+\frac{s^2}{c^2}}\right)^3}{c\sqrt{1+\frac{s^2}{c^2}}}$$

The same rule applies for a scale change concerning the image variable.

```
InverseLaplaceTransform[Exp[- Sqrt[s]] // ScaleChange[c, s], s, t]
```

$$\frac{e^{-\frac{c}{4t}}}{2c\sqrt{\pi}\,\left(\frac{t}{c}\right)^{3/2}}$$

The inverse Laplace transform of $e^{-\sqrt{cs}}$ was computed.

■ 3.2.6 The t-Multiplication Rule

This rule of Section 1.1.2.7 is automatically applied.

```
LaplaceTransform[t^2 Cos[ω t], t, p]
```

$$-\frac{4p}{(p^2+\omega^2)^2}+p\left(\frac{8p^2}{(p^2+\omega^2)^3}-\frac{2}{(p^2+\omega^2)^2}\right)$$

Let us control the result.

```
InverseLaplaceTransform[%, p, t] // Together
```

```
t^2 Cos[t ω]
```

■ 3.2.7 The Image Integration Rule

This rule of Section 1.1.2.8 can be applied under the assumption that the $f(t)/t$ has a finite limit as $t \to 0+$. If this assumption is satisfied `f[t]//DivisonBy[t]` computes the Laplace transform of $f[t]/t$. For example:

```
LaplaceTransform[Sin[x] // DivisionBy[x], x, s]
```

$$\text{ArcTan}\left[\frac{1}{s}\right]$$

which is the Laplace transform of $\sin x\,/x$.

■ 3.2.8 Convolutions

The convolution $f*g$ of two originals, defined in Section 1.1.2.9, is represented as `Convolution[f[u],g[u],u]`. If the convolution product should be computed, you have two possibilities: by direct integration using the definition of the convolution, or by the method of Laplace transformation, by making use of the convolution theorem. By use of the Named Rule `EvaluateConvolution`, *Mathematica* computes the convolution product by direct integration:

```
Convolution[Cos[u], Exp[u], u] /. EvaluateConvolution
```

$$\frac{1}{2}\,(-\text{Cos}[\#1]+\text{Sin}[\#1])+\frac{1}{2}\,e^{\#1}\,(\text{Cos}[\#1]^2+\text{Sin}[\#1]^2)\;\&$$

The result is obtained in the form of a pure function. If an expression in t is wanted, use

```
Convolution[Cos[u], Exp[u], u][t] /. EvaluateConvolution
```

$$\frac{1}{2}\,(-\text{Cos}[t]+\text{Sin}[t])+\frac{1}{2}\,e^{t}\,(\text{Cos}[t]^2+\text{Sin}[t]^2)$$

The use of the Named Rule `ComputeConvolution` has the effect that the convolution product is computed by making use of the convolution theorem.

```
Convolution[Cos[u], Exp[u], u][t] /. ComputeConvolution
```

$$\frac{e^{t}}{2}+2\left(-\frac{\text{Cos}[t]}{4}+\frac{\text{Sin}[t]}{4}\right)$$

■ 3.2.9 Efros' Rule or the Substitution Rule

Efros' theorem of Section 1.1.2.10 is a powerful tool. Several special cases are implemented in the Package. Given a correspondence $F(s)\;\blacksquare\!-\!\circ\;f(t)$, it is sometimes possible to compute the inverse Laplace transform of expressions such as $F[q(s)]$, $F[q(s)]\,q(s)$, $F[q(s)]/q(s)$ by applying Efros' theorem. The original function comes up as an integral which in most cases cannot be integrated to a closed expression, but sometimes the powerful built-in integration procedures of *Mathematica* succeed to evaluate the integral. In other cases the integral, which depends on a parameter t, can be the starting point for obtaining an asymptotic expansion of the original function. The substitution rule is applied by use of `Subst[s,q[s]]` or the use of `Subst[s,q[s],u]`. The second form will use the specified symbol u for the integration variable, avoiding by that the ugly form of an indexed integration variable in the integrand.

A) From $F(s)\;\blacksquare\!-\!\circ\;f(t)$ compute the inverse Laplace transform of $F\left[\sqrt{s}\right]$:

```
Clear[F];

InverseLaplaceTransform[F[s] // Subst[s, Sqrt[s], x], s, t]
```

$$\frac{\int_0^{\infty} e^{-\frac{x^2}{4t}}\,x\,\text{InverseLaplaceTransform}[F[s],\,s,\,x]\,dx}{2\sqrt{\pi}\;t^{3/2}}$$

This is the general formula. If an image function is specified, for example $F(s) = 1/(s+1)$, one obtains

```
InverseLaplaceTransform[1 / (s + 1) // Subst[s, Sqrt[s], x], s, t]
```

$$\frac{1}{2\sqrt{\pi}\,t^{3/2}}$$

$$\text{If}\left[\frac{\text{Re}[t]}{\text{Im}[t]^2+\text{Re}[t]^2} > 0,\ \frac{2\left(-e^t\sqrt{\pi}+\sqrt{\frac{1}{t}}+e^t\sqrt{\pi}\,\text{Erf}\left[\frac{1}{\sqrt{\frac{1}{t}}}\right]\right)}{\left(\frac{1}{t}\right)^{3/2}},\ \int_0^\infty e^{-x-\frac{x^2}{4t}}\,x\,dx\right]$$

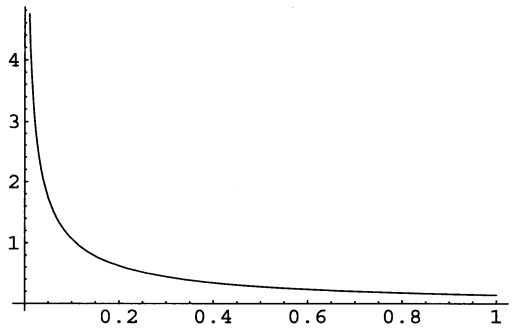
In this case *Mathematica* succeeded in the evaluation of the integral. The inverse Laplace transform of $1/(\sqrt{s}+1)$ is obtained when the assumption is verified:

```
InverseLaplaceTransform[1 / (s + 1) // Subst[s, Sqrt[s], x],
  s, t, Assumptions -> Re[t]/(Im[t]^2 + Re[t]^2) > 0] // PowerExpand
```

$$\frac{-e^t\sqrt{\pi}+\frac{1}{\sqrt{t}}+e^t\sqrt{\pi}\,\text{Erf}\left[\sqrt{t}\right]}{\sqrt{\pi}}$$

The complicated expression involving a special function can readily be plotted.

```
Plot[%, {t, 0.01, 1}];
```



B) From $F(s) \bullet\!\!-\!\!\circ f(t)$ compute the inverse Laplace transform of $F\left[\sqrt{s}\right]/\sqrt{s}$:

```
InverseLaplaceTransform[(1 / (s + 1) // Subst[s, Sqrt[s], x]) / Sqrt[s],
  s, t, Assumptions -> Re[t]/(Im[t]^2 + Re[t]^2) > 0] // PowerExpand
```

$$-e^t\left(-1+\text{Erf}\left[\sqrt{t}\right]\right)$$

In this case *Mathematica* succeeded in the evaluation of the integral. You obtained the inverse Laplace transform of $1/\left(\sqrt{s}\left(\sqrt{s}+1\right)\right) = 1/\left(s+\sqrt{s}\right)$. The same result is returned by applying directly `InverseLaplaceTransform` because the Package knows correspondences like this one.

```
InverseLaplaceTransform[1 / (Sqrt[s] (1 + Sqrt[s])), s, t]
```

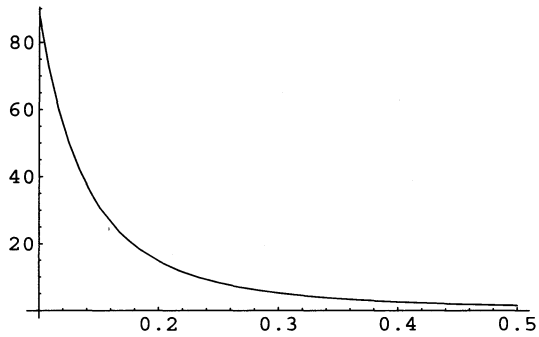
$$e^t\,\text{Erfc}\left[\sqrt{t}\right]$$

C) From $F(s) \blacksquare\!-\!\circ\ f(t)$ compute the inverse Laplace transform of $F\left[\sqrt{s}\right]\left(\sqrt{s}\right)^n$:

```
InverseLaplaceTransform[(1 / (s + 1) // Subst[s, Sqrt[s], x]) (Sqrt[s])^3,
  s, t, Assumptions -> Re[t] / (Im[t]^2 + Re[t]^2) > 0] // PowerExpand
```

$$-\frac{1}{\sqrt{\pi}\, t^{3/2}}\left(16 - \frac{3\, e^t \sqrt{\pi}}{\sqrt{t}} - 24\, e^t \sqrt{\pi}\sqrt{t} + 16\, t - 16\, e^t \sqrt{\pi}\, t^{3/2} + \frac{3\, e^t \sqrt{\pi}\, \text{Erf}\left[\sqrt{t}\right]}{\sqrt{t}} + 8\, e^t \sqrt{\pi}\sqrt{t}\,(3 + 2t)\,\text{Erf}\left[\sqrt{t}\right]\right)$$

```
Plot[%, {t, 0.1, 0.5}, AxesOrigin -> {0.1, 0}];
```



Important remark: You have obtained an expression but this function would be the inverse Laplace transform of $s^{3/2}/\left(\sqrt{s}+1\right)$ which does not exist, at least as an ordinary function, because images of ordinary functions must tend to zero as $s \to \infty$, which is not the case of $s^{3/2}/\left(\sqrt{s}+1\right)$. Please note: *The Package can do valuable and very helpful things but also stupid things, as for example computing a result that does not exist. In general it cannot decide whether a certain computation makes sense or not, hence the user should know what he is doing.*

But even in a case like this one the above "result" can help. The graph shows that there is something like a Dirac Impulse at the origin. Indeed, the given image function can be written as

$$\frac{s\sqrt{s}}{\sqrt{s}+1} = s - \sqrt{s} + 1 - \frac{1}{\sqrt{s}+1}.$$

The original which corresponds to the last term we have computed above. Furthermore, we have the generalized correspondences $s \blacksquare\!-\!\circ\ \delta'(t)$, $1 \blacksquare\!-\!\circ\ \delta(t)$, and

```
InverseLaplaceTransform[Sqrt[s], s, t]
```

$$\frac{\text{Pf}\left[\frac{1}{t^{3/2}}\right]}{2\sqrt{\pi}}$$

which is a generalized function of Hadamard's finite part type. We finally have found the generalized correspondence

$$\frac{s\sqrt{s}}{\sqrt{s}+1} \blacksquare - \square$$

$$\delta'(t) - \frac{1}{2\sqrt{\pi}}\,\mathrm{Pf}\left[\frac{1}{t^{3/2}}\right] + \delta(t) - \frac{-e^t\sqrt{\pi} + \frac{1}{\sqrt{t}} + e^t\sqrt{\pi}\,\mathrm{Erf}\left[\sqrt{t}\right]}{\sqrt{\pi}}.$$

D) From $F(s) \blacksquare - \square f(t)$ compute the inverse Laplace transform of $F[1/s]/\sqrt{s}$:

The general formula is

```
InverseLaplaceTransform[(F[s] // Subst[s, 1 / s, x]) / Sqrt[s], s, t]
```

$$\frac{\int_0^\infty \mathrm{Cos}\left[2\sqrt{t\,x}\right]\,\mathrm{InverseLaplaceTransform}[F[s],\,s,\,x]\,dx}{\sqrt{\pi}\,\sqrt{t}}$$

Let us take $F(s) = 1/\sqrt{1+s^2}$, for example.

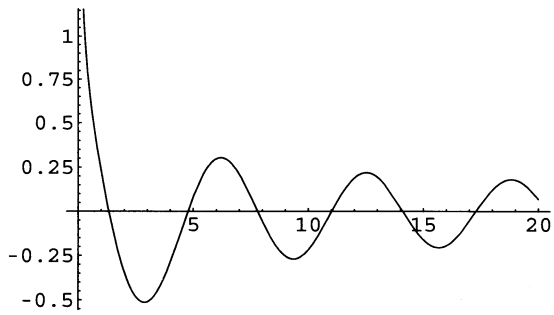
```
InverseLaplaceTransform[
 ((1 / Sqrt[1 + s^2]) // Subst[s, 1 / s, x]) / Sqrt[s], s, t]
```

$$\frac{1}{\sqrt{\pi}\,\sqrt{t}}\,\mathrm{If}\left[\mathrm{Im}\left[\sqrt{t}\right] == 0,\right.$$
$$\left.\mathrm{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\},\left\{\frac{1}{4},\frac{3}{4}\right\},-\frac{t^2}{4}\right],\int_0^\infty \mathrm{BesselJ}[0,\,x]\,\mathrm{Cos}\left[2\sqrt{t\,x}\right]\,dx\right]$$

By this we found the correspondence

$$\sqrt{\frac{s}{s^2+1}} \blacksquare - \square\ \mathrm{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\},\left\{\frac{1}{4},\frac{3}{4}\right\},-\frac{t^2}{4}\right]\Big/\sqrt{\pi t}$$

```
Plot[HypergeometricPFQ[{1/2}, {1/4, 3/4}, -t^2/4] / Sqrt[π t], {t, 0.01, 20}];
```



E) From $F(s) \blacksquare - \square f(t)$ compute the inverse Laplace transform of $F[s+1/s]/s$:

```
InverseLaplaceTransform[
 ((1 / Sqrt[a^2 + s^2]) // Subst[s, s + 1 / s, x]) / s, s, t]
```

$$\int_0^t \mathrm{BesselJ}\left[0,\,\sqrt{a^2}\,x\right]\,\mathrm{BesselJ}\left[0,\,2\sqrt{(t-x)\,x}\right]\,dx$$

This is an example where the integral could not be evaluated to a closed expression. But the integral in the obtained correspondence

$$\frac{1}{\sqrt{s^4+(2+a^2)s^2+1}} \blacksquare\!-\!\square \int_0^t J_0(a\,x)\; J_0\left(2\sqrt{(t-x)\,x}\right) dx$$

can numerically be evaluated, provided a numerical value is given to a.

The reader may explore himself the following substitutions:

$F[s+a/s]/s^2$, $F\left[s+\sqrt{s}\right]$, $F\left[s+\sqrt{s}\right]/\sqrt{s}$, $F\left[\sqrt{s^2+a^2}\right]$,

$F\left[\sqrt{s^2+a^2}\right]/\sqrt{s^2+a^2}$, $F\left[\sqrt{s^2+a^2}\right]s/\sqrt{s^2+a^2}$,

$F\left[\sqrt{s^2+a^2}\right]\left(\sqrt{s^2+a^2}-s\right)^3/\sqrt{s^2+a^2}$, $F\left[\sqrt{s^2-a^2}\right]$,

$F\left[\sqrt{s^2-a^2}\right]/\sqrt{s^2-a^2}$, $F\left[\sqrt{s^2-a^2}\right]s/\sqrt{s^2-a^2}$,

$F\left[\sqrt{s^2-a^2}\right]\left(s-\sqrt{s^2-a^2}\right)^5/\sqrt{s^2-a^2}$,

$F[\ln s]$, $F[\ln s^a]$, $F[a\,s+b]$.

■ 3.2.10 Other Rules

3.2.10.1 Initial Values

In Section 1.1.2.11 we presented the Initial Value Theorem. If the given image function $F(s)$ is a proper rational function, the corresponding original $f(t)$ consists of elementary functions such as polynomials, exponential und trigonometric terms. For these functions the limits to the right $f(0+)$, $f'(0+)$, ... in general will exist. By use of the `InitialValues[F[s],s,n]` command the first n initial values of the original $f(0+)$, $f'(0+)$, ..., $f^{(n-1)}(0+)$ corresponding to $F(s) = \mathcal{L}[f(t)](s)$ are computed.

```
InitialValues[(A s^2 + B s + C) / ((s - 1)^2 (a s^2 + b s + c)), s, 3]
```

$$\left\{0,\ \frac{A}{a},\ \frac{2A}{a}-\frac{Ab}{a^2}+\frac{B}{a}\right\}$$

3.2.10.2 Transformation of Rational Image Functions

AmplifyBy: *Mathematica* output has not always the form that you want. For example, you may wish to transform a rational expression by amplification. The command *(num/denom)* // `AmplifyBy[expr]` amplifies the rational function *(num/denom)* by *expr* and expands numerator and denominator. For example:

```
(s + Sqrt[s^2 + a^2]) / Sqrt[s^2 + a^2] // AmplifyBy[-s + Sqrt[a^2 + s^2]]
```

$$\frac{a^2}{a^2+s^2-s\sqrt{a^2+s^2}}$$

SimplifyRImageFunction: Often *Mathematica* does not simplify rational image functions as one wishes. For example:

```
F = (s + a) ^ 2 / (s ^ 2 + b s ^ 2 - c s + d s + 1)
```

$$\frac{(a+s)^2}{1-c\,s+d\,s+s^2+b\,s^2}$$

Use `SimplifyRImageFunction[F[s], s]` to simplify the expression.

```
SimplifyRImageFunction[F, s]
```

$$\frac{a^2+2\,a\,s+s^2}{1+(-c+d)\,s+(1+b)\,s^2}$$

NormalizeRImageFunction: Often you want to get rid of too many parameters in a given expression. If

$$F(s) = \frac{a_0 + a_1\,s + \cdots + a_m\,s^m}{b_0 + b_1\,s + \cdots + b_n\,s^n}, \quad (b_0,\ b_n,\ a_0,\ a_m \neq 0),$$

we write first

$$F(s) = \frac{1 + (a_1/a_0)\,s + \cdots + (a_m/a_0)\,s^m}{1 + (b_1/b_0)\,s + \cdots + (b_n/b_0)\,s^n} \cdot \frac{a_0}{b_0}.$$

Then, under the assumption that $b_n/b_0 > 0$, we set $\omega := (b_n/b_0)^{-1/n} > 0$ and we have

$$F(s) = \frac{1 + A_1\,(s/\omega) + \cdots + A_m\,(s/\omega)^m}{1 + B_1\,(s/\omega) + \cdots + (s/\omega)^n} \cdot \frac{a_0}{b_0},$$

with

$$A_1 = (a_1/a_0)\,\omega,\ A_2 = (a_2/a_0)\,\omega^2,\ \ldots,\ A_m = (a_m/a_0)\,\omega^m,$$
$$B_1 = (b_1/b_0)\,\omega,\ A_2 = (b_2/b_0)\,\omega^2,\ \ldots,\ A_{n-1} = (b_{n-1}/b_0)\,\omega^{n-1}.$$

Finally,

$$F(s) = \left(\frac{a_0}{b_0}\ \frac{1 + A_1\,s + \cdots + A_m\,s^m}{1 + B_1\,s + \cdots + s^n}\right)_{s \to s/\omega}.$$

The rational function

$$G(s) := \frac{1 + A_1\,s + \cdots + A_m\,s^m}{1 + B_1\,s + \cdots + s^n}$$

is now free from physical dimensions and has three coefficients normalized to unity. If $g(t) = \mathcal{L}^{-1}[G(s)]\,(t)$ is the inverse Laplace transform of $G(s)$, then, by making use of the scale change property, the inverse Laplace transform of the originally given $F(s)$ is given by

$$f(t) = \frac{a_0}{b_0}\,\omega\ g(\omega\,t).$$

The command `NormalizeRImageFunction[F[s], s]` does all that for you.

```
NormalizeRImageFunction[(a0 + a1 s + a2 s ^ 2) / (b0 + b1 s + b2 s ^ 2 + b3 s ^ 3), s]
```

$$\left\{\frac{a0\left(1+\frac{a1\,s}{a0\left(\frac{b3}{b0}\right)^{1/3}}+\frac{a2\,s^2}{a0\left(\frac{b3}{b0}\right)^{2/3}}\right)}{b0\left(1+\frac{b1\,s}{b0\left(\frac{b3}{b0}\right)^{1/3}}+\frac{b2\,s^2}{b0\left(\frac{b3}{b0}\right)^{2/3}}+s^3\right)},\ \texttt{ScaleChange}\left[\left(\frac{b3}{b0}\right)^{1/3},\ s\right]\right\}$$

If `InverseLaplaceTransform[First[%]//Last[%],s,t]` is then executed, the inverse Laplace transform of the originally given image function is obtained. For example:

```
F = (30 + s + s^2) / (2 + 2 s + 3 s^2 + 100 s^3);

Fnorm = NormalizeRImageFunction[F, s] // N
```

$$\left\{\frac{15.\ (1.+0.00904806\,s+0.00245602\,s^2)}{1.+0.271442\,s+0.110521\,s^2+s^3},\ \text{ScaleChange}[3.68403,\ s]\right\}$$

```
InverseLaplaceTransform[First[Fnorm] // Last[Fnorm], s, t] // Chop //
 Expand
```

$$1.47752\,e^{-0.256342\,t} - 1.46752\,e^{0.113171\,t}\,\text{Cos}[0.255369\,t] + 2.17149\,e^{0.113171\,t}\,\text{Sin}[0.255369\,t]$$

But in most cases you will prefer to stay with the scaled original.

```
InverseLaplaceTransform[First[Fnorm], s, t] // Chop
```

$$5.44323\,e^{-0.944371\,t} + 2\,e^{0.416925\,t}\,(-2.70319\,\text{Cos}[0.940786\,t] + 3.99992\,\text{Sin}[0.940786\,t])$$

3.3 The Finite Laplace Transform

■ 3.3.1 The Finite Laplace Integral

The finite Laplace integral of $f(t)$ defined on the interval $[0, T]$ is

$$F_T(s) = F(s, T) := \int_0^T e^{-st} f(t)\,dt.$$

It always defines an entire analytic function of the complex variable s, i.e. a function which has no singularities in the finite complex plane. The finite Laplace transform $F_T(s)$ can be regarded as the ordinary Laplace transform of the original function

$$f_T(t) := \begin{cases} f(t), & 0 \le t \le T \\ 0, & t < 0,\ t > T \end{cases}.$$

The finite Laplace transform

$$F_{a,b}(s) = F(s, a, b) := \int_a^b e^{-st} f(t)\,dt$$

is computed by

`FiniteLaplaceTransform[f[t],{t,a,b},s]`.

```
FiniteLaplaceTransform[x Cos[x], {x, Pi / 4, Pi / 2}, s]
```

$$-\frac{e^{-\frac{\pi s}{2}}\left(-\frac{\pi}{2}-2s-\frac{\pi s^2}{2}\right)}{(-i+s)^2(i+s)^2}+\left(e^{-\frac{\pi s}{4}}\left(-\frac{1}{\sqrt{2}}-\frac{\pi}{4\sqrt{2}}-\sqrt{2}\,s+\frac{\pi s}{4\sqrt{2}}+\frac{s^2}{\sqrt{2}}-\frac{\pi s^2}{4\sqrt{2}}+\frac{\pi s^3}{4\sqrt{2}}\right)\right)\Big/\left((-i+s)^2(i+s)\right)$$

In general `FiniteLaplaceTransform` produces rather large expressions which can often be simplified. If `InverseLaplaceTransform` is executed the original $f_{a,b}(t)$ is obtained, but simplify first the image function.

```
SimplifyRImageFunction[%, s] // Apart
```

$$\frac{e^{-\frac{\pi s}{2}}(\pi+4s+\pi s^2)}{2(1+s^2)^2}+\frac{e^{-\frac{\pi s}{4}}(-4-\pi-8s+\pi s+4s^2-\pi s^2+\pi s^3)}{4\sqrt{2}(1+s^2)^2}$$

```
InverseLaplaceTransform[%, s, t] // Simplify
```

```
t Cos[t] (-UnitStep[-Pi/2 + t] + UnitStep[-Pi/4 + t])
```

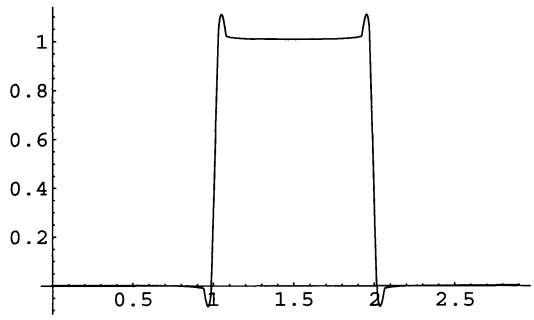
■ 3.3.2 The Numerical Inversion of a Finite Laplace Transform

A finite Laplace image function $F(s, T)$ can be numerically inverted by using the FFT algorithm. This is done by calling the command`NFiniteInverseLaplaceTransform[F[s,T],{s,0,T}]`. The output will be in the form of a `InterpolatingFunction` object. The Laplace trans-form of a unit-rectangle impulse over [1, 2] is given, for example, by

$$\frac{-e^{-2s}+e^{-s}}{s}.$$

The same result is obtained by calling

```
FiniteLaplaceTransform[1, {t, 1, 2}, s];

NFiniteInverseLaplaceTransform[%, {s, 0, 3}]

InterpolatingFunction[{{0., 2.95313}}, <>]

Plot[%[t], {t, 0, 2.9}];
```



Note the small horns (Gibb's phenomenon).

According to Section 1.2.2.7 the Laplace transform of a periodic original $f(t)$ with period T, obtained by the periodic extension of an expression $f_T(t)$ defined on $[0, T]$, is given by

$$F(s) = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f_T(t)\, dt .$$

Therefore, it is also obtained by

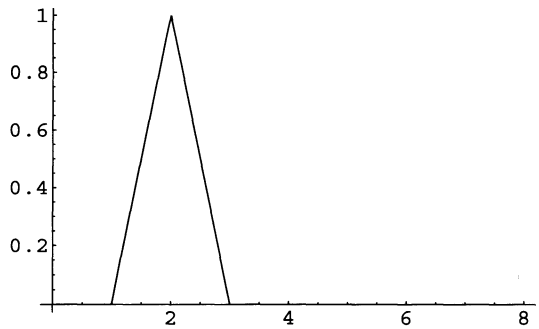
`FiniteLaplaceTransform[`f_T`[t],{t,0,T},s]/(1-Exp[-sT])`.

3.4 Special Functions

■ 3.4.1 Piecewise Defined Functions

Linear differential equations with constant coefficients can be solved by several methods. If the right-hand side of a differential equation is a piecewise defined function, the method of Laplace transformation is very efficient. A piecewise defined original is defined by different expressions on a finite number of adjacent intervals. The command `PiecewiseFunction[{{a1,b1,expr1},...,{an,bn,exprn}}]` defines the function which is specified by the expression `expr1` on the interval `[a1,b1)`, by `expr2` on the interval `[a2,b2)`, ..., by `exprn` on the interval `[an,bn)`. The various expressions must be given in the form of pure functions. The defined piecewise function will also be in the form of a pure function. In the exterior of the union of the specified intervals the defined function is assumed to be zero. For example:

```
f = PiecewiseFunction[{{0, 1, 0 &}, {1, 2, (# - 1) &}, {2, 3, 1 + (2 - #) &}}];

Plot[f[t], {t, 0, 8}];
```



The Laplace transform of this unit-triangle original is

```
F = LaplaceTransform[f[t], t, s] // Together // ExpandNumerator
```

$$\frac{e^{-3s}\,(1 - 2e^{s} + e^{2s})}{s^2}$$

The inverse Laplace transform of this image will be expressed by unit-step functions, which is another method of representing a piecewise defined function.

```
InverseLaplaceTransform[F, s, t]
```

```
(-3 + t) UnitStep[-3 + t] - 2 (-2 + t) UnitStep[-2 + t] + (-1 + t) UnitStep[-1 + t]
```

Suppose that the above piecewise function $f(t)$ is the right-hand side of the differential equation

$$y'' + 2y' + 5y(t) = f(t),$$
$$y(0+) = 0,\ y'(0+) = 0.$$

The Laplace transform of the solution is

```
Y = F / (s^2 + 2 s + 5)
```

$$\frac{e^{-3s}\,(1 - 2\,e^{s} + e^{2s})}{s^2\,(5 + 2\,s + s^2)}$$

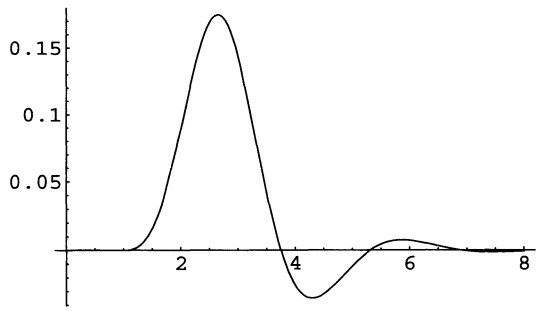
and the solution itself is

```
y = InverseLaplaceTransform[Y // Apart, s, t]
```

$$\left(-\frac{2}{25} + \frac{1}{5}(-3+t) + 2\,e^{3-t}\left(\frac{1}{25}\text{Cos}[2(-3+t)] - \frac{3}{100}\text{Sin}[2(-3+t)]\right)\right)\text{UnitStep}[-3+t] -$$
$$2\left(-\frac{2}{25} + \frac{1}{5}(-2+t) + 2\,e^{2-t}\left(\frac{1}{25}\text{Cos}[2(-2+t)] - \frac{3}{100}\text{Sin}[2(-2+t)]\right)\right)\text{UnitStep}[-2+t] +$$
$$\left(-\frac{2}{25} + \frac{1}{5}(-1+t) + 2\,e^{1-t}\left(\frac{1}{25}\text{Cos}[2(-1+t)] - \frac{3}{100}\text{Sin}[2(-1+t)]\right)\right)\text{UnitStep}[-1+t]$$

Notice the `Apart` inserted in the command above.

```
Plot[%, {t, 0, 8}];
```



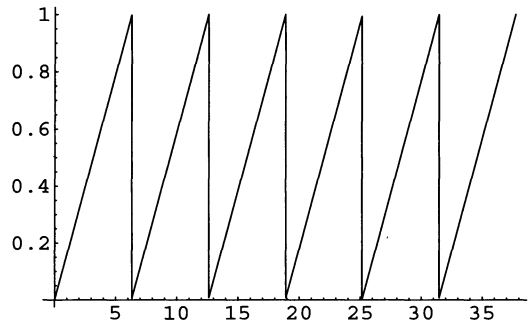
■ 3.4.2 Periodic Originals and Step Functions

In Section 1.2.2.7 we studied the Laplace transforms of periodic originals. Usually a periodic function $f(t)$ is defined by an expression $f_T(t)$ on the interval of period $[0, T]$ and the original is obtained by periodic extension to the right. As for piecewise defined functions, the expression $f_T(t)$ has to be specified in the form of a pure function. Then the defined periodic function will also be of this form.

3.4.2.1 Periodic Functions

`PeriodicFunction[`f_T`, T]` defines the periodic function which is specified by the expression $f_T(t)$ on the interval of period T, for example:

```
f = PeriodicFunction[# / (2 Pi) &, 2 Pi];

Plot[f[t], {t, 0, 12 Pi}];
```



The Laplace transform of this periodic function is

```
F = LaplaceTransform[f[t], t, s]
```

$$-\frac{1 - e^{2\pi s} + 2\pi s}{2\,(-1 + e^{2\pi s})\,\pi s^2}$$

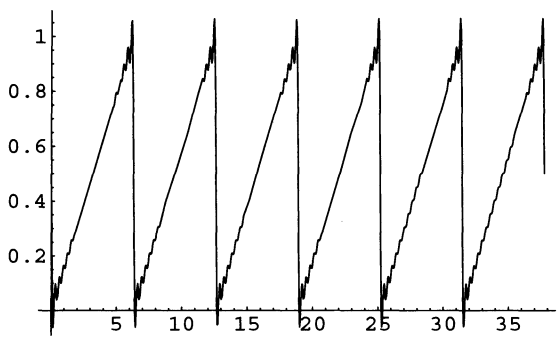
This is an example of a meromorphic image function. If the inverse Laplace transform is computed, an infinite series occurs which is the Fourier series of the given periodic function.

```
InverseLaplaceTransform[F, s, t]
```

$$\frac{1}{2} + \text{Summation}\left[-\frac{\text{Sin[k1 t]}}{\text{k1}\,\pi}, \{\text{k1}, 1, \infty\}\right]$$

You can truncate this series at the 20th term, for example, and plot it. Use the NamedRule `TruncateSummation[20]` or just replace ∞ by 20.

```
f1 = % /. TruncateSummation[20];

Plot[f1, {t, 0, 12 Pi}];
```



If more than one single expression is needed for defining the periodic function on the interval $[0, T]$ use `FiniteLaplaceTransform` (see Section 3.3.1), or the `PiecewiseFunction` command. For example, if the periodic function of period $T = 3$ is given by

$$f_3(t) := \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 0, & 2 < t < 3 \end{cases},$$

its Laplace transform is

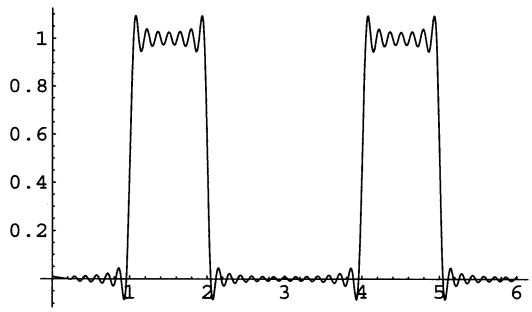
```
F3 = FiniteLaplaceTransform[1, {t, 1, 2}, s] / (1 - Exp[-3 s])
```

$$\frac{-\frac{e^{-2s}}{s} + \frac{e^{-s}}{s}}{1 - e^{-3s}}$$

```
InverseLaplaceTransform[F3 // Expand, s, t]
```

$$\frac{2-t}{3} + \frac{1}{3}(-1+t) - \text{Summation}\left[-\frac{1}{2\,k2^2\,\pi^2}\left(3\left(\frac{2}{3}\,k2\,\pi\,\text{Cos}\left[\frac{2\,k2\,\pi\,t}{3}\right]\text{Sin}\left[\frac{4\,k2\,\pi}{3}\right] - \frac{2}{3}\,k2\,\pi\,\text{Cos}\left[\frac{4\,k2\,\pi}{3}\right]\text{Sin}\left[\frac{2\,k2\,\pi\,t}{3}\right]\right)\right), \{k2, 1, \infty\}\right] - \text{Summation}\left[-\frac{1}{2\,k3^2\,\pi^2}\left(3\left(-\frac{2}{3}\,k3\,\pi\,\text{Cos}\left[\frac{2\,k3\,\pi\,t}{3}\right]\text{Sin}\left[\frac{2\,k3\,\pi}{3}\right] + \frac{2}{3}\,k3\,\pi\,\text{Cos}\left[\frac{2\,k3\,\pi}{3}\right]\text{Sin}\left[\frac{2\,k3\,\pi\,t}{3}\right]\right)\right), \{k3, 1, \infty\}\right]$$

```
% /. Infinity → 20 // N;

Plot[%, {t, 0, 6}];
```

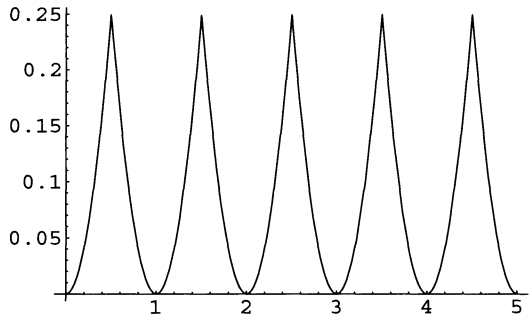


3.4.2.2 Even Periodic Functions

`PeriodicEvenFunction[`f_H`,` H`]` defines the even periodic original, specified by the expression $f_H(t)$ on the interval of the half-period $[0, H]$.

```
f = PeriodicEvenFunction[#^2 &, 1/2];
```

```
Plot[f[t], {t, 0, 5}];
```




```
F = LaplaceTransform[f[t], t, s]
```

$$\frac{e^{-s/2}\,(-1+e^{s}-e^{s/2}\,s)\,\mathrm{Csch}[\frac{s}{2}]}{s^3}$$

Suppose that this periodic function $f(t)$ is the right-hand side of the differential equation

$$y'' + 2y' + 5y(t) = f(t),$$
$$y(0+) = 0,$$
$$y'(0+) = 0,$$

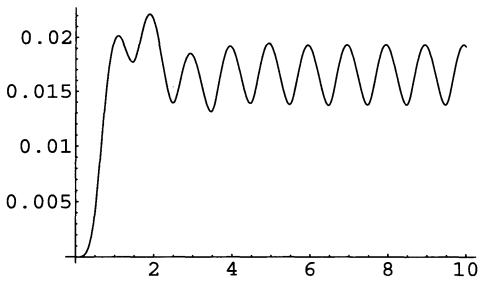
then the Laplace transform of the solution is given by

```
Y = F / (s^2 + 2 s + 5)
```

$$\frac{e^{-s/2}\,(-1+e^{s}-e^{s/2}\,s)\,\mathrm{Csch}[\frac{s}{2}]}{s^3\,(5+2\,s+s^2)}$$

The solution (not displayed) is

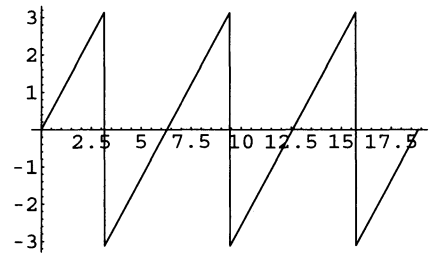
```
y = InverseLaplaceTransform[Y, s, t];

y1 = y /. TruncateSummation[8];

Plot[y1, {t, 0, 10}, PlotRange → All];
```



3.4.2.3 Odd Periodic Functions

`PeriodicOddFunction[`f_H`, `H`]` defines the odd periodic original, specified by the expression $f_H(t)$ on the interval of the half period $[0, H]$.

```
f = PeriodicOddFunction[# &, Pi];

Plot[f[t], {t, 0, 6 Pi}];
```



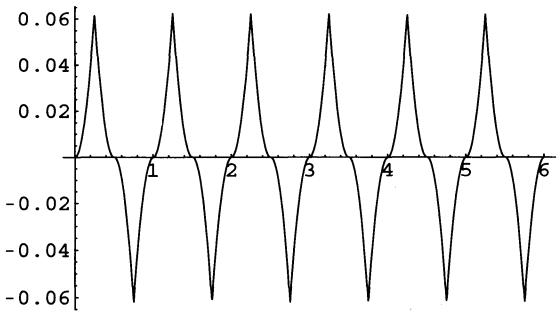

```
LaplaceTransform[f[t], t, s]
```

$$\frac{e^{-\pi s}\,(-1+e^{2\pi s}-2\,e^{\pi s}\,\pi s)\,\mathrm{Csch}[\pi s]}{2\,s^2}$$

3.4.2.4 Periodic Functions with a Symmetry of the Third Kind

These originals are special cases of odd periodic functions with a supplementary symmetry (see Section 1.2.2.7). The command `Periodic3dsFunction[`f_Q`, Q]` defines the periodic function with a third-kind-symmetry (3ds) by specifying the expression $f_Q(t)$ on the interval of the quarter period [0, Q].

```
f = Periodic3dsFunction[#^2 &, 1 / 4];

Plot[f[t], {t, 0, 6}];
```




```
LaplaceTransform[f[t], t, s]
```

$$\frac{e^{-s/4}\,(-2+2\,e^{s/2}-e^{s/4}\,s)\,\mathrm{Sech}[\frac{s}{4}]}{2\,s^3}$$

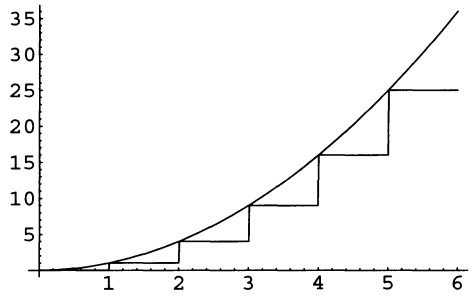
3.4.2.5 Step Functions

A regularly spread step function is defined by

$$f(t) := f_T(k),\ kT \le t < (k+1)T, \quad (k = 0, 1, 2, \ldots)$$

where $f_T(k)$ is a given sequence. Use `StepFunction[`f_T`, T]` to define it. The expression f_T has to be specified in the form of a pure function. Then the defined step function is also of this form. For example:

```
f = StepFunction[#^2 &, 1];

Plot[{f[t], t^2}, {t, 0, 6}];
```




```
F = LaplaceTransform[f[t], t, s]
```

$$\frac{1 + e^s}{(-1 + e^s)^2\, s}$$

```
InverseLaplaceTransform[F, s, t]
```

```
Floor[t]^2
```

There is a generalized form of a step function, defined as

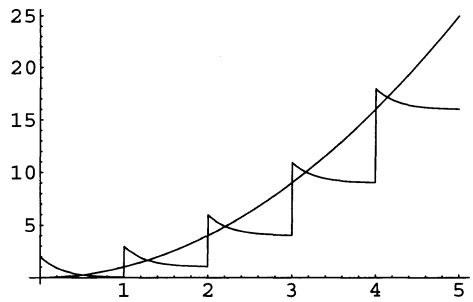
$$f(t) = f_T(k) + g(t - kT),\ kT \le t < (k+1)T, \quad (k = 0, 1, 2, \ldots)$$

where $f_T(k)$ and $g(t)$ are given expressions. The latter expression g needs only be defined on the interval $[0, T]$. Use `StepFunction[`f_T`, T, g]` where f_T and g are to be specified as pure functions.

```
f = StepFunction[#^2 &, 1, 2 Exp[-4 #] &]
```

```
StepFunction[#1^2 &, 1, 2 e^(-4 #1) &]
```

```
Plot[{t^2, f[t]}, {t, 0, 5}];
```



■ 3.4.3 Generalized Originals

In Section 1.2.3 the Dirac Impulse and its derivatives were introduced in a heuristic way. In *Mathematica* the Dirac Impulse (or Dirac Functional) at $t = a$ is represented by `DiracDelta[t-a]`. If $f(t)$ is continuous at $t = a$, then

$$f(t)\,\delta(t-a) \;=\; f(a)\,\delta(t-a).$$

In order to apply this simplification rule use the NamedRule `DiracDeltaRules`.

```
(t + Cos[t]) DiracDelta[t - a] /. DiracDeltaRules
```

```
(a + Cos[a]) DiracDelta[a - t]
```

Also,

```
f[t] DiracDelta[a t - b] /. DiracDeltaRules
```

$$\frac{\text{DiracDelta}\left[\frac{b}{a} - t\right] f\left[\frac{b}{a}\right]}{\text{Abs}[a]}$$

The one-sided Laplace transform of a Dirac Impulse and its derivatives are

```
LaplaceTransform[DiracDelta[t - a], t, s]
```

$$e^{-a\,s}\ \text{UnitStep}[a]$$

```
LaplaceTransform[DiracDelta[t], t, s]
```

```
1
```

```
LaplaceTransform[DiracDelta[t - 3]'', t, s]
```

$$e^{-3\,s}\ s^2$$

```
InverseLaplaceTransform[s^4 Exp[-a s], s, t]
```

$$\text{UnitStep}[-a + t]\ \text{DiracDelta}[a - t]^{(4)}$$

Another generalized original is a *Dirac Row*, a finite or infinite linear combination of evenly spread Dirac impulses:

$$\delta_f(t,\,T) \;:=\; \sum_{k=0}^{\infty} f(k)\,\delta(t - k\,T)\,,$$

which is represented by `DiracDeltaRow[f,T]` where f is given by a pure function. The Laplace transform of

$$\sum_{k=0}^{\infty} a^k\,\delta(t - k\,T),$$

for example, is

```
LaplaceTransform[DiracDeltaRow[a^# &, T][t], t, s]
```

$$\frac{e^{s\,T}}{-a + e^{s\,T}}$$

Other generalized originals also have Laplace transforms while originals like

$$\frac{1}{t},\ \frac{1}{t^2},\ \frac{1}{t-2},\ \ldots$$

do not have a Laplace transform in the ordinary sense, because the Laplace integral does not converge. It is possible, however, to define generalized functions that are represented by these symbols. How this is done in detail depends on the underlying theory of generalized functions. One way is to define these generalized functions by Hadamard's principal value of a divergent integral, see for example [Zemanian I (1987)]. In the framework of a theory of generalized functions, these generalized functions turn out to have Laplace transforms in a generalized sense. In order to distinguish these generalized functions from their ordinary counterparts, the symbol *Pf* is used. It stands for pseudo function, and is reminiscent of how it came into existence, as Hadamard's finite part . Thus one writes

$$Pf\,\frac{1}{t},\ Pf\,\frac{1}{t^2},\ Pf\,\frac{1}{t-2},\ \ldots$$

for these generalized functions. In the Package they are represented as `PF[...]`. For a non-integral exponent the Laplace transform is:

```
LaplaceTransform[Pf[1/t^(5/3)], t, s]
```

$$s^{2/3}\ \text{Gamma}\left[-\frac{2}{3}\right]$$

If the exponent is an integer, things change.

```
DefPositive[n]; DefInteger[n];
LaplaceTransform[Pf[1/t^n], t, s]
```

$$-\frac{1}{(-1+n)!}\left((-s)^{-1+n}\left(\text{EulerGamma} + \text{Log}[s] - \text{Summation}\left[\frac{1}{k1},\ \{k1,\ 1,\ -1+n\}\right]\right)\right)$$

```
LaplaceTransform[Pf[1/t^5], t, s]
```

$$-\frac{1}{24}\,s^4\left(-\frac{25}{12} + \text{EulerGamma} + \text{Log}[s]\right)$$

Or, for example,

```
DefPositive[b];
LaplaceTransform[Pf[1/(t-b)], t, s]
```

$$-e^{-b\,s}\ \text{ExpIntegralEi}[b\,s]$$

Generalized functions can also result from applying inverse Laplace transforms, for example:

```
InverseLaplaceTransform[ArcTan[s], s, t]
```

$$\frac{1}{2}\,\pi\,\text{DiracDelta}[t] - \frac{\text{Sin}[t]}{t}$$

■ 3.4.4 Special Functions of Mathematical Physics

The Package handles certain problems by computing others from a table of correspondences. The Package contains a large table of Laplace transforms and an even larger one of inverse Laplace transforms. If the Laplace transform or the inverse Laplace transform of a special function from mathematical physics is needed, the Package will consult these tables. The tables are extended with every new version of the Package.

```
LaplaceTransform[Sqrt[t] BesselJ[1 / 2, 3 t], t, s]
```

$$\frac{\sqrt{\frac{6}{\pi}}}{9 + s^2}$$

```
LaplaceTransform[LaguerreL[5, x], x, s]
```

$$\frac{1}{120}\left(-\frac{120}{s^6} + \frac{600}{s^5} - \frac{1200}{s^4} + \frac{1200}{s^3} - \frac{600}{s^2} + \frac{120}{s}\right)$$

```
% // Together // Factor
```

$$\frac{(-1 + s)^5}{s^6}$$

```
InverseLaplaceTransform[%, s, x]
```

$$1 - 5x + 5x^2 - \frac{5x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{120}$$

which is Laguerre's polynomial of degree 5.

```
f = (-t + Sqrt[a^2 + t^2])^(3 / 2) / (Sqrt[t] Sqrt[a^2 + t^2])
```

$$\frac{\left(-t + \sqrt{a^2 + t^2}\right)^{3/2}}{\sqrt{t}\,\sqrt{a^2 + t^2}}$$

Define the quantity a to be positive.

```
DefPositive[a]; LaplaceTransform[f, t, s]
```

$$\frac{1}{\sqrt{2}}\left(\pi\left(\text{BesselJ}\left[1, \frac{a\,s}{2}\right]\text{Cos}\left[\frac{a\,s}{2}\right] + \text{BesselY}\left[1, \frac{a\,s}{2}\right]\text{Sin}\left[\frac{a\,s}{2}\right]\right)\right)$$

The Laplace transforms of the examples in this book have rarely been computed by integrating the defining Laplace integral. But you may do this by benefitting from the strong integration capacities of *Mathematica*. A Laplace transform may be directly computed by `DirectLaplaceTransform[f[t],t,s]`.

```
DirectLaplaceTransform[1 / (t^4 + 1), t, s]
```

$$\text{If}\left[\text{Re}[s] > 0,\ \frac{\text{MeijerG}\left[\left\{\left\{\frac{3}{4}\right\}, \{\}\right\}, \left\{\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right\}, \{\}\right\}, \frac{s^4}{256}\right]}{4\sqrt{2}\,\pi^{3/2}},\ \int_0^\infty \frac{e^{-s\,t}}{1 + t^4}\,dt\right]$$

```
DirectLaplaceTransform[1 / (t^4 + 1), t, s, Assumptions → Re[s] > 0]
```

$$\frac{\text{MeijerG}\left[\left\{\left\{\frac{3}{4}\right\}, \{\}\right\}, \left\{\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right\}, \{\}\right\}, \frac{s^4}{256}\right]}{4\sqrt{2}\,\pi^{3/2}}$$

3.5 Inverse Laplace Transformation

Applying Laplace transformation in general is straightforward until the point where one faces the problem of finding the inverse Laplace transform. Thus techniques and methods for transforming back image functions are very important.

■ 3.5.1 The Inversion of Rational Image Functions

If ordinary differential equations or systems are solved by means of Laplace transformation, the image of the solution often is a rational image function. Only proper rational image functions correspond to ordinary originals. If the degree of the numerator polynomial is greater than or equal to the degree of the denominator polynomial, the rational function has to be decomposed into a polynomial and a proper rational function. A linear combination of Dirac impulses and their derivatives plus an ordinary function will result.

```
InverseLaplaceTransform[(s + b) / ((s^2 + 2 s + 26) (s + 1)), s, t]
```

$$\frac{1}{25}(-1+b)\,e^{-t}+2\,e^{-t}\left(\left(\frac{1}{50}-\frac{b}{50}\right)\text{Cos}[5\,t]+\frac{1}{10}\text{Sin}[5\,t]\right)$$

```
InverseLaplaceTransform[(s + b)^4 / ((s^2 + 2 s + 26) (s + 1)), s, t]
```

$$\frac{1}{25}(-1+b)^4\,e^{-t}-3\,\text{DiracDelta}[t]+4\,b\,\text{DiracDelta}[t]+$$
$$2\,e^{-t}\left(\left(-\frac{238}{25}-\frac{148\,b}{25}+\frac{72\,b^2}{25}+\frac{2\,b^3}{25}-\frac{b^4}{50}\right)\text{Cos}[5\,t]-\right.$$
$$\left.\left(-\frac{48}{5}+\frac{44\,b}{5}+\frac{6\,b^2}{5}-\frac{2\,b^3}{5}\right)\text{Sin}[5\,t]\right)+\text{DiracDelta}[t]'$$

As long as the denominator of a rational function is composed only of powers of linear or quadratic factors exact results are returned.

```
InverseLaplaceTransform[1 / ((s^2 + s + 1)^2 (s + 1)^3), s, t]
```

$$e^{-t}+2\,e^{-t}\,t+\frac{1}{2}e^{-t}\,t^2+\frac{2}{3}e^{-t/2}\,t\,\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right]+$$
$$2\,e^{-t/2}\left(-\frac{1}{2}\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right]-\frac{13\,\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{6\sqrt{3}}\right)$$

If the denominator contains a factor of degree 3 or higher, then it must contain only numerical coefficients because the roots of such a polynomial must all be computed. Although there are exact formulas for computing the roots of a polynomial of degree 3 or 4, they are unhandy and of little use for our purpose. So are the Root objects of *Mathematica* dealing with the roots of a higher degree polynomial in an exact way. They are finally of little practical use for our goals. The following example is a lucky exception because an exact factorization can be found.

```
InverseLaplaceTransform[1 / (s^3 + s^2 + s + 1)^2, s, t]
```

$$\frac{e^{-t}}{2}+\frac{e^{-t}\,t}{4}+2\left(-\frac{\text{Cos}[t]}{4}+\frac{\text{Sin}[t]}{8}\right)-\frac{1}{4}\,t\,\text{Sin}[t]$$

```
InverseLaplaceTransform[1 / (- s^3 + 4 s^2 + 2 s + 1)^2, s, t] // Chop
```

$$-0.00163326\, e^{4.49449\, t} + 0.00195001\, e^{4.49449\, t}\, t + 2\, e^{-0.247246\, t}\, t\, (-0.0674403 \cos[0.401701\, t] + 0.0115091 \sin[0.401701\, t]) + 2\, e^{-0.247246\, t}\, (0.000816628 \cos[0.401701\, t] + 0.175099 \sin[0.401701\, t])$$

The latter of the two examples above is a typical case that needs the `Chop` command in order to chop off the small roundings. The result is certainly of a practical value but it is not exact in a strict sense. If the denominator contains a quadratic factor with symbolic coefficients, the corresponding original depends on the discriminant of the quadratic factor. If there is just one single power of a quadratic factor, the original appears inside a `Which` construct.

```
InverseLaplaceTransform[(s + 2) / (a s^2 + b s + c)^2, s, t] // Short
```

$$\text{Which}\left[b^2 - 4\,a\,c < 0,\ \frac{e^{\text{<<1>>}}\ \text{<<1>>}\ (\text{<<1>>})}{a^2},\ \text{<<2>>},\ \text{<<1>>},\ \frac{e^{-\text{<<1>>}}\ (\frac{\text{<<1>>}}{2} + \text{<<1>>})}{a^2}\right.$$

Restricting the sign of the discriminant, yields

```
InverseLaplaceTransform[
 (s + 2) / (a s^2 + b s + c)^2, s, t, Assumptions → b^2 - 4 a c < 0]
```

$$\frac{1}{a^2}\left(e^{-\frac{b\,t}{2a}}\left(-\frac{2\,a^2\,(-4\,a+b)\,\sin\left[\frac{\sqrt{-b^2+4\,a\,c}\ t}{2\,a}\right]}{(-b^2+4\,a\,c)^{3/2}} + 2\,t\left(\frac{a\,(-4\,a+b)\,\cos\left[\frac{\sqrt{-b^2+4\,a\,c}\ t}{2\,a}\right]}{2\,(-b^2+4\,a\,c)} + \frac{a\,\sin\left[\frac{\sqrt{-b^2+4\,a\,c}\ t}{2\,a}\right]}{2\sqrt{-b^2+4\,a\,c}}\right)\right)\right)$$

If there are factors other than the quadratic one, the situation is less comfortable:

```
InverseLaplaceTransform[(s + 2) / ((a s^2 + b s + c) (s + 3)), s, t]
```

$$-\frac{e^{-3\,t}}{9\,a-3\,b+c} + \frac{\left(4\,a-b-\sqrt{b^2-4\,a\,c}\right)\,e^{\frac{\left(-b-\sqrt{b^2-4\,a\,c}\right)\,t}{2\,a}}}{b^2-4\,a\,c-6\,a\sqrt{b^2-4\,a\,c}+b\sqrt{b^2-4\,a\,c}} - \frac{\left(4\,a-b+\sqrt{b^2-4\,a\,c}\right)\,e^{\frac{\left(-b+\sqrt{b^2-4\,a\,c}\right)\,t}{2\,a}}}{-b^2+4\,a\,c-6\,a\sqrt{b^2-4\,a\,c}+b\sqrt{b^2-4\,a\,c}}$$

In the case of $b^2 - 4\,ac \le 0$ the result has to be processed by hand. One way out of such a situation is to use the incomplete partial fraction decomposition first.

```
IncompletePartialFraction[(s + 2) / ((a s^2 + b s + c) (s + 3)), h, k, s]
```

$$\left\{\frac{h[1]}{3+s} + \frac{s\,k[1]+k[2]}{c+b\,s+a\,s^2},\ \left\{h[1] \to -\frac{1}{9\,a-3\,b+c},\ k[1] \to \frac{a}{9\,a-3\,b+c},\ k[2] \to 1 + \frac{-3\,a+b}{9\,a-3\,b+c}\right\}\right\}$$

```
InverseLaplaceTransform[First[%], s, t, Assumptions → b^2 - 4 a c < 0];

% /. Last[%%]
```

$$-\frac{e^{-3\,t}}{9\,a-3\,b+c} + \frac{1}{a}\left(2\,e^{-\frac{b\,t}{2a}}\left(\frac{a\,\cos\left[\frac{\sqrt{-b^2+4\,a\,c}\ t}{2\,a}\right]}{2\,(9\,a-3\,b+c)} - \left(\frac{a\,b}{2\,(9\,a-3\,b+c)\sqrt{-b^2+4\,a\,c}} - \frac{a\,\left(1+\frac{-3\,a+b}{9\,a-3\,b+c}\right)}{\sqrt{-b^2+4\,a\,c}}\right)\sin\left[\frac{\sqrt{-b^2+4\,a\,c}\ t}{2\,a}\right]\right)\right)$$

■ 3.5.2 Meromorphic Image Functions

A function of the complex variable s is meromorphic if the only singularities in the finite complex plane are poles. A rational function always has a finite number of poles and no other singularities, thus it is an example of a meromorphic function. This section deals with meromorphic image functions that have an infinite number of poles. Often the poles are evenly spaced on, or parallel to, the imaginary axis. The Laplace transform of a periodic original is a meromorphic image function with evenly spaced poles on the imaginary axis. The original of such a meromorphic image appears in the form of an infinite series, a Fourier series or a generalized form of it. Almost all meromorphic image functions $F(s)$ encountered in applications are real in the sense that $F(\bar{s}) = \overline{F(s)}$. This relation implies that pairs of complex conjugate poles combine to real quantities. A rational function with real coefficients in both the numerator and the denominator polynomials is a simple example of a real meromorhic function. In the following we consider only real meromorphic functions.

```
F = Sinh[x s] / (s^5 Cosh[a s]);
```

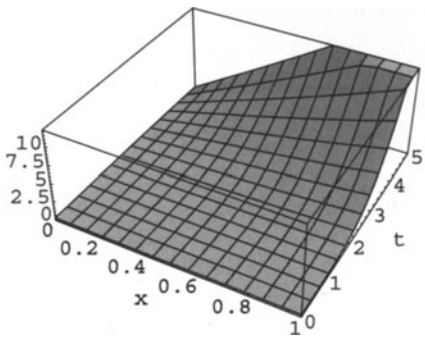
where $0 < x < a$.

```
InverseLaplaceTransform[F, s, t]
```

$$\frac{1}{24}\left(4\left(-3a^2t+t^3\right)x+4tx^3\right)+\text{Summation}\left[\frac{64(-1)^{k1}a^4\,\text{Sin}\left[\frac{\left(\frac{1}{2}+k1\right)\pi t}{a}\right]\text{Sin}\left[\frac{\left(\frac{1}{2}+k1\right)\pi x}{a}\right]}{(1+2k1)^5\pi^5},\{k1,0,\infty\}\right]$$

The series converges very rapidly. Thus it suffices to take just a few terms in order to obtain a good approximation. If a numerical value is assigned to the parameter a, the original as a function of the two variables t and x can be plotted. Image functions of this type frequently arise when boundary value problems are solved by means of Laplace transformation.

```
f1 = % /. TruncateSummation[5] /. a → 1;

Plot3D[f1, {x, 0, 1}, {t, 0, 5}, AxesLabel → {"x", "t", ""}];
```




```
F = Sinh[x Sqrt[s]] / (Sqrt[s] Cosh[a Sqrt[s]])
```

$$\frac{\text{Sech}\left[a\sqrt{s}\right]\text{Sinh}\left[\sqrt{s}\,x\right]}{\sqrt{s}}$$

where again it is assumed that $0 < x < a$. Image functions of this type arise in heat conduction problems, for example. At first glance it seems that we have an image function with a branch point at $s = 0$ due to the presence of the square root. But this is not the case, because expanding the numerator and the denominator into Taylor series shows that the square root is canceled and a quotient of two entire functions remains.

```
InverseLaplaceTransform[F, s, t]
```

$$\frac{1}{a}\left(2\,\text{Summation}\left[(-1)^{-1+n1}\,e^{-\frac{(-1+2\,n1)^2\,\pi^2\,t}{4\,a^2}}\,\text{Sin}\left[\frac{(-1+2\,n1)\,\pi\,x}{2\,a}\right],\,\{n1,\,1,\,\infty\}\right]\right)$$

In order to inspect the first few terms of the series, replace `Infinity` by a positive integer.

```
% /. Infinity → 3
```

$$\frac{1}{a}\left(2\left(e^{-\frac{\pi^2\,t}{4\,a^2}}\,\text{Sin}\left[\frac{\pi\,x}{2\,a}\right]-e^{-\frac{9\,\pi^2\,t}{4\,a^2}}\,\text{Sin}\left[\frac{3\,\pi\,x}{2\,a}\right]+e^{-\frac{25\,\pi^2\,t}{4\,a^2}}\,\text{Sin}\left[\frac{5\,\pi\,x}{2\,a}\right]\right)\right)$$

In Section 3.4.2.2 we studied an example with a periodic input acting on a linear system described by a differential equation. Another example of this type is exciting a pendulum by evenly applied strikes of a hammer. If the pendulum is modeled by

$$y'' + y(t) = f(t),$$
$$y(0+) = 0,$$
$$y'(0+) = 0,$$

and the action of the evenly applied strikes, each with intensity a, by a Dirac Row

$$f(t) = \sum_{k=0}^{\infty} a\,\delta(t - k\,T),$$

then

```
F = LaplaceTransform[DiracDeltaRow[a &, T][t], t, s]
```

$$\frac{a\,e^{s\,T}}{-1+e^{s\,T}}$$

The Laplace transform of the pendulum's response is

```
Y = % / (s^2 + 1)
```

$$\frac{a\,e^{s\,T}}{(-1+e^{s\,T})\,(1+s^2)}$$

This is a meromorphic image function. The response of the pendulum depends on the relation between the angular frequency $2\pi/T$ of the hammer strikes and the eigenfrequency of the pendulum oscillation, which is assumed to be 1.

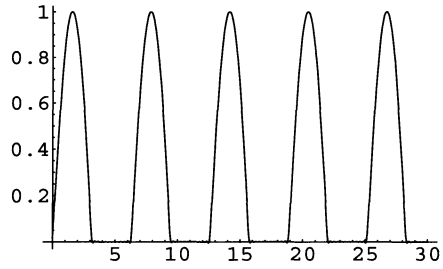
```
y = InverseLaplaceTransform[Y, s, t];
```

A) $T = \pi$: the period of the hammer strikes is half the period of free oscillation of the pendulum:

```
y /. {T → Pi, a → 1}
```

$$\frac{1}{\pi}+\frac{\text{Sin}[t]}{2}+\text{Summation}\left[\frac{2\,\text{Cos}[2\,k2\,t]}{(1-4\,k2^2)\,\pi},\,\{k2,\,1,\,\infty\}\right]$$

```
Plot[% /. Infinity → 10, {t, 0, 30}];
```



B) $T = 2\pi$: the period of the hammer strikes is equal to the period of free oscillation; (resonance case):

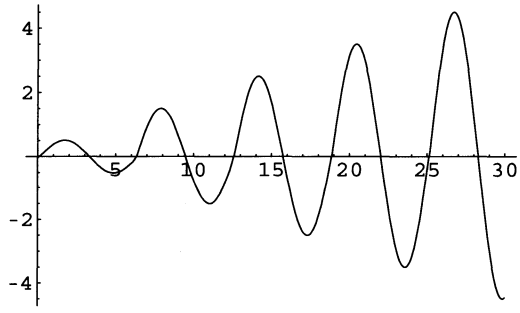
```
Y /. {T → 2 Pi, a → 1}
```

$$\frac{e^{2\pi s}}{(-1+e^{2\pi s})\,(1+s^2)}$$

```
InverseLaplaceTransform[%, s, t]
```

$$\frac{1}{2\pi} + \frac{t\,\text{Sin}[t]}{2\pi} + \text{Summation}\left[\frac{\text{Cos}[k3\,t]}{(1-k3^2)\,\pi},\ \{k3,\ 2,\ \infty\}\right]$$

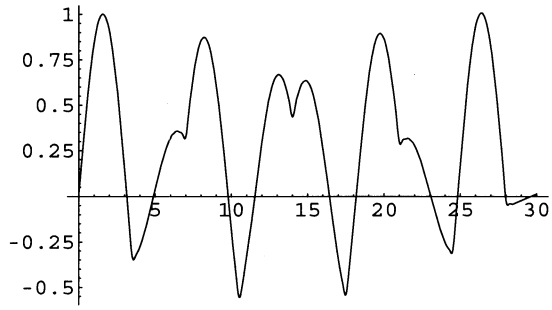
```
% /. TruncateSummation[10];

Plot[% , {t, 0, 30}];
```



C) $T \neq \pi, 2\pi$ (for example T = 3.5) this produces an erratic motion.

```
y /. {a → 1, T → 3.5 , Infinity → 10 } // Chop;
```

```
Plot[% , {t, 0, 30}];
```



A given meromorphic expression with exponential terms must first be transformed into one with hyperbolic terms before applying `InverseLaplaceTransform`. Use the Named Rule `ToHyperbolicFunctions`. For example:

```
F = 1 / (s^2 (s^2 + Pi^2 / a^2) (1 - Exp[-a s]));

F = F  /. ToHyperbolicFunctions
```

$$\frac{e^{\frac{a s}{2}}\ \text{Csch}\left[\frac{a s}{2}\right]}{2 s^2 \left(\frac{\pi^2}{a^2} + s^2\right)}$$

```
InverseLaplaceTransform[F, s, t]
```

$$\frac{a \left(\frac{a}{2} + t\right)^2}{2 \pi^2} - \frac{a^3 \text{Sin}\left[\frac{\pi t}{a}\right]}{2 \pi^3} + \text{Summation}\left[\frac{a^3 \text{Cos}\left[\frac{2 k4 \pi t}{a}\right]}{2 k4^2 \left(-1 + 4 k4^2\right) \pi^4}, \{k4, 1, \infty\}\right]$$

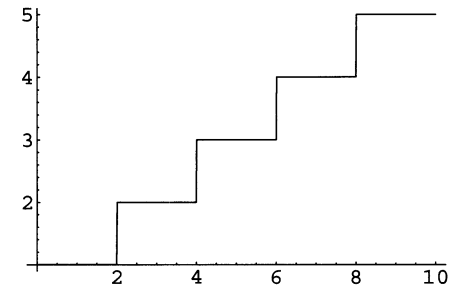
For the following class of meromorphic functions the initial behavior is studied:

$$Y(s) = \frac{F(s)}{(1 \pm b \exp(\pm a s))^m}, \quad (m \in \mathbb{N})$$

```
InverseLaplaceTransform[1 / (s (1 - Exp[-2 s])), s, t, Transient → True]
```

```
Summation[(-1)^n2 Binomial[-1, n2], {n2, 0, Floor[t/2]}]
```

```
Plot[%, {t, 0, 10}];
```



Let us take this function as the input in a system described by its transfer function

$$G(s) = \frac{1}{s^2 + 2s + 17}.$$

Then the output of the system is the inverse Laplace transform of

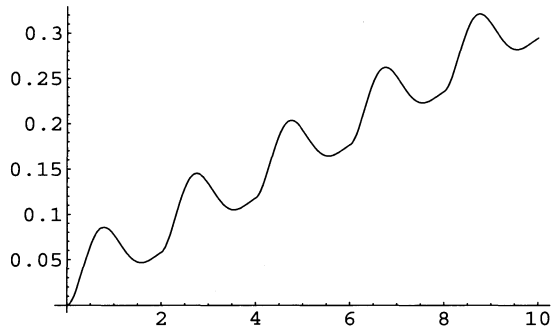
```
1 / (s (1 - Exp[-2 s]) (s^2 + 2 s + 17));
```

and the initial behavior is obtained by

```
InverseLaplaceTransform[%, s, t, Transient → True]
```

$$\text{Summation}\left[(-1)^{n3}\ \text{Binomial}[-1,\ n3]\left(\frac{1}{17}+2\,e^{2\,n3-t}\left(-\frac{1}{34}\,\text{Cos}[4\,(-2\,n3+t)]-\frac{1}{136}\,\text{Sin}[4\,(-2\,n3+t)]\right)\right),\ \left\{n3,\ 0,\ \text{Floor}\left[\frac{t}{2}\right]\right\}\right]$$

```
Plot[%, {t, 0, 10}];
```



Further techniques for the inversion of a Laplace transform of meromorphic image functions will be treated in Chapter 6.

3.6 Differential Equations

■ 3.6.1 Ordinary Differential Equations with Constant Coefficients

There are two ways of specifying differential equations that are to be solved by means of Laplace transformation. The first uses classical notation, similar to the built-in `DSolve` of *Mathematica*, the second is an operational notation. The specification of the initial values are dealt with differently between the two ways.

3.6.1.1 Classical Notations (LinearIDSolve)

In order to specify a differential equation, the same syntax is used as for the *Mathematica* built-in `DSolve` command. For example:

```
Clear[y];

eqn = y''[t] + 5 y[t] == -2 y'[t] + t ;
```

The order of the terms, i.e. what is on the left-hand side or on the right-hand side does not matter. By use of `LinearIDSolve[eqn, y[t],t]` the general solution is obained.

```
LinearIDSolve[eqn, y[t], t]
```

$$\left\{y[t] \to -\frac{2}{25} + \frac{t}{5} + 2\,e^{-t}\left(\text{Cos}[2\,t]\left(\frac{1}{25} + \frac{y[0]}{2}\right) - \text{Sin}[2\,t]\left(\frac{1}{100}\,(3 - 25\,y[0]) - \frac{y'[0]}{4}\right)\right)\right.$$

The initial values not specified appear explicitly. Specified initial values are put into a replacement list.

```
iv = {y[0] → 2, y'[0] → -1};

LinearIDSolve[eqn, iv, y[t], t]
```

$$\left\{y[t] \to -\frac{2}{25} + \frac{t}{5} + 2\,e^{-t}\left(\frac{26}{25}\,\text{Cos}[2\,t] + \frac{11}{50}\,\text{Sin}[2\,t]\right)\right\}$$

This solves the initial value problem. Thus when using the `LinearIDSolve` command, the initial values that are not specified appear explicitly in the solution. If in the second to the last argument in the command, the independent variable is omitted, the solution is produced in the form of a pure function.

```
LinearIDSolve[eqn, iv, y, t]
```

$$\left\{y \to \left(-\frac{2}{25} + 2\,e^{-\#1}\left(\frac{26}{25}\,\text{Cos}[2\,\#1] + \frac{11}{50}\,\text{Sin}[2\,\#1]\right) + \frac{\#1}{5}\;\&\right)\right\}$$

This is a convenient form to check whether the solution is correct.

```
Expand[ y''[t] + 5 y[t] /. %] == Expand[-2 y'[t] + t /. %]

True
```

Perhaps the Package may not successfully compute the solution in a complicated situation, or, for some other reasons, you want to take a look at the image function of the solution. This can be done by using the option `{GoFor->Image[Y,s]}`.

```
Clear[Y];

LinearIDSolve[5 y[t] + y''[t] == t - 2 y'[t],
 {y[0] → 2, y'[0] → -1}, y[t], t, GoFor → Image[Y, s]]
```

$$\left\{Y[s] \to \frac{1 + 3\,s^2 + 2\,s^3}{s^2\,(5 + 2\,s + s^2)}\right\}$$

Systems of differential equations are handled in the same way. For example,

```
sys = {x''[t] + 3 x[t] == -y'[t] + 15 Exp[-t],
   y''[t] + 3 y[t] - 4 x'[t] == 15 Sin[2 t]};

ivs = {x[0] → 35, x'[0] → -48, y[0] → 27, y'[0] → -5};

LinearIDSolve[sys, ivs, {x[t], y[t]}, t]
```

$$\left\{x[t] \to 3\,e^{-t} + \frac{95\,\text{Cos}[t]}{4} + 2\,\text{Cos}[2\,t] + 2\left(\frac{25}{8}\,\text{Cos}[3\,t] - \frac{15}{2}\,\text{Sin}[3\,t]\right),\right.$$
$$\left.y[t] \to -3\,e^{-t} - \frac{95\,\text{Sin}[t]}{2} + \text{Sin}[2\,t] + 2\left(15\,\text{Cos}[3\,t] + \frac{25}{4}\,\text{Sin}[3\,t]\right)\right\}$$

In order to obtain the solutions in the form of pure functions, drop the independent variable in the second to the last list of arguments:

```
sol = LinearIDSolve[sys, ivs, {x, y}, t]
```

$$\left\{x \to \left(3\, e^{-\#1} + \frac{95\,\text{Cos}[\#1]}{4} + 2\,\text{Cos}[2\,\#1] + 2\left(\frac{25}{8}\,\text{Cos}[3\,\#1] - \frac{15}{2}\,\text{Sin}[3\,\#1]\right)\&\right),\right.$$
$$\left. y \to \left(-3\, e^{-\#1} - \frac{95\,\text{Sin}[\#1]}{2} + \text{Sin}[2\,\#1] + 2\left(15\,\text{Cos}[3\,\#1] + \frac{25}{4}\,\text{Sin}[3\,\#1]\right)\&\right)\right\}$$

Now the solution can be checked.

```
{Expand[ x''[t] + 3 x[t] /. sol] = Expand[ -y'[t] + 15 Exp[-t] /. sol],
 Expand[y''[t] + 3 y[t] - 4 x'[t] /. sol] = Expand[15 Sin[2 t] /. sol]}
```

```
{True, True}
```

Let us also check whether the solution satisfies the initial conditions.

```
{(x /. sol)[t], (x /. sol)'[t], (y /. sol)[t], (y /. sol)'[t]} /. t → 0
```

```
{35, -48, 27, -5}
```

If you want to take a look at the image functions (just one or all of them) use the option {GoFor->Image[{Y[1],Y[2]},s]} (you must use an indexed symbol name for the variables, for example Y[1],Y[2] or X[1],X[2] rather than X and Y).

```
LinearIDSolve[sys, ivs, {x[t], y[t]}, t, GoFor → Image[{Y[1], Y[2]}, s]]
```

$$\left\{Y[1][s] \to \frac{-72 + 718\,s + 820\,s^2 + 135\,s^3 + 357\,s^4 - 13\,s^5 + 35\,s^6}{(1+s)\,(1+s^2)\,(4+s^2)\,(9+s^2)},\right.$$
$$\left. Y[2][s] \to \frac{-1650 - 1422\,s - 437\,s^2 - 260\,s^3 + 100\,s^4 + 22\,s^5 + 27\,s^6}{(1+s)\,(1+s^2)\,(4+s^2)\,(9+s^2)}\right\}$$

Another example:

```
sys2 = {x''[t] + y[t] + x[t] == 10, x''[t] + 2 x[t] - y[t] == t};

ivs2 = {x[0] → 0, x'[0] → 0, y[0] → 0, y'[0] → 0};

LinearIDSolve[sys2, ivs2, {x[t], y[t]}, t]
```

$$\left\{x[t] \to \frac{10}{3} + \frac{t}{3} + 2\left(-\frac{5}{3}\,\text{Cos}\left[\sqrt{\frac{3}{2}}\,t\right] - \frac{\text{Sin}\left[\sqrt{\frac{3}{2}}\,t\right]}{3\sqrt{6}}\right),\right.$$
$$\left. y[t] \to \frac{20}{3} - \frac{t}{3} + 2\left(-\frac{5}{6}\,\text{Cos}\left[\sqrt{\frac{3}{2}}\,t\right] - \frac{\text{Sin}\left[\sqrt{\frac{3}{2}}\,t\right]}{6\sqrt{6}}\right)\right\}$$

Again, let us check the solution:

```
sol = LinearIDSolve[sys2, ivs2, {x, y}, t];

{Expand[x''[t] + y[t] + x[t] /. sol] == 10,
 Expand[x''[t] + 2 x[t] - y[t] /. sol] == t }
```

```
{True, True}
```

Thus the computed expressions satisfy the system of differential equations. Let us check the initial conditions

```
{(x /. sol)[t], (x /. sol)'[t], (y /. sol)[t], (y /. sol)'[t]} /. t → 0
```

$$\left\{0,\ 0,\ 5,\ -\frac{1}{2}\right\}$$

Whoops ! The solution does not satisfy the initial conditions. Is there a mistake in the Package? No, things lie deeper. The first of the above examples is a normal system of differential equations, while the second is called an *anomalous system.* The system in the second example reads in matrix notation

$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x''\\ y''\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} + \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}10\\ t\end{pmatrix}.$$

Here the coefficient matrix of the highest order derivative has a vanishing determinant, which defines an anomalous system. Anomalous systems, in general, produce solutions that do not satisfy all the initial conditions. This is annoying considering the fact that most systems of differential equations encountered in applications are anomalous. This, of course, is not a shortcoming of the method of Laplace transformation. It is rather an intrinsic fact of the classical modeling process of a physical reality. A remedy can be achieved only in the framework of generalized functions. We will revisit anomalous systems in a later chapter (see Section 6.3).

3.6.1.2 The Operational Notation (LinearDOpSolve)

A differential equation of the type

$$\left(a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1\frac{d}{dt} + a_0\right) y(t)$$
$$= \left(b_m \frac{d^m}{dt^m} + b_{m-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + b_1\frac{d}{dt} + b_0\right) f(t)$$

can be written as

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0)\, y(t)$$
$$= (b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0)\, f(t)$$

with the differential operator $p := d/dt$. This form is used in the `LinearDOpSolve[P(p)[y]==Q[p][f(t)],p,y[t],t]` command, where

$$P(p) = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0$$
$$Q(p) = b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0.$$

For example,

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 5\right) y(t) = \left(\frac{d}{dt} - 1\right) f(t)$$

```
LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p - 1)[Cos[t]], p, y[t], t]
```

$$\left\{y[t] \to 2\left(-\frac{\text{Cos}[t]}{20} - \frac{3\,\text{Sin}[t]}{20}\right) + 2\,e^{-t}\left(\frac{1}{20}\,\text{Cos}[2\,t] + \frac{1}{10}\,\text{Sin}[2\,t]\right)\right\}$$

One obtains the response, i.e. the solution corresponding to zero initial values of $y(t)$: $y(0+) = 0$, $y'(0+) = 0$. Note that this is not the same as

$$\mathcal{L}^{-1}[G(s)\,F(s)]\,(t),$$

where $G(s)$ is the transfer function of the system, and $F(s)$ is the image of the input function.

```
G = (s - 1)/(s^2 + 2 s + 5); F = s/(s^2 + 1);

InverseLaplaceTransform[G F, s, t]
```

$$2\left(-\frac{\text{Cos}[t]}{20} - \frac{3\,\text{Sin}[t]}{20}\right) + 2\,e^{-t}\left(\frac{1}{20}\,\text{Cos}[2\,t] + \frac{7}{20}\,\text{Sin}[2\,t]\right)$$

The last expression is the forced response as defined in Section 1.3.1.1. If there is no differential operator on the right-hand side, the two answers are the same.

Non-zero initial values are substituted into the replacement list. For example

```
iv = {y[0] → A, y'[0] → B};

LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p - 1)[Cos[t]], iv, p, y[t], t]
```

$$\left\{y[t] \to -\frac{1}{10}\,e^{-t}\,(e^{t}\,\text{Cos}[t] - \text{Cos}[2\,t] - 10\,A\,\text{Cos}[2\,t] + 3\,e^{t}\,\text{Sin}[t] - 2\,\text{Sin}[2\,t] - 5\,A\,\text{Sin}[2\,t] - 5\,B\,\text{Sin}[2\,t])\right\}$$

By omitting the independent variable in the second to the last argument, the solution is given as a pure function.

```
LinearDOpSolve[(p^2 + 2 p + 5)[y] == Cos[t], iv, p, y, t]
```

$$\left\{y \to \left(\frac{1}{20}\,e^{-\#1}\,(4\,e^{\#1}\,\text{Cos}[\#1] - 4\,\text{Cos}[2\,\#1] + 20\,A\,\text{Cos}[2\,\#1] + 2\,e^{\#1}\,\text{Sin}[\#1] - 3\,\text{Sin}[2\,\# 10\,A\,\text{Sin}[2\,\#1] + 10\,B\,\text{Sin}[2\,\#1])\ \&\right)\right\}$$

If you want to take a look at the image function of the solution, use the option `GoFor->Image[Y,s]`:

```
LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p - 1)[Cos[t]],
 iv, p, y[t], t, GoFor → Image[Y, s]]
```

$$\left\{Y[s] \to \frac{-1 + 2\,A + B - s + A\,s + 2\,A\,s^2 + B\,s^2 + A\,s^3}{(1 + s^2)\,(5 + 2\,s + s^2)}\right\}$$

Systems in operator form are dealt with similarly. For example, if the left-hand matrix differential operator is

```
L = {{17 + 2 p + p^2, -5 + p, -3}, {-1 + 2 p, p, -p^2}, {3, 4 - p, -p}};
MatrixForm[L]
```

$$\begin{pmatrix} 17 + 2\,p + p^2 & -5 + p & -3 \\ -1 + 2\,p & p & -p^2 \\ 3 & 4 - p & -p \end{pmatrix}$$

the right-hand differential operator is

```
R = {{-1 + 3 p, p, 1}, {0, p^3, 0}, {p, 0, -5}}; MatrixForm[R]
```

$$\begin{pmatrix} -1+3p & p & 1 \\ 0 & p^3 & 0 \\ p & 0 & -5 \end{pmatrix}$$

and the input vector is

```
f = {Sin[t], Exp[-t], 1 + t}; MatrixForm[f]
```

$$\begin{pmatrix} \text{Sin}[t] \\ e^{-t} \\ 1+t \end{pmatrix}$$

and all initial values are zero, then the system is

$$\begin{pmatrix} 17+2\frac{d}{dt}+\frac{d^2}{dt^2} & -5+\frac{d}{dt} & -3 \\ -1+2\frac{d}{dt} & \frac{d}{dt} & -\frac{d^2}{dt^2} \\ 3 & 4-\frac{d}{dt} & -\frac{d}{dt} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$= \begin{pmatrix} -1+3\frac{d}{dt} & \frac{d}{dt} & 1 \\ 0 & \frac{d^3}{dt^3} & 0 \\ \frac{d}{dt} & 0 & -5 \end{pmatrix} \begin{pmatrix} \sin t \\ e^{-t} \\ 1+t \end{pmatrix}$$

$$y_1(0+) = y_1'(0+) = y_2(0+) = y_3(0+) = y_3'(0+) = 0$$

```
LinearDOpSolve[L[{y1, y2, y3}] = R[f],
   p, {y1[t], y2[t], y3[t]}, t] // Chop // Short
```

```
{y1[t] → -1.25 + 0.05 e^-t + <<2>> + 2 (<<1>>) + <<1>>, <<1>>, y3[t] → <<1>>}
```

If you are interested in the image function of, say `y3[t]`, then use the option

`{GoFor->Image[{Y[3]},s]`

```
LinearDOpSolve[L[{y1, y2, y3}] = R[f], p,
 {y1[t], y2[t], y3[t]}, t, GoFor → Image[{Y[3]}, s]]
```

$$\left\{Y[3][s] \to \frac{-21+92\,s+137\,s^2+272\,s^3+145\,s^4+125\,s^5+13\,s^6+5\,s^7}{s^2\,(1+s)\,(1+s^2)\,(-12+13\,s-61\,s^2+12\,s^3-s^4+s^5)}\right\}$$

3.6.1.3 Special Solutions

The general solution of a homogeneous differential equation is obtained via `LinearIDSolve` or `LinearDOpSolve` by assigning general symbols to all initial values. For example:

```
LinearDOpSolve[(p^3 - a^3)[y] = 0,
 {y[0] → a, y'[0] → b, y''[0] → c}, p, y[x], x]
```

$$\left\{y[x] \to \frac{1}{3\,a^2}\left(e^{-\frac{a\,x}{2}}\left(a^3\,e^{\frac{3\,a\,x}{2}}+a\,b\,e^{\frac{3\,a\,x}{2}}+c\,e^{\frac{3\,a\,x}{2}}+2\,a^3\,\text{Cos}\left[\frac{1}{2}\sqrt{3}\,a\,x\right]-a\,b\,\text{Cos}\left[\frac{1}{2}\sqrt{3}\,a\,x\right]\right.\right.\right.$$
$$\left.\left.\left.c\,\text{Cos}\left[\frac{1}{2}\sqrt{3}\,a\,x\right]+\sqrt{3}\,a\,b\,\text{Sin}\left[\frac{1}{2}\sqrt{3}\,a\,x\right]-\sqrt{3}\,c\,\text{Sin}\left[\frac{1}{2}\sqrt{3}\,a\,x\right]\right)\right)\right\}$$

This is a form of a general solution depending on three integration constants. But it is in an unnecessarily complicated form. `SpecialDOpSolve` gives simpler expressions.

```
SpecialDOpSolve[(p^3 - a^3)[y] == 0, p, y[x], x] /. Arg[a] → 0
```

$$\{y[x] \to c1\, e^{a x} + c2\, e^{-\frac{a x}{2}} \operatorname{Cos}\left[\frac{1}{2}\sqrt{3}\, a x\right] + c3\, e^{-\frac{a x}{2}} \operatorname{Sin}\left[\frac{1}{2}\sqrt{3}\, a x\right]\}$$

You may also use this command for obtaining the general solution of a non- homogeneous equation.

```
SpecialDOpSolve[(p^3 - a^3)[y] == x^2, p, y[x], x] /. Arg[a] → 0
```

$$\{y[x] \to c4\, e^{a x} - \frac{x^2}{a^3} + c5\, e^{-\frac{a x}{2}} \operatorname{Cos}\left[\frac{1}{2}\sqrt{3}\, a x\right] + c6\, e^{-\frac{a x}{2}} \operatorname{Sin}\left[\frac{1}{2}\sqrt{3}\, a x\right]\}$$

If you just want a particular solution of a non-homogeneous equation, use the option

`{GoFor->OnlyParticular[]}`

```
SpecialDOpSolve[(p^2 + a^2)[y] == Cos[a x] + Sin[2 x],
  p, y[x], x, GoFor → OnlyParticular[]] /. Arg[a] → 0
```

$$\{y[x] \to -2\left(-\frac{1}{8\,(-2+a)} + \frac{1}{8\,(2+a)}\right) \operatorname{Sin}[2x] + \frac{x \operatorname{Sin}[a x]}{2a}\}$$

3.6.1.4 The Resolvent

In Section 1.3.1.4 the resolvent of a given constant square matrix **A** was introduced:

$$\mathbf{R}(s, \mathbf{A}) := (s\mathbf{I} - \mathbf{A})^{-1}.$$

The resolvent is closely related to the solution of a first order system of differential equations in standard form:

$$\mathbf{y}'(t) - \mathbf{A}\mathbf{y}(t) = \mathbf{f}(t),$$
$$y(0) = y_0.$$

The Laplace transform of the solution is (see Section 1.3.1.4)

$$\mathbf{Y}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}_0 = \mathbf{R}(s, \mathbf{A})\mathbf{F}(s) + \mathbf{R}(s, \mathbf{A})\mathbf{y}_0.$$

Thus, if the resolvent is available, the solution can be obtained as

$$\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{R}(s, \mathbf{A})\mathbf{F}(s)](t) + \mathcal{L}^{-1}[\mathbf{R}(s, \mathbf{A})](t)\,\mathbf{y}_0.$$

The resolvent of a matrix is computed by `Resolvent[s,A]`. For example

```
A = {{a, b}, {c, d}}; MatrixForm[A]
```

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

```
Resolvent[s, A]
```

$$\left\{\left\{\frac{-d+s}{-bc+ad+(-a-d)s+s^2}, \frac{b}{-bc+ad+(-a-d)s+s^2}\right\},\right.$$
$$\left.\left\{\frac{c}{-bc+ad+(-a-d)s+s^2}, \frac{-a+s}{-bc+ad+(-a-d)s+s^2}\right\}\right\}$$

Example. By taking the inverse Laplace transform of the resolvent of a square matrix **A**, the exponential matrix $e^{t\mathbf{A}}$ can be computed (see Problem 1.32). For example:

```
A = {{-1, -4}, {1, -1}}; MatrixForm[A]
```

$$\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

```
Resolvent[s, A]
```

$$\left\{\left\{\frac{1+s}{5+2s+s^2}, -\frac{4}{5+2s+s^2}\right\}, \left\{\frac{1}{5+2s+s^2}, \frac{1+s}{5+2s+s^2}\right\}\right\}$$

```
InverseLaplaceTransform[%, s, t] // MatrixForm
```

$$\begin{pmatrix} e^{-t}\,\text{Cos}[2t] & -2e^{-t}\,\text{Sin}[2t] \\ \frac{1}{2}e^{-t}\,\text{Sin}[2t] & e^{-t}\,\text{Cos}[2t] \end{pmatrix}$$

Chapter 4
z-Transformation with the Package

This chapter introduces the *Mathematica* package **LaplaceAndzTransforms V5.0** concerning z-transformations. It can be read in parallel to Chapter 2. Readers already familiar with the theory of z-transformation may start with this chapter. It is assumed that the reader has a sufficient level of expertise with *Mathematica*. At any case we recommend to first read Section 3.1 of the foregoing chapter.

4.1 Basics

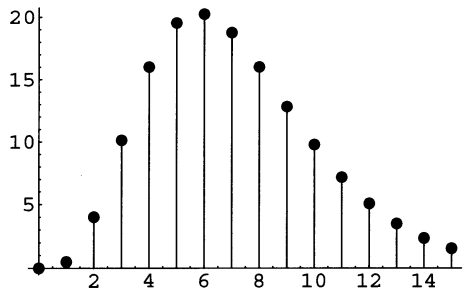
■ 4.1.1 Loading the Package

For the installation and loading of the Package see Section 3.1.1 and Section 3.1.2.

■ 4.1.2 Basic Syntax and a Survey

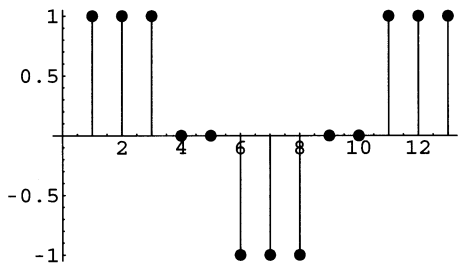
The z-transformation is applied to a sequence of a nonnegative integer variable, also called a discrete signal. By use of `DiscreteSignal[f[k],{k,k0,k1}]`, followed by the `Show` command, a graphic representation of the discrete signal $f(k)$ is obtained where the integer variable k runs from k_0 to k_1.

```
DiscreteSignal[k^4 2^-k, {k, 0, 15}] // Show;
```



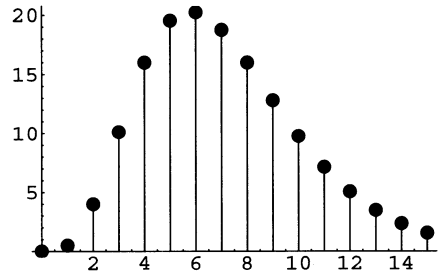
If the signal is specified by a list of values rather than a formula, use

```
DiscreteSignal[ {1, 1, 1, 0, 0, -1, -1, -1, 0, 0, 1, 1, 1}] // Show;
```



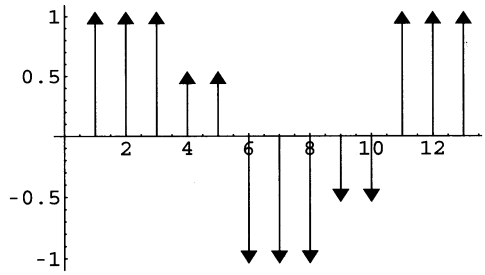
The size of the filled circles on the top can be changed by use of the option {`TopSize -> value`}.

```
DiscreteSignal[k^4 2^-k, {k, 0, 15}, TopSize → 0.05] // Show;
```



If arrows are preferred, use the option {`Arrow -> True`}.

```
DiscreteSignal[ {1, 1, 1, 0.5, 0.5, -1, -1, -1, -0.5, -0.5, 1, 1, 1},
   Arrow → True] // Show;
```



In order to compute the z-transform of a given sequence $f(k)$ use the command `ZTransform[f[k],k,z]` , for example:

```
ZTransform[k^4 2^-k, k, z]
```

$$\frac{2\,z\,(1+22\,z+44\,z^2+8\,z^3)}{(-1+2\,z)^5}$$

To compute the inverse z-transform of a given image function use the command `InverseZTransform[F[z],z,k]`

```
InverseZTransform[%, z, k]
```

$$2\left(\frac{1}{4}\,\text{GenFac}\left[2,\,1,\,\frac{1}{2}\right][k] + \frac{7}{4}\,\text{GenFac}\left[3,\,1,\,\frac{1}{2}\right][k] + \frac{9}{4}\,\text{GenFac}\left[4,\,1,\,\frac{1}{2}\right][k] + \frac{3}{4}\,\text{GenFac}\left[5,\,1,\,\frac{1}{2}\right][k]\right)$$

The computed original sequence is often expressed in terms of the generalized factorial function `GenFac[n,m,a][k]` which is defined as

$$\text{GenFac}[n, m, a][k] := \begin{cases} a^{k+m-n}\binom{k+m-1}{n-1}, & m = 1, 2, 3, \ldots, n \\ u(k-1)\,a^{k-n}\binom{k-1}{n-1}, & m = 0 \end{cases}$$

($u(k)$ is the discrete unit-step function). The NamedRule `ExpandGenFac` expands a `GenFac` term.

```
% /. ExpandGenFac
```

$$2\,(2^{-1-k}\,k + 7\,2^{-1-k}\,(-1+k)\,k + 3\,2^{-k}\,(-2+k)\,(-1+k)\,k + 2^{-1-k}\,(-3+k)\,(-2+k)\,(-1+k)\,k)$$

Often a result can be simplified.

```
% // Expand
```

$$2^{-k}\,k^4$$

In certain cases an `Unevaluated` has to be wrapped around the sequence to prevent evaluation.

```
ZTransform[Sum[i^3, {i, 0, k}] // Unevaluated, k, z]
```

$$\frac{z^2\,(1 + 4\,z + z^2)}{(-1+z)^5}$$

```
InverseZTransform[%, z, k] /. ExpandGenFac // Factor
```

$$\frac{1}{4}\,k^2\,(1+k)^2$$

which is the formula for the sum of the first k cubes.

```
ZTransform[Product[(k - i), {i, 0, 7}] // Unevaluated, k, z]
```

$$\frac{40320\,z}{(-1+z)^9}$$

```
ZTransform[Unevaluated[Binomial[k + 5, 5]], k, z]
```

$$\frac{z^6}{(-1+z)^6}$$

The z-transformation can be used to compute closed-form solutions of linear difference equations. The method is analogous to the use of Laplace transformation for solving differential equations. Let us consider, for example, the following initial value problem.

```
eqn = y[k + 3] - 3 y[k + 2] == -3 y[k + 1] + y[k] + 24 (k + 2);
iv = {y[0] → 1, y[1] → 6, y[2] → 39};
```

```
DiscreteSystemSolve[eqn, iv, y[k], k]
```

$$\{y[k] \to 1 + 5\,\text{GenFac}[2, 1, 1][k] + 28\,\text{GenFac}[3, 1, 1][k] + 48\,\text{GenFac}[4, 1, 1][k] + 24\,\text{GenFac}[5, 1, 1][k]\}$$

```
y[k] /. % /. ExpandGenFac // Expand
```

$$1 + k + k^2 + 2\,k^3 + k^4$$

Example. By using Problem 2.8 of Chapter 2 the power $\mathbf{A}^k$ of a constant square matrix $\mathbf{A}$ can be computed as the inverse z-transform of the resolvent of $\mathbf{A}$, multiplied by z. For example:

```
A = {{1, -4}, {1, 1}}; MatrixForm[A]
```

$$\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$$

```
z Resolvent[z, A]
```

$$\left\{\left\{\frac{(-1+z)\,z}{5-2\,z+z^2}, -\frac{4\,z}{5-2\,z+z^2}\right\}, \left\{\frac{z}{5-2\,z+z^2}, \frac{(-1+z)\,z}{5-2\,z+z^2}\right\}\right\}$$

```
InverseZTransform[%, z, k]
```

$$\left\{\left\{\text{Cos}[k\,\text{ArcTan}[2]]\,\text{GenFac}\left[1, 1, \sqrt{5}\right][k], -2\,\text{Sin}[k\,\text{ArcTan}[2]]\,\text{GenFac}\left[1, 1, \sqrt{5}\right][k]\right\}, \left\{\frac{1}{2}\,\text{Sin}[k\,\text{ArcTan}[2]]\,\text{GenFac}\left[1, 1, \sqrt{5}\right][k], \text{Cos}[k\,\text{ArcTan}[2]]\,\text{GenFac}\left[1, 1, \sqrt{5}\right][k]\right\}\right\}$$

```
% /. ExpandGenFac // MatrixForm
```

$$\begin{pmatrix} 5^{k/2}\,\text{Cos}[k\,\text{ArcTan}[2]] & -2\,5^{k/2}\,\text{Sin}[k\,\text{ArcTan}[2]] \\ \frac{1}{2}\,5^{k/2}\,\text{Sin}[k\,\text{ArcTan}[2]] & 5^{k/2}\,\text{Cos}[k\,\text{ArcTan}[2]] \end{pmatrix}$$

4.2 Use of Transformation Rules

■ 4.2.1 Linearity

The linearity properties of the z-transformation and inverse z-transformation are automatically applied.

```
ZTransform[c1 k^2 - c2 /k!, k, z]
```

$$-c2\,e^{\frac{1}{z}} + \frac{c1\,z\,(1+z)}{(-1+z)^3}$$

```
InverseZTransform[%, z, k] /. ExpandGenFac // Expand
```

$$c1\,k^2 - \frac{c2}{k!}$$

■ 4.2.2 Right and Left Shifts

The two original shift rules explained in Section 2.1.2.2 are applied by use of `RightShift[m,k]` and `LeftShift[m,k]`, where m is a specified positive integer. In the first case the z-transform of the sequence $u(k-m)\,f(k-m)$ is computed. When applying the `LeftShift` the z-transform of the sequence $u(k)\,f(k+m)$ is evaluated.

```
ZTransform[a^k // RightShift[3, k], k, z]
```

$$\frac{1}{z^2\,(-a+z)}$$

This the z-transform of the sequence $u(k-3)\,a^{k-3}$. The following example

```
ZTransform[1 / k! // LeftShift[4, k], k, z]
```

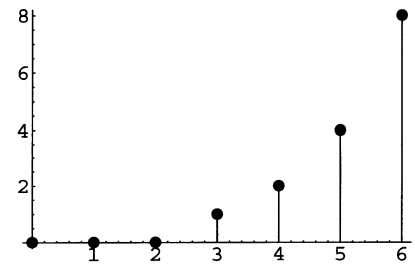
$$-\frac{z}{6}-\frac{z^2}{2}-z^3-z^4+e^{\frac{1}{z}}\,z^4$$

computes the z-transform of the sequence

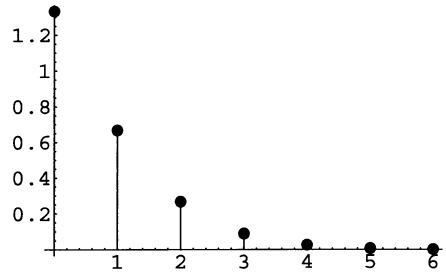
$$f(k) := \begin{cases} 1/(k+4)!, & k = 0, 1, 2, \ \ldots \\ 0, & k = -1, -2, -3, -4, \ \ldots \end{cases}$$

In order to plot a shifted signal use `RightShift[m,k]//N//Evaluate` and `LeftShift[m,k]//N//Evaluate` inside `DiscreteSignal`.

```
DiscreteSignal[2^k // RightShift[3, k] // N // Evaluate, {k, 0, 6}] // Show;
```




```
DiscreteSignal[
   2^k/k! // LeftShift[3, k] // N // Evaluate, {k, 0, 6}] // Show;
```



■ 4.2.3 The Image Scale Change or Damping Rule

Changing the independent variable z into $a\,z$ in the image function corresponds to multiplying the original sequence by a^k. In order to apply this rule, use `ScaleChange[a,z]` wrapped over the image function.

```
InverseZTransform[z (z + 1) / (z^2 + 2 z + 5), z, k]
```

```
Cos[k (π - ArcTan[2])] GenFac[1, 1, √5][k]
```

```
InverseZTransform[z (z + 1) / (z^2 + 2 z + 5) // ScaleChange[a, z], z, k]
```

```
a^-k Cos[k (π - ArcTan[2])] GenFac[1, 1, √5][k]
```

■ 4.2.4 The Image Shift Property

Section 2.1.2.5 delt with the image shift property. `ImageShift[c,z]` computes the inverse z-transform of $F(z-c)$. For example:

```
F = z / (z - a); f = InverseZTransform[F, z, k]
```

```
a^k
```

```
g = InverseZTransform[F // ImageShift[c, z], z, k]
```

```
KroneckerDelta[k] +
 Summation[a^i1 c^(-i1+k) Binomial[-1 + k, -1 + i1], {i1, k}] UnitStep[-1 + k]
```

Try to evaluate the finite sum by replacing `Summation` by `Sum`.

```
g = g /. Summation → Sum
```

$$\text{KroneckerDelta}[k] + \frac{a\,(1+\frac{a}{c})^k\,c^k\,\text{UnitStep}[-1+k]}{a+c}$$

This should be the same as the inverse z-transform of

```
F1 = F /. z → z - c
```

$$\frac{-c+z}{-a-c+z}$$

```
g1 = InverseZTransform[F1, z, k]
```

```
(a + c)^k - c (a + c)^(-1+k) UnitStep[-1 + k]
```

Is g1(k) the same as $g(k)$?

```
Table[g1 // Expand, {k, 0, 5}]
```

```
{1, a, a^2 + a c, a^3 + 2 a^2 c + a c^2,
 a^4 + 3 a^3 c + 3 a^2 c^2 + a c^3, a^5 + 4 a^4 c + 6 a^3 c^2 + 4 a^2 c^3 + a c^4}
```

```
Table[g // Simplify // Expand, {k, 0, 5}]
```

$$\{1,\ a,\ a^2 + a\,c,\ a^3 + 2\,a^2\,c + a\,c^2,$$
$$a^4 + 3\,a^3\,c + 3\,a^2\,c^2 + a\,c^3,\ a^5 + 4\,a^4\,c + 6\,a^3\,c^2 + 4\,a^2\,c^3 + a\,c^4\}$$

Indeed it is. What is the inverse z-transform of $\exp[1/(z+a)]$?

```
InverseZTransform[Exp[1 / z] // ImageShift[-a, z], z, k]
```

$$\text{KroneckerDelta}[k] + \text{Summation}\left[\frac{(-a)^{-i2+k}\ \text{Binomial}[-1+k,\ -1+i2]}{i2!},\ \{i2,\ k\}\right]\ \text{UnitStep}[-1+k]$$

```
% /. Summation → Sum
```

$$\text{KroneckerDelta}[k] - \frac{(-a)^k\ \text{Hypergeometric1F1}[1-k,\ 2,\ \frac{1}{a}]\ \text{UnitStep}[-1+k]}{a}$$

The first six values of the corresponding original sequence are:

```
Table[%, {k, 0, 5}]
```

$$\left\{1,\ 1,\ -\left(1-\frac{1}{2\,a}\right)a,\ \left(1+\frac{1}{6\,a^2}-\frac{1}{a}\right)a^2,\right.$$
$$\left.-\left(1-\frac{1}{24\,a^3}+\frac{1}{2\,a^2}-\frac{3}{2\,a}\right)a^3,\ \left(1+\frac{1}{120\,a^4}-\frac{1}{6\,a^3}+\frac{1}{a^2}-\frac{2}{a}\right)a^4\right\}$$

This result can be checked by using the command introduced in next Section.

■ 4.2.5 Initial Behavior of an Inverse z-Transform

`InverseZTransformInitialData[F[z],z,n]` computes the first $n+1$ terms of the original sequence.

```
InverseZTransformInitialData[Exp[1 / (z + a)], z, 5]
```

$$\left\{1,\ 1,\ \frac{1}{2}(1-2\,a),\ \frac{1}{3}\left(\frac{1}{2}(1-2\,a)-2\,a+3\,a^2\right),\right.$$
$$\frac{1}{4}\left(-(1-2\,a)\,a+3\,a^2-4\,a^3+\frac{1}{3}\left(\frac{1}{2}(1-2\,a)-2\,a+3\,a^2\right)\right),$$
$$\frac{1}{5}\left(\frac{3}{2}(1-2\,a)\,a^2-4\,a^3+5\,a^4-\frac{2}{3}\,a\left(\frac{1}{2}(1-2\,a)-2\,a+3\,a^2\right)+\right.$$
$$\left.\left.\frac{1}{4}\left(-(1-2\,a)\,a+3\,a^2-4\,a^3+\frac{1}{3}\left(\frac{1}{2}(1-2\,a)-2\,a+3\,a^2\right)\right)\right)\right\}$$

```
Simplify[%% - %]
```

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

■ 4.2.6 Division by k

The k-Division Rule of Section 2.1.2.6 is implemented via the `DivisionBy[k]` command. When applied to a given sequence $f(k)$, it computes the z-transform of $f(k)/k$. Note that $f(0) = 0$ must be satisfied. Furthermore, an integral of the image function is involved, which in certain cases *Mathematica* may fail to evaluate.

```
ZTransform[UnitStep[k - 1] a^k // DivisionBy[k], k, z]
```

```
Log[z] - Log[-a + z]
```

This is the z-transform of $u(k-1)\,a^k/k$.

```
ZTransform[UnitStep[k - 1] (k - 1) a^k // DivisionBy[k], k, z] // Together
```

$$\frac{-a - a\,\text{Log}[\frac{z}{a}] + z\,\text{Log}[\frac{z}{a}] + a\,\text{Log}[-1 + \frac{z}{a}] - z\,\text{Log}[-1 + \frac{z}{a}]}{a - z}$$

This is the z-transform of $u(k-1)\,(k-1)\,a^k/k$.

■ 4.2.7 Convolution of Sequences and the Summation Rule

The convolution of two sequences, explained in Section 2.1.2.7, is represented by `DiscreteConvolution[f[n],g[n],n]` in the form of a pure function. The NamedRule `ComputeDiscreteConvolution` computes the convolution product of the two sequences by means of z-transformation, i.e. by using the convolution theorem.

```
DiscreteConvolution[n^2, 2^n, n][k] /. ComputeDiscreteConvolution /.
 ExpandGenFac
```

```
-6 + 3 2^(1+k) - 5 k - (-1 + k) k
```

If the NamedRule `EvaluateDiscreteConvolution` is used, then *Mathematica* tries to evaluate the defining sum of the convolution product by built-in methods.

```
DiscreteConvolution[i^2, 2^i, i][k] /. EvaluateDiscreteConvolution
```

```
-6 + 3 2^(1+k) - 4 k - k^2
```

4.3 Difference Equations

■ 4.3.1 Linear Difference Equations

There are two ways to specify difference equations, similar to the syntax of differential equations, either by classical notation or by operational notation. Again there is a difference in how to take the initial conditions into account.

4.3.1.1 Classical Notation (DiscreteSystemSolve)

Look at the forward difference equation

```
eq1 = y[k + 2] - 6 y[k + 1] + 8 y[k] == 0;

DiscreteSystemSolve[eq1, y[k], k]
```

```
{y[k] → 2^(-1+k) (4 y[0] - y[1]) + 2^(-1+2 k) (-2 y[0] + y[1])}
```

Because no initial conditions were given, the initial values appear in the solution explicitly. Initial conditions can be specified in a replacement list, for example:

```
ic = {y[0] → 3, y[1] → 2};

DiscreteSystemSolve[eq1, ic, y[k], k]
```

$$\{y[k] \to 5\,2^k - 2^{1+2k}\}$$

If the independent discrete variable k in the second to the last argument is omitted, the solution is produced in the form of a pure function.

```
DiscreteSystemSolve[eq1, ic, y, k]
```

$$\{y \to (5\,2^{\#1} - 2^{1+2\,\#1}\ \&)\}$$

This is a convenient form to check the solution.

```
eq1 /. % // Simplify

True
```

The {GoFor->Image[Y,z]} option returns the z-transform of the solution. In transform analysis, it is usually the inverse transformation that is difficult, therefore it may be useful to look at the image function first, before taking the inverse transform.

```
DiscreteSystemSolve[eq1, ic, y[k], k, GoFor → Image[Y, z]]
```

$$\left\{Y[z] \to \frac{-16\,z + 3\,z^2}{(-4+z)\,(-2+z)}\right\}$$

Systems of difference equations are treated similarly. For example:

```
sys = {a[k + 1] == 3 a[k] + 4 c[k], b[k + 1] == 3 b[k] + 4 d[k],
   c[k + 1] == a[k] + 3 c[k], d[k + 1] == b[k] + 3 d[k]};
ics = {a[0] → 1, b[0] → 0, c[0] → 0, d[0] → 1};

DiscreteSystemSolve[sys, ics, {a[k], b[k], c[k], d[k]}, k]
```

$$\left\{a[k] \to \frac{1}{2} + \frac{5^k}{2},\ b[k] \to 4\left(-\frac{1}{4} + \frac{5^k}{4}\right),\ c[k] \to -\frac{1}{4} + \frac{5^k}{4},\ d[k] \to \frac{1}{2} + \frac{5^k}{2}\right\}$$

Use the pure function form of the solution in order to check the solution.

```
DiscreteSystemSolve[sys, ics, {a, b, c, d}, k]
```

$$\left\{a \to \left(\frac{1}{2} + \frac{5^{\#1}}{2}\ \&\right),\ b \to \left(4\left(-\frac{1}{4} + \frac{5^{\#1}}{4}\right)\ \&\right),\ c \to \left(-\frac{1}{4} + \frac{5^{\#1}}{4}\ \&\right),\ d \to \left(\frac{1}{2} + \frac{5^{\#1}}{2}\ \&\right)\right\}$$

```
sys /. % // Simplify

{True, True, True, True}
```

If, for example, you are interested in the image functions of $c(k)$ and $d(k)$, use the {GoFor -> Image[{A[3],A[4]},z]} option (you must use an indexed variable for the image functions).

```
DiscreteSystemSolve[sys, ics,
 {a[k], b[k], c[k], d[k]}, k, GoFor → Image[{A[3], A[4]}, z]]
```

$$\left\{A[3][z] \to \frac{z}{(-5+z)\,(-1+z)},\ A[4][z] \to \frac{-3\,z+z^2}{(-5+z)\,(-1+z)}\right\}$$

4.3.1.2 Operational Notation (DiscreteDOpSolve)

Often it is more convenient to use an operational notation for a difference equation. By use of the shift operator $T^m\ y(k) := y(k+m)$ the difference equation

$$y(k+2) - 6\,y(k+1) + 8\,y(k) = 3^{k+1} + 3^k$$

can be written in the operational form

$$(T^2 - 6\,T + 8)\,y(k) = (T+1)\,3^k.$$

```
DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[3^k], T, y[k], k]
```

$$\{y[k] \to 4\ (2^{-1+k} + 2^{-1+2\,k} - 3^k)\}$$

Here we obtained the solution corresponding to the zero initial conditions $y[0] = 0,\ y[1] = 0$. Note that this is not the same as the forced response in Section 2.2.1. The latter is obtained by

```
F = ZTransform[3^k, k, z]
```

$$\frac{z}{-3+z}$$

```
InverseZTransform[(z + 1) / (z^2 - 6 z + 8) * F, z, k]
```

$$3\,2^{-1+k} + 5\,2^{-1+2\,k} - 4\,3^k$$

Because this system is not stable, it does not make sense to compute the forced response. However, if a system is stable, it does make sense, since the difference between the forced response and the reponse corresponding to zero initial conditions of the output is a transient when a difference operator acts on the input sequence. If there is no difference operator acting on the input sequence, the two ways give the same result.

If the initial conditions are not assumed to vanish, specify them in a replacement list.

```
ic = {y[0] → a, y[1] → b};

DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[3^k], ic, T, y[k], k]
```

$$\left\{y[k] \to \frac{1}{2}\ (2^{2+k} + 2^{2+2\,k} - 8\,3^k + 2^{2+k}\,a - 2^{1+2\,k}\,a - 2^k\,b + 2^{2\,k}\,b)\right\}$$

To take a look at the image function of the solution, use

```
DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[3^k],
 ic, T, y[k], k, GoFor → Image[Y, z]]
```

$$\left\{Y[z] \to \frac{4\,z + 18\,a\,z - 3\,b\,z - 9\,a\,z^2 + b\,z^2 + a\,z^3}{(-4+z)\,(-3+z)\,(-2+z)}\right\}$$

One may switch between the classical notation and the operational notation via `ToDiscreteDOpForm[eqn,T,y,k]` and `ToShiftedForm[eqn,T,y,k]`.

```
ToDiscreteDOpForm[c2 y[k + 2] + c1 y[k + 1] + c0 y[k] == f[k], T, y, k]
```

$$(c0 + c1\,T + c2\,T^2)\,[y] == a^k\,[k]$$

```
ToShiftedForm[%, T, y, k]
```

$$c0\,y[k] + c1\,y[1 + k] + c2\,y[2 + k] == a^k\,[k]$$

Let us now consider a coupled system of difference equations in operational notation. We take an anomalous system. A system is anomalous if the determinant of the coefficient matrix of the highest difference operator on the left-hand side vanishes.

```
DopL = T {{1, 1}, {1, 1}} + {{-1, -2}, {3, 4}}; MatrixForm[DopL]
```

$$\begin{pmatrix} -1 + T & -2 + T \\ 3 + T & 4 + T \end{pmatrix}$$

```
DopR = T^2 {{0, 0}, {1, 0}} + T {{1, 0}, {0, 0}} + {{0, 0}, {0, 1}};
MatrixForm[DopR]
```

$$\begin{pmatrix} T & 0 \\ T^2 & 1 \end{pmatrix}$$

```
DiscreteDOpSolve[DopL[{y1, y2}] == DopR[{k^2, 2^k}],
  T, {y1[k], y2[k]}, k] /. ExpandGenFac
```

$$\left\{y1[k] \to \frac{1}{2}\,(-4\,(-1)^k + 7\,k + 3\,(-1 + k)\,k),\right.$$
$$\left. y2[k] \to \frac{1}{2}\left(1 + \frac{8\,(-1)^k}{3} + \frac{2^k}{3} - 4\,k - 2\,(-1 + k)\,k\right)\right\}$$

```
{y1[k], y2[k]} /. % /. k → 0
```

$$\{-2, 2\}$$

We assumed the initial conditions y1(0) = y2(0) = 0, but you see that they are not taken on by the computed solution. This is not a breakdown of the method of z-transformation, it is rather a typical characteristic of anomalous systems. If we change the left-hand difference operator in order to get a non-vanishing determinant, i.e. a normal system, the initial conditions will be satisfied by the computed solution.

```
DopL1 = T {{1, 1}, {0, 1}} + {{-1, -2}, {3, 4}};

DiscreteDOpSolve[DopL1[{y1, y2}] == DopR[{k^2, 2^k}],
   T, {y1[k], y2[k]}, k] /. ExpandGenFac // Simplify
```

$$\left\{y1[k] \to \frac{1}{6}\left(2\,(-5 + 6\,k + 6\,k^2) + 5\,2^{\frac{2+k}{2}}\,\mathrm{Cos}\left[\frac{k\,\pi}{2}\right] - 19\,2^{\frac{1+k}{2}}\,\mathrm{Sin}\left[\frac{k\,\pi}{2}\right]\right),\right.$$
$$\left. y2[k] \to \frac{1}{6}\left(12 + 2^k - 6\,k^2 - 13\,2^{k/2}\,\mathrm{Cos}\left[\frac{k\,\pi}{2}\right] + 11\,2^{\frac{1+k}{2}}\,\mathrm{Sin}\left[\frac{k\,\pi}{2}\right]\right)\right\}$$

```
{y1[k], y2[k]} /. % /. k → 0
```

$$\{0, 0\}$$

Chapter 5
Applications To Automatic Control

This book cannot be a textbook on automatic control theory. However the methods of Laplace and z-transformation has widespread applications in the field of automatic control theory. Hence a special chapter is dedicated to these applications gathering the subjects for which the Package can be helpful. For references on automatic control theory, see for example, [Kuo (1995)], [Sarachik (1996)], [Soliman and Mandyam (1998)], [Distefano, Stuberrud and Williams (1967)], [Bühler H. (1982)], [Ramirez W.F. (1994)], [Hsu J.C. and Meyer A.U. (1968)].

5.1 Controller Configurations

Assume that a process, or simply a linear time invariant input-output system is represented by the block diagram shown in Fig. 5.1.

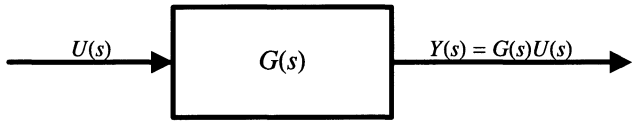


Fig.5.1: Linear time invariant input-output system.

The process or the plant is represented by its transfer function $G(s)$. $U(s)$ is the Laplace transform of the input variable called the control variable. $Y(s)$ is the Laplace transform of the output variable which is the variable to be controlled. Input and output variables are possibly vector-valued. The design objective for an automatic control system is to achieve a desirable behavior of the variable $y(t)$ by adding a feedback configuration to the system. Often the controlled process is unstable or has a poor stability and one of the aims of the feedback configuration is to stabilize or to improve the stability of the system. The design methods in control systems often rely on a fixed-configuration design. This means that at the outset the designer decides the basic configuration of the overall system and the place where the controller is to be positioned relative to the controlled process. We assume that the controllers are characterized by their transfer functions denoted by $H(s)$ or $C(s)$. Fig.5.3 and Fig.5.4. show two simple controller configurations. The original system, without the added feedback configuration, is also called an open-loop system. The resulting feedback or closed-loop system can be regarded as being equivalent to another open-loop system with a new transfer function, called the closed-loop *transfer function.* For example, by referring to Fig.5.3, we can write

$$Y(s) = G(s)\,U(s); \quad U(s) = R(s) \mp Y(s)\,H(s).$$

By eliminating $U(s)$ we arrive at

$$Y(s) = \frac{G(s)}{1 \pm H(s)\,G(s)}\,R(s),$$

where the upper sign stands for a negative and the lower sign for a positive feedback. The closed-loop system regarded as a direct or open-loop system now is (Fig.5.2)

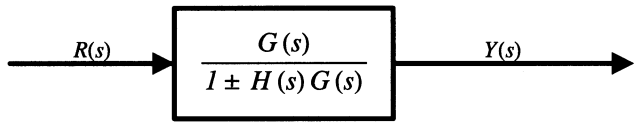


Fig.5.2: Closed loop system as an open-loop system

and $G(s)/(1 \pm H(s)\,G(s))$ is the closed-loop transfer function for the chosen controller configuration. The former transfer function $G(s)$, which describes the plant or process being controlled, is then called the *forward-path transfer function.*

We mostly consider control systems with rational transfer functions. If the closed-loop transfer function has all poles located in the left half-plane, the closed-loop system is stable (in the sense of asymptotic stability). It is one of the principal objectives of automatic control design to render the closed-loop system stable, or to improve its stability. This must be achieved even in the case where the forward-path transfer function $G(s)$ has poles on the imaginary axis, or in the complex right half-plane. For a given $G(s)$ and a chosen controller configuration, the coefficients of the controller transfer function $H(s)$ must be determined such that the poles of the closed-loop transfer function are in the left half-plane. The *characteristic function of the* closed-loop system is the denominator polynomial of the closed-loop transfer function. The zeros of this polynomial are the poles of the closed-loop transfer function.

The following controller configurations are standardized [Kuo (1995)]:

1. Standard Controller Configuration (`StandardCC`) shown in Fig.5.3:

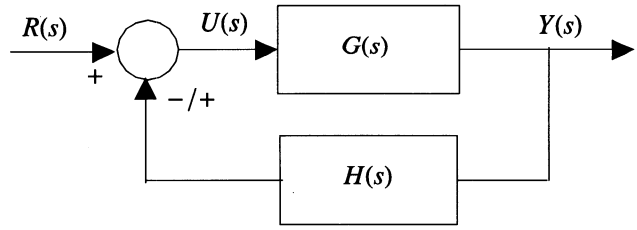


Fig.5.3: Standard controller configuration

2. Cascade Controller Configuration (`CascadeCC`) shown in Fig.5.4:

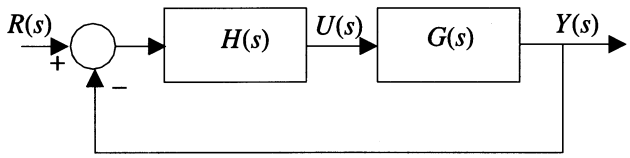


Fig.5.4: Cascade controller configuration

3. Feedback Controller Configuration (`FeedbackCC`) shown in Fig.5.5:

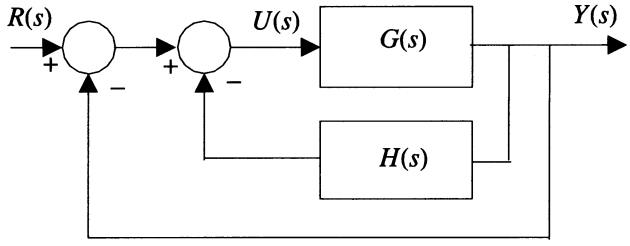


Fig.5.5: Feedback controller configuration

4. Series-Feedback Controller Configuration (`SeriesFeedbackCC`) with two degrees of freedom shown in Fig.5.6:

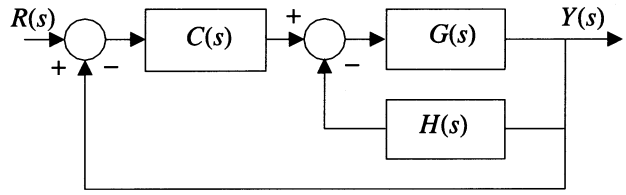


Fig.5.6: Series-feedback configuration

5. Series-Series Controller Configuration (`SeriesSeriesCC`) with two degrees of freedom shown in Fig.5.7:

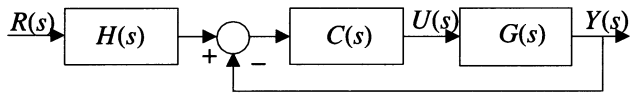


Fig.5.7: Series-series controller configuration

6. Feed forward Controller Configuration (FeedForwardCC) with two degrees of freedom shown in Fig.5.8:

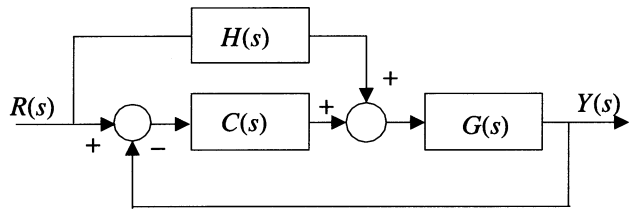


Fig.5.8: Feed forward controller configuration

7. Perturbation Cascade Controller Configuration (PerturbationCascadeCC) is a cascade controller configuration which takes into account a perturbation $W(s)$. It is shown in Fig.5.8.1.

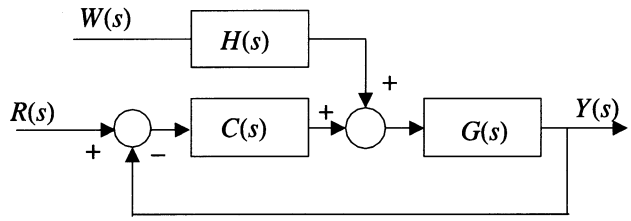


Fig.5.8.1: Perturbation cascade controller configuration

■ 5.1.1 Use of the Package

If the Package LaplaceAndzTransforms has not been loaded yet, do so with

```
<< LaplaceAndzTransforms`LZSession`;
InitLZPackage[]

InitOK!
```

5.1.1.1 Closed-loop Transfer and Characteristic Functions

Assume that the process or plant is described by the transfer function $G(s)$. Then choose for one of the above controller configurations, for example for the cascade configuration. Then try a PD-controller, for example, whose transfer function is given by

```
H = k1 + k2 s;
```

Suppose, for example, that the process has the transfer function

```
G = 1 / (s (s + 2));
```

You get the closed-loop transfer function of the chosen controller configuration by using `ClosedLoopTransferFunction[`*controllerconfig*`]`.

```
M = ClosedLoopTransferFunction[CascadeCC[-1, {H}, {G}, s]]
```

$$\frac{k1 + k2\ s}{k1 + s\ (2 + k2 + s)}$$

The characteristic function of the closed-loop system is obtained by

`ClosedLoopCharacteristicFunction[`*controllerconfig*`]`.

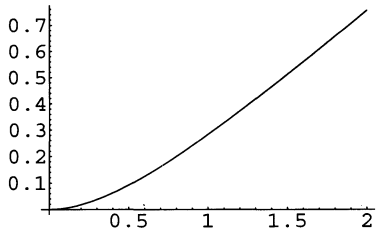
```
ClosedLoopCharacteristicFunction[CascadeCC[-1, {H}, {G}, s]]

k1 + s (2 + k2 + s)
```

In this case, any positive values for k1 and k2 imply a stabilized system, while the original open-loop system is not stable. Compare for example the unit-step response for the original open-loop and for the stabilized closed-loop system:

```
InverseLaplaceTransform[G / s, s, t]
```

$$-\frac{1}{4} + \frac{e^{-2\,t}}{4} + \frac{t}{2}$$

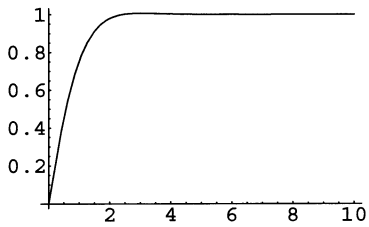
```
Plot[%, {t, 0, 2}];
```



```
InverseLaplaceTransform[(M /. {k1 → 3, k2 → 1}) / s, s, t]
```

$$1 + 2\,e^{-3t/2}\left(-\frac{1}{2}\,\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] - \frac{\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{2\sqrt{3}}\right)$$

```
Plot[%, {t, 0, 10}, PlotRange → All];
```



The cascade controller configuration with a perturbation (PerturbationCascadeCC) needs some comments. If `ClosedLoopTransferFunction` is called, a list of two transfer functions is returned. The first entry is the closed-loop transfer function between the reference and the output, the second entry between the perturbation and the output.

```
Clear[G];

ClosedLoopTransferFunction[
 PerturbationCascadeCC[-1, {C[s]}, {H[s]}, {G[s]}, s]]
```

$$\left\{\frac{C[s]\,G[s]}{1 + C[s]\,G[s]},\ \frac{G[s]\,(k1 + k2\,s)\,[s]}{1 + C[s]\,G[s]}\right\}$$

The same occurs for `ClosedLoopCharacteristicFunction`. For example,

```
ClosedLoopCharacteristicFunction[
 PerturbationCascadeCC[-1, {1 + 1 / s}, {1 / (s + b)}, {1 / (s (s + a))}, s]]
```

$$\{1 + s + a\,s^2 + s^3,\ (b + s)\,(1 + s + a\,s^2 + s^3)\}$$

5.2 State-Variable Analysis

■ 5.2.1 State-Variable and Transfer Functions

A control problem is often represented in the form of dynamic equations. The dynamic equations are specified by six matrices **A**, **B**, **C**, **D**, **E**, and **H**:

$$\begin{aligned}\frac{d\boldsymbol{x}}{dt} &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{w}(t),\\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) + \mathbf{H}\,\mathbf{w}(t),\end{aligned} \tag{5.1}$$

where the n-vector $\boldsymbol{x}(t)$ is called the system state vector, the p-vector $\mathbf{u}(t)$ is the input vector, the q-vector $\mathbf{y}(t)$ is the output vector and the v-vector $\mathbf{w}(t)$ is the disturbance vector . Thus **A** is an $n \times n$–matrix, **B** an $n \times p$-matrix, **C** a $q \times n$-matrix, **D** a $q \times p$-matrix, **E** a $n \times v$-matrix and **H** a $q \times v$-matrix. We assume a time invariant system which implies that all six matrices are constant. By assuming

$$\mathbf{x}(t) \circ\!-\!\bullet\, \mathbf{X}(s);\quad \mathbf{u}(t) \circ\!-\!\bullet\, \mathbf{U}(s);\quad \mathbf{w}(t) \circ\!-\!\bullet\, \mathbf{W}(s);$$

and $\mathbf{x}(0) = \mathbf{0}$, i.e. a zero initial state, we can write

$$s\,\mathbf{X}(s) = \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) + \mathbf{E}\,\mathbf{W}(s);\quad \mathbf{Y}(s) = \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s) + \mathbf{H}\,\mathbf{W}(s),$$

and, by eliminating the state vector,

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}[\mathbf{B}\,\mathbf{U}(s) + \mathbf{E}\,\mathbf{W}(s)] + \mathbf{D}\,\mathbf{U}(s) + \mathbf{H}\,\mathbf{W}(s)\\ &= \mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,[\mathbf{B}\,\mathbf{U}(s) + \mathbf{E}\,\mathbf{W}(s)] + \mathbf{D}\,\mathbf{U}(s) + \mathbf{H}\,\mathbf{W}(s)\\ &= [\mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,\mathbf{B} + \mathbf{D}]\,\mathbf{U}(s) + [\mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,\mathbf{E} + \mathbf{H}]\,\mathbf{W}(s).\end{aligned}$$

The $n \times n$-matrix $\mathbf{R}(s, \mathbf{A}) = (s\,\mathbf{I} - \mathbf{A})^{-1}$ is the resolvent introduced in Section 1.3.1.4. The $q \times p$-matrix $\mathbf{G_u}(s) := \mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,\mathbf{B} + \mathbf{D}$ is the *transfer matrix* between the input $\mathbf{U}(s)$ and the output $\mathbf{Y}(s)$ and $\mathbf{G_w}(s) := \mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,\mathbf{E} + \mathbf{H}$ is the transfer matrix between the disturbance $\mathbf{W}(s)$ and the output $\mathbf{Y}(s)$. We also have $\mathbf{Y}(s) = \mathbf{G_u}(s)\,\mathbf{U}(s) + \mathbf{G_w}(s)\,\mathbf{W}(s)$. We can also say that $\mathbf{G_u}(s)$ is the transfer matrix under the absence of disturbance and $\mathbf{G_w}(s)$ is the transfer matrix under the absence of input. If we interpret the input vector $\mathbf{u}(t)$ as the control vector, and if we introduce a full state feedback in the form of

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\,\mathbf{x}(t),\quad \mathbf{U}(s) = \mathbf{R}(s) - \mathbf{K}\,\mathbf{X}(s)$$

where $\mathbf{K}$ is a $p \times n$-matrix, and where $\mathbf{r}(t)$ is the reference input, we can write

$$\mathbf{Y}(s) = \mathbf{G}_r(s)\,\mathbf{R}(s) + \mathbf{G}_w(s)\,\mathbf{W}(s).$$

The transfer matrix between the reference input and the output is given by

$$\mathbf{G}_r(s) = \mathbf{G}(s)\,\mathbf{B} + \mathbf{D},$$

and the transfer matrix between the perturbance and the output is

$$\mathbf{G}_w(s) = \mathbf{G}(s)\,\mathbf{E} + \mathbf{H},\ \text{with}\ \mathbf{G}(s) = (\mathbf{C} - \mathbf{D}\,\mathbf{K})\,\mathbf{R}(s, \mathbf{A} + \mathbf{B}\,\mathbf{K})\,\mathbf{E} + \mathbf{H}.$$

■ 5.2.2 Use of the Package

5.2.2.1 Dynamic System Response

Let us consider, for example, the following state control problem:

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\0 & -4 & 3\\-1 & -1 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} + \begin{pmatrix}0 & 0\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix} + \begin{pmatrix}0\\2\\0\end{pmatrix} w(t),$$

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

Comparison with (5.1) gives the matrices

```
Am = {{0, 1, 0}, {0, -4, 3}, {-1, -1, -2}};
Bm = {{0, 0}, {1, 0}, {0, 1}};
Em = {{0}, {2}, {0}};
Cm = {{1, 0, 0}, {0, 0, 1}};
Dm = NullMatrix[2];
Hm = NullMatrix[2, 1];
```

Suppose we want to find the response of the system to the input vector

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}$$

and to a disturbance given by a Dirac impulse $w(t) = \delta(t)$. Use the command `DynamicSystemResponse[sys,E,H,][u[t],w[t],t,s]`, where *sys* stands for the state control system `StateControlSystem[A,B,C,D]`. For the above example we have

```
sys = StateControlSystem[Am, Bm, Cm, Dm];

DynamicSystemResponse[sys, Em, Hm][{1, t}, {DiracDelta[t]}, t, s]
```

$$\left\{\frac{3 + 2\,s + 5\,s^2 + 2\,s^3}{s^2\,(3 + 11\,s + 6\,s^2 + s^3)},\ -\frac{-3 + 2\,s + 2\,s^2}{s\,(3 + 11\,s + 6\,s^2 + s^3)}\right\}$$

The Laplace transform of the response vector is obtained by

```
InverseLaplaceTransform[%, s, t] // Chop

{-3. + 3.53285 e^-0.3283 t + 1. t +
  2 e^-2.83585 t (-0.266424 Cos[1.04687 t] + 0.309855 Sin[1.04687 t]),
 1. - 1.41952 e^-0.3283 t +
  2 e^-2.83585 t (0.209761 Cos[1.04687 t] - 0.609593 Sin[1.04687 t])}
```

This is the response corresponding to a zero initial state. If a non-zero initial state is given, for example by the vector {1, 2, 3}, use

```
DynamicSystemResponse[sys, Em, Hm, {1, 2, 3}][
 {1, t}, {DiracDelta[t]}, t, s]
```

$$\left\{\frac{3 + 2\,s + 29\,s^2 + 10\,s^3 + s^4}{s^2\,(3 + 11\,s + 6\,s^2 + s^3)},\ \frac{3 - 8\,s + 7\,s^2 + 3\,s^3}{s\,(3 + 11\,s + 6\,s^2 + s^3)}\right\}$$

5.2.2.2 Transfer Matrices

The two transfer matrices

$$\mathbf{G_u}(s) := \mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,\mathbf{B} + \mathbf{D};\quad \mathbf{G_w}(s) := \mathbf{C}\,\mathbf{R}(s, \mathbf{A})\,\mathbf{E} + \mathbf{H}$$

between the input vector $\mathbf{u}(t)$ and the output vector $\mathbf{y}(t)$, and between the disturbance vector $\mathbf{w}(t)$ and the output vector $\mathbf{y}(t)$, are obtained by `TransferMatrices[sys,s]`, where *sys* stands for the `StateControlSystem[A,B,C,D]`. For the above example we obtain

```
TransferMatrices[sys, Em, Hm, s]
```

$$\left\{\left\{\left\{\frac{2 + s}{3 + 11\,s + 6\,s^2 + s^3},\ \frac{3}{3 + 11\,s + 6\,s^2 + s^3}\right\},\right.\right.$$
$$\left.\left\{-\frac{1 + s}{3 + 11\,s + 6\,s^2 + s^3},\ \frac{s\,(4 + s)}{3 + 11\,s + 6\,s^2 + s^3}\right\}\right\},$$
$$\left\{\left\{\frac{2\,(2 + s)}{3 + 11\,s + 6\,s^2 + s^3}\right\},\ \left\{-\frac{2\,(1 + s)}{3 + 11\,s + 6\,s^2 + s^3}\right\}\right\}\Big\}$$

A list of the two transfer matrices is returned. If you want to find the response to a sinusoidal disturbance, for example, $w(t) = \cos t$, and no input, use

```
InverseLaplaceTransform[%[[2]] s / (s^2 + 1), s, t] // Chop

{{-0.134193 e^-0.3283 t + 2 (0.0366972 Cos[t] + 0.211009 Sin[t]) +
    2 e^-2.83585 t (0.0303991 Cos[1.04687 t] - 0.140256 Sin[1.04687 t])},
 {0.0539195 e^-0.3283 t + 2 (-0.0642202 Cos[t] - 0.119266 Sin[t]) +
    2 e^-2.83585 t (0.0372604 Cos[1.04687 t] + 0.223315 Sin[1.04687 t])}}
```

5.2.2.3 State Feedback

Let us consider a state control system with a state feedback as shown in Fig.5.9.

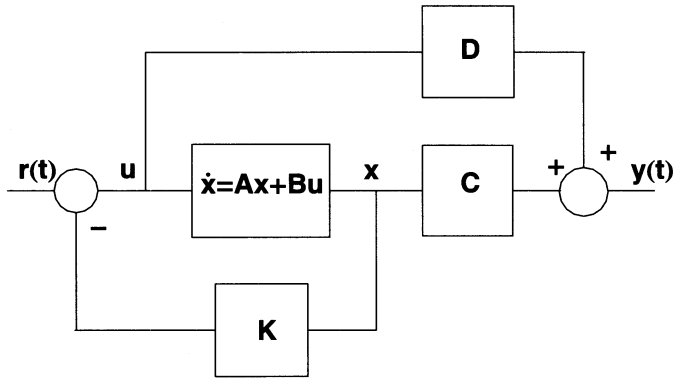


Fig.5.9: Closed-loop system with a state feedback

The dynamic equations are

$$\frac{dx}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t);\quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t),$$

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\,\mathbf{x}(t),$$

where the feedback matrix has the dimension $p \times n$. We consider the example

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix};\quad u(t) = r(t) - (k_1, k_2)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We have the matrices:

```
Am = {{0, 1}, {-2, -3}};
Bm = {{0}, {1}};
Cm = IdentityMatrix[2];
Dm = NullMatrix[2, 1];
Km = {{k1, k2}};

sys = StateControlSystem[Am, Bm, Cm, Dm];
```

The closed-loop transfer matrix between the reference vector $\mathbf{r}(t)$ and the output vector $\mathbf{y}(t)$ is obtained via `ClosedLoopTransferMatrix[`*Controlfig*`]` where *Controlfig* stands for the state control configuration `StateFeedbackCC[`*sys*`,K,s]`. *sys* stands for the `State-ControlSystem[A,B,C,D]` and `K` is the feedback matrix.

```
cf = StateFeedbackCC[sys, Km, s];

M = ClosedLoopTransferMatrix[cf]
```

$$\left\{\left\{\frac{1}{2 + k1 + 3\,s + k2\,s + s^2}\right\}, \left\{\frac{s}{2 + k1 + 3\,s + k2\,s + s^2}\right\}\right\}$$

The closed-loop characteristic function is obtained by

`ClosedLoopCharacteristicFunction[StateFeedbackCC[`*sys*`,K,s]`,

where *sys* stands for `StateControlSystem[A,B,C,D]` .

```
ClosedLoopCharacteristicFunction[cf]

2 + k1 + (3 + k2) s + s^2
```

If $k_1 > -2$ and $k_2 > -3$, we see that the closed-loop system is stable. The response of $y_1(t)$ with respect to a unit-step input is, if we choose for example, k1 = −1, k2 = −2,

```
InverseLaplaceTransform[(M[[1, 1]] /. {k1 → -1, k2 → -2}) / s, s, t]
```

$$1 + 2\,e^{-t/2}\left(-\frac{1}{2}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] - \frac{\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{2\sqrt{3}}\right)$$

Let us consider a state control system with a state feedback and a disturbance, i.e.

$$\frac{d\boldsymbol{x}}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{w}(t);\quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) + \mathbf{H}\,\mathbf{w}(t),$$

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\,\mathbf{x}(t),$$

where the feedback matrix is of dimension $p \times n$. We consider the example:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(t),$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix};\quad u(t) = (k_1, k_2)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We have the matrices:

```
Am = {{0, 1}, {-2, -3}};
Bm = {{0}, {1}};
Em = {{1 / 10}, {0}};
Cm = IdentityMatrix[2];
Dm = {{2}, {-3}};
Hm = {{1 / 5}, {1 / 4}};
Km = {{k1, k2}};

sys = StateControlSystem[Am, Bm, Cm, Dm];
```

The closed-loop transfer matrix between the reference vector $\mathbf{r}(t)$ and the output vector $\mathbf{y}(t)$ is obtained via `ClosedLoopTransferMatrices[`*controlfig*`,E,H][[1]]` where *controlfig* stands for the state control configuration `StateFeedbackCC[`*sys*`,K,s]`, where *sys* is for the `StateControlSystem[A,B,C,D]` and `K` is the feedback matrix.

```
cf = StateFeedbackCC[sys, Km, s];
```

```
Gr = ClosedLoopTransferMatrices[cf, Em, Hm][[1]]
```

$$\left\{\left\{\frac{5 + 6\,s + 2\,s^2}{2 + k1 + 3\,s + k2\,s + s^2}\right\},\ \left\{\frac{-6 - 8\,s - 3\,s^2}{2 + k1 + 3\,s + k2\,s + s^2}\right\}\right\}$$

The closed-loop transfer matrix between the disturbance vector $\mathbf{w}(t)$ and the output vector $\mathbf{y}(t)$ is obtained via the `ClosedLoopTransferMatrices[cf,E,H]][[2]]` command.

```
Gw = ClosedLoopTransferMatrices[cf, Em, Hm][[2]]
```

$$\left\{\left\{\frac{7 - 4\,k1 + 5\,k2 + 7\,s - 2\,k1\,s + 2\,k2\,s + 2\,s^2}{10\,(2 + k1 + 3\,s + k2\,s + s^2)}\right\},\right.$$
$$\left.\left\{\frac{6 + 21\,k1 - 12\,k2 + 15\,s + 6\,k1\,s + 5\,k2\,s + 5\,s^2}{20\,(2 + k1 + 3\,s + k2\,s + s^2)}\right\}\right\}$$

Thus if $k_1 > -2$ and $k_2 > -3$, the system is stable. The response of the closed-loop system to a reference input $r(t)$ and a disturbance $w(t)$ then is

$$\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{G}_r(s)\,\mathbf{R}(s) + \mathbf{G}_w(s)\,\mathbf{W}(s)]\,(t).$$

For example, for

$$r(t) = u(t),\ w(t) = \sin t$$

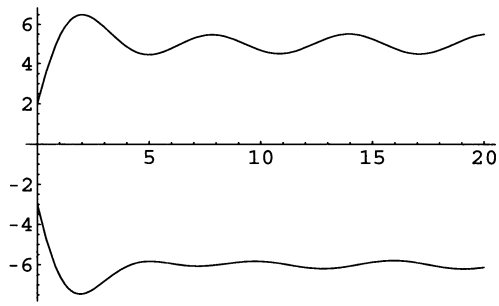
```
Y = Together[Gr.{1 / s} + Gw.{1 / (s^2 + 1)}] /. { k1 → -1, k2 → -2}
```

$$\left\{\frac{50 + 61\,s + 75\,s^2 + 62\,s^3 + 20\,s^4}{10\,s\,(1 + s^2)\,(1 + s + s^2)},\ \frac{-120 - 151\,s - 181\,s^2 - 155\,s^3 - 60\,s^4}{20\,s\,(1 + s^2)\,(1 + s + s^2)}\right\}$$

```
InverseLaplaceTransform[Y, s, t]
```

$$\left\{5 + 2\left(\frac{\mathrm{Cos}[t]}{20} + \frac{\mathrm{Sin}[t]}{4}\right) + 2\,e^{-t/2}\left(-\frac{31}{20}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \frac{43\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{20\sqrt{3}}\right),\right.$$
$$\left.-6 + 2\left(-\frac{\mathrm{Cos}[t]}{10} - \frac{\mathrm{Sin}[t]}{40}\right) + 2\,e^{-t/2}\left(\frac{8}{5}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] - \frac{31\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{10\sqrt{3}}\right)\right\}$$

```
Plot[Evaluate[%], {t, 0, 20}, PlotRange → All];
```



■ 5.2.3 State-Variable and Transition Analysis

The *state-transition matrix* of the dynamic system (5.1) is defined by

$$\mathbf{\Phi}(t) := \mathcal{L}^{-1}[\mathbf{R}(s, \mathbf{A})]\,(t), \tag{5.2}$$

i.e. by the inverse Laplace transform of the resolvent $\mathbf{R}(s, \mathbf{A}) = (s\,\mathbf{I} - \mathbf{A})^{-1}$. Assume that the dynamic system is at $t = 0$ in the state $\mathbf{x}(0)$, then

$$s\,\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) + \mathbf{E}\,\mathbf{W}(s)$$

which implies

$$\mathbf{X}(s) = \mathbf{R}(s, \mathbf{A})\,\mathbf{x}(0) + \mathbf{R}(s, \mathbf{A})[\mathbf{B}\,\mathbf{U}(s) + \mathbf{E}\,\mathbf{W}(s)].$$

By taking the inverse Laplace transform we obtain, by the convolution theorem,

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0) + \Phi(t) * [\mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{w}(t)]$$

or, explicitly,

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau. \tag{5.3}$$

The state transition matrix can also be regarded as the solution of the initial value problem

$$\frac{d\Phi}{dt} = \mathbf{A}\Phi(t)\,;\ \Phi(0) = \mathbf{I}.$$

The reason is that

$$s\,\mathcal{L}[\Phi(t)](s) - \mathbf{I} = \mathbf{A}\,\mathcal{L}[\Phi(t)](s) \Longrightarrow (s\ \mathbf{I} - \mathbf{A})\,\mathcal{L}[\Phi(t)](s) = \mathbf{I}$$
$$\Longrightarrow \mathcal{L}[\Phi(t)](s) = (s\ \mathbf{I} - \mathbf{A})^{-1} = \mathbf{R}(s, \mathbf{A}).$$

We now show that the transition matrix $\Phi(t)$ commutes with the matrix $\mathbf{A}$:

$$\Phi(t)\,\mathbf{A} = \mathbf{A}\,\Phi(t) \quad \forall\, t.$$

Proof. Let $\Omega(t) := \mathbf{A}\,\Phi(t) - \Phi(t)\,\mathbf{A}$. We have

$$\Omega'(t) := \mathbf{A}\,\Phi'(t) - \Phi'(t)\,\mathbf{A} = \mathbf{A}\,\mathbf{A}\Phi(t) - \mathbf{A}\,\Phi(t)\,\mathbf{A}$$
$$= \mathbf{A}\,(\mathbf{A}\Phi(t) - \Phi(t)\,\mathbf{A}) = \mathbf{A}\,\Omega(t),$$

and also

$$\Omega(0) = \mathbf{A}\,\mathbf{I} - \mathbf{I}\,\mathbf{A} = \mathbf{A} - \mathbf{A} = \mathbf{0}.$$

But the initial value problem $\Omega'(t) = \mathbf{A}\,\Omega(t)\,;\ \Omega(0) = \mathbf{0}$ has the unique solution $\Omega(t) \equiv \mathbf{0}$ proving that $\Phi(t)\,\mathbf{A} = \mathbf{A}\,\Phi(t)$.

Next, we prove that $\Phi(t + \tau) = \Phi(t)\ \Phi(\tau)$.

Let $\Psi(t) = \Phi(t + \tau)$ and $\Upsilon(t) = \Phi(t)\ \Phi(\tau)$. We have immediately

$$\Psi(0) = \Phi(\tau) = \mathbf{I}\,\Phi(\tau) = \Phi(0)\ \Phi(\tau) = \Upsilon(0).$$

Also,

$$\frac{d\Psi(t)}{dt} = \frac{d\Phi(t+\tau)}{dt} = \frac{d\Phi(t+\tau)}{d(t+\tau)} = \mathbf{A}\,\Phi(t+\tau) = \mathbf{A}\,\Psi(t),$$

$$\frac{d\Upsilon(t)}{dt} = \frac{d\Phi(t)}{dt}\,\Phi(\tau) = \mathbf{A}\,\Phi(t)\,\Phi(\tau) = \mathbf{A}\,\Phi(t)\,\Phi(\tau) = \mathbf{A}\,\Upsilon(t).$$

Since these two auxiliary matrix functions $\Psi(t)$ and $\Upsilon(t)$ satisfy the same initial value problem, they must be identical, thus proving the proposition.

From the two properties

1. $\Phi(0) = \mathbf{I}$,

2. $\Phi(t+\tau) = \Phi(t)\,\Phi(\tau)$,

it follows that

3. $\Phi(-t) = \Phi(t)^{-1}$

and also

4. $\Phi(t_2 - \tau)\,\Phi(\tau - t_1) = \Phi(t_2 - t_1)$.

Equation (5.3) is now solved for the initial state $\mathbf{x}(0)$:

$$\mathbf{x}(0) = \Phi(-t)\,\mathbf{x}(t) - \Phi(-t)\int_0^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau$$

or, by replacing t by t_0,

$$\mathbf{x}(0) = \Phi(-t_0)\,\mathbf{x}(t_0) - \Phi(-t_0)\int_0^{t_0} \Phi(t_0-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau.$$

By substituting the last expression into (5.3), we obtain

$$\begin{aligned}
\mathbf{x}(t) &= \Phi(t)\,\Phi(-t_0)\,\mathbf{x}(t_0) - \Phi(t)\,\Phi(-t_0)\int_0^{t_0} \Phi(t_0-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&+ \int_0^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&= \Phi(t-t_0)\,\mathbf{x}(t_0) - \int_0^{t_0} \Phi(t-t_0)\,\Phi(t_0-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&+ \int_0^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&= \Phi(t-t_0)\,\mathbf{x}(t_0) - \int_0^{t_0} \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&+ \int_0^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&= \Phi(t-t_0)\,\mathbf{x}(t_0) + \int_{t_0}^0 \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&+ \int_0^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau \\
&= \Phi(t-t_0)\,\mathbf{x}(t_0) + \int_{t_0}^t \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau\ .
\end{aligned}$$

This yields the general transition formulas

$$\mathbf{x}(t) = \Phi(t-t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau\,, \tag{5.4}$$

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{C}\,\Phi(t-t_0)\,\mathbf{x}(t_0)\\ &+ \int_{t_0}^{t} \mathbf{C}\,\Phi(t-\tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau + + \mathbf{D}\,\mathbf{u}(t) + \mathbf{H}\,\mathbf{w}(t).\end{aligned} \tag{5.5}$$

The last two formulas are of rather theoretical value. In order to compute the state at the instant t from the given state at $t = 0$, use directly

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\,\mathbf{R}(s,\,\mathbf{A})\,\mathbf{x}(0) + \mathbf{R}(s,\,\mathbf{A})\,\{\mathbf{B}\,\mathbf{U}(s) + \mathbf{E}\,\mathbf{W}(s)\}]\,(t). \tag{5.6}$$

■ 5.2.4 Discrete Control Systems

The discrete counterpart of (5.1) is the fixed discrete state-variable control system

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k) + \mathbf{E}\,\mathbf{w}(k),\\ \mathbf{y}(k) &= \mathbf{C}\,\mathbf{x}(k) + \mathbf{D}\,\mathbf{u}(k) + \mathbf{H}\,\mathbf{w}(k),\end{aligned} \tag{5.7}$$

where the discrete variable k varies over the nonnegative integers. These dynamic equations are specified by the six matrices **A**, **B**, **C**, **D**, **E**, and **H**. The n-vector $\boldsymbol{x}(k)$ is called the system state vector, the p-vector $\mathbf{u}(k)$ is the input vector, the q-vector $\mathbf{y}(k)$ is the output vector and the v-vector $\mathbf{w}(k)$ is the disturbance vector. Thus **A** is an $n\times n$-matrix, **B** an $n\times p$-matrix, **C** a $q\times n$-matrix, **D** a $q\times p$-matrix, **E** an $n\times v$-matrix and **H** a $q\times v$-matrix. We assume a time invariant system, implying that all six matrices are constant.

The dynamic equations (5.7) are often replaced by

$$\begin{aligned}\mathbf{x}[(k+1)\,T] &= \mathbf{A}\,\mathbf{x}(k\,T) + \mathbf{B}\,\mathbf{u}(k\,T) + \mathbf{E}\,\mathbf{w}(k\,T)\,,\\ \mathbf{y}(k\,T) &= \mathbf{C}\,\mathbf{x}(k\,T) + \mathbf{D}\,\mathbf{u}(k\,T) + \mathbf{H}\,\mathbf{w}(k\,T)\,,\end{aligned} \tag{5.8}$$

where $T > 0$ is a given constant. The following deduction is quite similar to that of the preceding section, except that the z-transform is used in place of the Laplace transform. Thus, assuming the z-transforms

$$\mathbf{x}(k) \circ\!-\!\diamond\ \mathbf{X}(z);\ \mathbf{u}(k) \circ\!-\!\diamond\ \mathbf{U}(z);\quad \mathbf{w}(k) \circ\!-\!\diamond\ \mathbf{W}(k);$$

and $\mathbf{x}(0) = \mathbf{0}$, i.e. a zero initial state, we have

$$z\,\mathbf{X}(z) = \mathbf{A}\,\mathbf{X}(z) + \mathbf{B}\,\mathbf{U}(z) + \mathbf{E}\,\mathbf{W}(z);\quad \mathbf{Y}(z) = \mathbf{C}\,\mathbf{X}(z) + \mathbf{D}\,\mathbf{U}(z) + \mathbf{H}\,\mathbf{W}(z)$$

and by eliminating the state vector with the z-transform,

$$\mathbf{X}(z) = (z\,\mathbf{I} - \mathbf{A})^{-1}[\mathbf{B}\,\mathbf{U}(z) + \mathbf{E}\ \mathbf{W}(z)]\ . \tag{5.9}$$

Hence the output is

$$\begin{aligned}\mathbf{Y}(z) &= \mathbf{C}\,(z\,\mathbf{I} - \mathbf{A})^{-1}[\mathbf{B}\,\mathbf{U}(z) + \mathbf{E}\ \mathbf{W}(z)] + \mathbf{D}\,\mathbf{U}(z) + \mathbf{H}\,\mathbf{W}(z)\\ &= \mathbf{C}\ \mathbf{R}(z,\,\mathbf{A})\ [\mathbf{B}\,\mathbf{U}(z) + \mathbf{E}\ \mathbf{W}(z)] + \mathbf{D}\,\mathbf{U}(z) + \mathbf{H}\,\mathbf{W}(z)\\ &= [\mathbf{C}\ \mathbf{R}(z,\,\mathbf{A})\ \ \mathbf{B} + \mathbf{D}]\,\mathbf{U}(z) + [\,\mathbf{C}\ \mathbf{R}(z,\,\mathbf{A})\,\mathbf{E} + \mathbf{H}]\,\mathbf{W}(z)\,.\end{aligned}$$

The $n \times n$-matrix $\mathbf{R}(z, \mathbf{A}) = (z\mathbf{I} - \mathbf{A})^{-1}$ is the resolvent introduced in Section 1.3.1.4. The $q \times p$-matrix

$$\mathbf{G_u}(z) := \mathbf{C}\,\mathbf{R}(z, \mathbf{A})\,\mathbf{B} + \mathbf{D}$$

is the *transfer matrix* between the input $\mathbf{U}(z)$ and the output $\mathbf{Y}(z)$ and

$$\mathbf{G_w}(z) := \mathbf{C}\,\mathbf{R}(z, \mathbf{A})\,\mathbf{E} + \mathbf{H}$$

is the transfer matrix between the disturbance $\mathbf{W}(z)$ and the output $\mathbf{Y}(z)$. Clearly,

$$\mathbf{Y}(z) = \mathbf{G_u}(s)\,\mathbf{U}(z) + \mathbf{G_w}(z)\,\mathbf{W}(z).$$

We can also say that $\mathbf{G_u}(z)$ is the transfer matrix under the absence of disturbance and $\mathbf{G_w}(z)$ is the transfer matrix under the absence of input.

If the input vector $\mathbf{u}(k)$ is regarded as the control vector, and if we introduce a full state feedback in the form of

$$\mathbf{u}(k) = \mathbf{r}(k) - \mathbf{K}\,\mathbf{x}(k), \quad \mathbf{U}(s) = \mathbf{R}(z) - \mathbf{K}\,\mathbf{X}(z),$$

where $\mathbf{K}$ is a $p \times n$-matrix, and $\mathbf{r}(k)$ is the reference input, we have

$$\mathbf{Y}(z) = \mathbf{G}_r(z)\,\mathbf{R}(z) + \mathbf{G}_w(z)\,\mathbf{W}(z).$$

The transfer matrix between the reference input and the output is

$$\mathbf{G}_r(z) = \mathbf{G}(z)\,\mathbf{B} + \mathbf{D}$$

and the transfer matrix between the perturbation and the output is

$$\mathbf{G}_w(z) = \mathbf{G}(z)\,\mathbf{E} + \mathbf{H}.$$

The matrix $\mathbf{G}(z)$ is given by

$$\mathbf{G}(z) = (\mathbf{C} - \mathbf{D}\,\mathbf{K})\,\mathbf{R}(z, \mathbf{A} + \mathbf{B}\,\mathbf{K})\,\mathbf{E} + \mathbf{H}.$$

■ 5.2.5 Use of the Package

5.2.5.1 State-Transition Matrix

The state-transition matrix of a given square matrix $\mathbf{A}$ is obtained by

```
StateTransitionMatrix[{{1, 2}, {-3, 2}}, t]
```

$$\left\{\left\{2\,e^{3t/2}\left(\frac{1}{2}\cos\left[\frac{\sqrt{23}\,t}{2}\right] - \frac{\sin\left[\frac{\sqrt{23}\,t}{2}\right]}{2\sqrt{23}}\right),\ \frac{4\,e^{3t/2}\sin\left[\frac{\sqrt{23}\,t}{2}\right]}{\sqrt{23}}\right\},\right.$$
$$\left.\left\{-\frac{6\,e^{3t/2}\sin\left[\frac{\sqrt{23}\,t}{2}\right]}{\sqrt{23}},\ 2\,e^{3t/2}\left(\frac{1}{2}\cos\left[\frac{\sqrt{23}\,t}{2}\right] + \frac{\sin\left[\frac{\sqrt{23}\,t}{2}\right]}{2\sqrt{23}}\right)\right\}\right\}$$

5.2.5.2 State at the Instant t respectively k

In order to compute the state $\mathbf{x}(t)$, apply formula (5.6) or use the command `DynamicSystemResponse[sys,E,H,x[0]][u[t],w[t],t,s]` with *sys* = `StateControlSystem[A,B,C,D]`. The matrices are: C, an $n \times n$–identity matrix; D, an $n \times p$-null matrix, H, an $n \times v$-null matrix. p is the dimension of the input vector $\mathbf{u}(t)$ and v is the dimension of the disturbance vector $\mathbf{w}(t)$.

For example:

```
Am = {{0, 1, 0}, {0, -4, 3}, {-1, -1, -2}};
Bm = {{0, 0}, {1, 0}, {0, 1}};
Em = {{0}, {2}, {0}};
Cm = IdentityMatrix[3];
Dm = NullMatrix[3, 2];
Hm = NullMatrix[3, 1];

sys = StateControlSystem[Am, Bm, Cm, Dm];
```

In order to compute the state at the instant t, corresponding to a given initial state $(1, 2, 3)^t$, an input $\mathbf{u}(t) = (1, t)$, and a disturbance $w(t) = \delta(t)$, use

```
DynamicSystemResponse[sys, Em, Hm, {1, 2, 3}][
 {1, t}, {DiracDelta[t]}, t, s]
```

$$\left\{\frac{3+2s+29s^2+10s^3+s^4}{s^2(3+11s+6s^2+s^3)}, \frac{3-s+18s^2+4s^3}{s(3+11s+6s^2+s^3)}, \frac{3-8s+7s^2+3s^3}{s(3+11s+6s^2+s^3)}\right\}$$

```
InverseLaplaceTransform[%, s, t] // Chop
```

```
{-3. + 6.44213 e^(-0.3283 t) + 1. t +
  2 e^(-2.83585 t) (-1.22106 Cos[1.04687 t] - 0.864751 Sin[1.04687 t]),
 1. - 2.11495 e^(-0.3283 t) + 2 e^(-2.83585 t)
   (2.55748 Cos[1.04687 t] + 3.7306 Sin[1.04687 t]), 1. - 2.58849 e^(-0.3283 t) +
  2 e^(-2.83585 t) (2.29424 Cos[1.04687 t] + 0.555211 Sin[1.04687 t])}
```

If a discrete control system is given by the above matrices, and we wish to find the state at the discrete instant k, we use the same procedure with the inverse z-transform replacing the inverse Laplace transform. Thus, in order to compute the discrete state $\mathbf{x}(k)$, use

`DiscreteDynamicSystemResponse[sys,E,H,x[0]][u[k],w[k],k,z]`

with *sys* = `StateControlSystem[A,B,C,D]`. The matrices are C, an $n \times n$-identity matrix ; D, an $n \times p$-null matrix, H, an $n \times v$-null matrix. p is the dimension of the input vector $\mathbf{u}(k)$ and v is that of the disturbance vector $\mathbf{w}(k)$.

```
DiscreteDynamicSystemResponse[sys, Em, Hm][{1, k}, {UnitStep[k]}, k, z]
```

$$\left\{\frac{3z(-1+z+z^2)}{(-1+z)^2(3+11z+6z^2+z^3)}, \right.$$
$$\left.\frac{3z^2(-1+z+z^2)}{(-1+z)^2(3+11z+6z^2+z^3)}, -\frac{z(-3-4z+2z^2)}{(-1+z)^2(3+11z+6z^2+z^3)}\right\}$$

The state at the discrete instant k is then obtained by

```
InverseZTransform[%, z, k] /. ExpandGenFac // Chop

{3 (0.0839002 - 0.0936862 (-0.3283)^k + 0.047619 k +
    2 3.02291^k (0.00489299 Cos[2.78796 k] - 0.064251 Sin[2.78796 k])),
 3 (0.131519 + 0.0307572 (-0.3283)^k + 0.047619 k +
    2 3.02291^k (-0.0811382 Cos[2.78796 k] + 0.177084 Sin[2.78796 k])),
 -0.294785 + 0.112931 (-0.3283)^k + 0.238095 k -
  2 3.02291^k (-0.0909267 Cos[2.78796 k] - 0.291093 Sin[2.78796 k])}
```

5.3 Second Order Differential Systems

The modeling process of a control problem often results in a system of second order differential equations. By doubling the number of variables, the second order system can be transformed into a first order system. However, for linear and time invariant systems it is not always necessary to include this step. By applying the method of Laplace transformation, higher order differential systems do not produce additional difficulties. Thus let us suppose that the dynamical system is modeled by the system of second order differential equations:

$$\mathbf{A}_2 \frac{d^2\,\mathbf{y}}{dt^2} + \mathbf{A}_1 \frac{d\mathbf{y}}{dt} + \mathbf{A}_0\,\mathbf{y}(t) = \mathbf{B}_0\,\mathbf{u}(t) + \mathbf{B}_1 \frac{d\mathbf{u}}{dt} + \mathbf{H}\,\mathbf{w}(t). \tag{5.10}$$

The vector-valued variable $\mathbf{u}(t)$ of dimension p is the input, $\mathbf{y}(t)$ of dimension n is the output and $\mathbf{w}(t)$ of dimension v is the disturbance vector-valued variable. The constant matrices are of the following dimensions: $\mathbf{A}_i$ of dimension $n \times n$, $\mathbf{B}_i$ of dimension $n \times p$ and $\mathbf{H}$ of dimension $n \times v$. The initial conditions are given by the two vectors $\mathbf{y}(0+)$ and $\mathbf{y}'(0+)$. There are two transfer matrices for this open-loop system. The transfer matrix between the input vector $\mathbf{u}(t)$ and the output vector $\mathbf{y}(t)$ is given by

$$\mathbf{G}_\mathbf{u}(s) = \mathbf{R}_0(s)\,(\mathbf{B}_0 + s\,\mathbf{B}_1),$$

and the transfer matrix between the disturbance vector $\mathbf{w}(t)$ and the output vector $\mathbf{y}(t)$ is

$$\mathbf{G}_\mathbf{w}(s) = \mathbf{R}_0(s)\,\mathbf{H}.$$

The matrix $\mathbf{R}_0(s)$ is

$$\mathbf{R}_0(s) := (s^2\,\mathbf{A}_2 + s\,\mathbf{A}_1 + \mathbf{A}_0)^{-1}.$$

Let us suppose that the given system is unstable or has poor stability. The aim is to stabilize the system by introducing a feedback of the form

$$\mathbf{u}(t) = \mathbf{K}_0\,\mathbf{y}(t) + \mathbf{K}_1 \frac{d\mathbf{y}}{dt}$$

where the matrices $\mathbf{K}_i$ are of dimensions $p \times n$. By introducing the feedback into the above differential equation, we obtain the system

$$\mathbf{C}_2 \frac{d^2\,\mathbf{y}}{dt^2} + \mathbf{C}_1 \frac{d\mathbf{y}}{dt} + \mathbf{C}_0\,\mathbf{y}(t) = \mathbf{H}\,\mathbf{w}(t)$$

with the constant $n \times n$-matrices

$$\mathbf{C}_2 := \mathbf{A}_2 - \mathbf{B}_1\,\mathbf{K}_1\,;\quad \mathbf{C}_1 := \mathbf{A}_1 - \mathbf{B}_0\,\mathbf{K}_1 - \mathbf{B}_1\,\mathbf{K}_0;\quad \mathbf{C}_0 := \mathbf{A}_0 - \mathbf{B}_0\,\mathbf{K}_0.$$

By taking the Laplace transform, we obtain

$$\mathbf{C}_2 \frac{d^2\,\mathbf{y}}{dt^2} \circ\!-\!\bullet\; \mathbf{C}_2\,[s^2\,\mathbf{Y}(s) - s\,\mathbf{y}(0+) - \mathbf{y}'(0+)]$$
$$= s^2\,\mathbf{C}_2\,\mathbf{Y}(s) - s\,\mathbf{C}_2\,\mathbf{y}(0+) - \mathbf{C}_2\,\mathbf{y}'(0+),$$
$$\mathbf{C}_1 \frac{d\mathbf{y}}{dt} \circ\!-\!\bullet\; \mathbf{C}_1\,[s\,\mathbf{Y}(s) - \mathbf{y}(0+)] = s\,\mathbf{C}_1\,\mathbf{Y}(s) - \mathbf{C}_1\,\mathbf{y}(0+),$$
$$\mathbf{C}_0\,\mathbf{y}(t) \circ\!-\!\bullet\; \mathbf{C}_0\,\mathbf{Y}(s),$$

and the image system becomes

$$(s^2\,\mathbf{C}_2 + s\,\mathbf{C}_1 + \mathbf{C}_0)\,\mathbf{Y}(s) = (s\,\mathbf{C}_2 + \mathbf{C}_1)\,\mathbf{y}(0+) + \mathbf{C}_2\,\mathbf{y}'(0+) + \mathbf{H}\,\mathbf{W}(s).$$

With the inverse matrix

$$\mathbf{R}(s) := (s^2\,\mathbf{C}_2 + s\,\mathbf{C}_1 + \mathbf{C}_0)^{-1}$$

we obtain

$$\mathbf{Y}(s) = [s\,\mathbf{R}(s)\,\mathbf{C}_2 + \mathbf{R}(s)\,\mathbf{C}_1]\,\mathbf{y}(0+) + \mathbf{R}(s)\,\mathbf{C}_2\,\mathbf{y}'(0+) + \mathbf{R}(s)\,\mathbf{H}\,\mathbf{W}(s)$$

for the response of the closed-loop system. The entries of the matrix $\mathbf{R}(s)$ are rational functions having the common denominator polynomial

$$P(s) := \det(s^2\,\mathbf{C}_2 + s\,\mathbf{C}_1 + \mathbf{C}_0).$$

Therefore,

$$P(s) := \det(s^2\,\mathbf{C}_2 + s\,\mathbf{C}_1 + \mathbf{C}_0) = 0$$

yields the characteristic equation of the closed-loop system. The entries of the two feedback matrices $\mathbf{K}_0$ and $\mathbf{K}_1$ must be chosen such that $P(s)$ becomes a Hurwitz polynomial. The closed-loop transfer matrix between the disturbance and the output is given by $\mathbf{G}(s) = \mathbf{R}(s)\mathbf{H}$.

■ 5.3.1 Use of the Package

5.3.1.1 Transfer and Characteristic Functions

Let us consider the example:

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1''\\ x_2''\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1'\\ x_2'\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$$
$$= \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_1'\\ u_2'\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} w(t).$$

```
A2 = {{1, 1}, {0, 1}};
A1 = {{1, 0}, {0, 0}};
A0 = {{0, 1}, {1, 0}};
B0 = {{0, 1}, {1, 1}};
B1 = {{0, 0}, {0, 1}};
Hm = {{0}, {1}};

sys = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}];
```

The two transfer matrices between the input vector $\mathbf{u}(t)$ and the output vector $\mathbf{y}(t)$, and between the disturbance vector $\mathbf{w}(t)$ and the output vector $\mathbf{y}(t)$, respectively, are obtained by

```
TransferMatrices[sys, Hm, s][[1]]
```

$$\left\{\left\{\frac{-1-s^2}{-1-s^2+s^3+s^4}, \frac{-1-s-s^3}{-1-s^2+s^3+s^4}\right\}, \left\{\frac{s+s^2}{-1-s^2+s^3+s^4}, \frac{-1+s+2s^2+s^3}{-1-s^2+s^3+s^4}\right\}\right\}$$

```
TransferMatrices[sys, Hm, s][[2]]
```

$$\left\{\left\{\frac{-1-s^2}{-1-s^2+s^3+s^4}\right\}, \left\{\frac{s+s^2}{-1-s^2+s^3+s^4}\right\}\right\}$$

A perturbation in form of a Dirac impulse $\delta(t)$ has the following repercussion on the output vector, provided no input is applied:

```
InverseLaplaceTransform[%, s, t] // Chop
```

```
{{0.459999 e^-1.75488 t - 0.4 e^t +
    2 e^-0.122561 t (-0.0299993 Cos[0.744862 t] + 0.134178 Sin[0.744862 t])},
 {-0.14937 e^-1.75488 t + 0.4 e^t +
    2 e^-0.122561 t (-0.125315 Cos[0.744862 t] + 0.206184 Sin[0.744862 t])}}
```

Now assume that there is a feedback.

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

```
K0 = {{k, 0}, {0, -1}}; K1 = IdentityMatrix[2];
```

Then the closed-loop transfer function between the disturbance and the output is obtained via

```
cf = FullFeedbackCC[sys, {K0, K1}, s];

ClosedLoopTransferMatrix[cf, Hm]
```

$$\left\{\left\{\frac{-2+s-s^2}{-2+2k+4s-ks-s^2+ks^2+s^3}\right\}, \left\{\frac{s+s^2}{-2+2k+4s-ks-s^2+ks^2+s^3}\right\}\right\}$$

The closed-loop characteristic function is obtained by

```
ClosedLoopCharacteristicFunction[cf]
```

$$-2+2k+(4-k)s+(-1+k)s^2+s^3$$

The Hurwitz test shows that we obtain a stable system by choosing $k = 1.5$, for example.

```
HurwitzTest[%, s]
```

```
{True, 4 - k > 0, -1 + k > 0, -2 + 2 k > 0, 4 - 10 k + 8 k^2 - 2 k^3 > 0}
```

```
HurwitzTest[%% /. k → 1.5, s]
```

```
{True}
```

5.3.1.2 Dynamic System Response

The dynamic system response of the above system with $k = 3/2$ and responding to a unit-step input and an initial zero state is

```
DynamicSystemResponse[sys, Hm,
 {K0 /. k → 3 / 2, K1}, {0, 0}, {0, 0}, {UnitStep[t]}, t, s]
```

$$\left\{-\frac{2\,(2-s+s^2)}{s\,(2+5\,s+s^2+2\,s^3)},\ \frac{2\,(1+s)}{2+5\,s+s^2+2\,s^3}\right\}$$

If the initial vectors are not zero, but for example {1, 2}and {0, 3}, then

```
DynamicSystemResponse[sys, Hm,
 {K0 /. k → 3 / 2, K1}, {1, 2}, {0, 3}, {UnitStep[t]}, t, s]
```

$$\left\{\frac{2\,(-2+5\,s+s^2+s^3)}{s\,(2+5\,s+s^2+2\,s^3)},\ \frac{4+7\,s+4\,s^2}{2+5\,s+s^2+2\,s^3}\right\}$$

5.4 Stability

■ 5.4.1 Analog Systems

Assume that a plant or a process is modeled by an input-output system described by its transfer function $G(s)$. Assume also that this system is not stable or has a poor stability. It should be stabilized or its stability should be improved in order to meet the required system specifications. If the system is unstable the transfer function will have at least one pole located on the imaginary axis or in the right-half plane (see Section 1.3.1.2). By adding an appropriate feedback, the given process can eventually be stabilized. This amounts to choosing a controller configuration. Normally, one tries first with the simplest one, for example with the standard controller configuration, where the controller is introduced in the feedback path, or with the cascade controller configuration, where the controller is in series in front of the plant. The controller is modeled by its transfer function $H(s)$ which, in general, is a simple rational function. Some or all of the coefficients of the controller transfer function will have to be tuned in order to meet the stability requirement. Or, the controller is considered to be fixed; and a gain factor k is explicitly introduced so that the transfer function of the control becomes $kH(s)$.

Once the feedback is installed, we have a closed-loop system. The input-output system originally given is an open-loop system. The closed-loop system again can be regarded as a direct input-output system described by a new transfer function called the closed-loop transfer function. The expression for the closed-loop transfer function depends on the chosen controller configuration and of the used controller. One principal goal of the design process is to tune the coefficients of the controller(s), or of the gain factor k (or both) in order to stabilize the closed-loop system, or to meet other system requirements, such as limiting the amplitude of an overshoot. This implies that the closed-loop transfer function must have all poles located in the left-half plane. For most automatic control problems the closed-loop transfer function is a rational function and its denominator polynomial $P(s)$ yields the characteristic equation $P(s) = 0$. The coefficients of the controller or the overall gain factor k have to be determined such that all zeros of $P(s)$ will be located in the left-half plane. The coefficients of the polynomial $P(s)$ will depend on the coefficients of the controller transfer function or on the

overall gain factor k. If it is not possible to tune the controller coefficients or the factor k, or both, such that all zeros of $P(s)$ are located in the left half-plane, another controller configuration must be tried out, redesigned and tested.

In the case of a state variable control problem the characteristic equation appears as the characteristic polynomial of a matrix; similarly, if the control problem is described by a second order differential system.

To determine the range for the coefficients of the controller transfer function or of the overall gain factor, such that the closed-loop system becomes stable, is one of the principal tasks for the control engineer. There are analytical methods (Routh-Hurwitz test) and semi-graphical methods (Nyquist and Root Locus procedures) to accomplish this task.

For a discrete or a sampled-data system the situation is quite analogous. There is a difference in as much as that for the stability of a discrete system, the roots of the characteristic polynomial $P(z)$ must lie inside the unit-circle of the complex z-plane rather than in the left-half plane (see Section 2.2.1.1).

■ 5.4.2 Use of the Package

5.4.2.1 StabilityQ, Hurwitz Test

The `StabilityQ[P[s],s]` command returns `True` or `False` depending on whether or not the given polynomial is stable, i.e. whether or not its zeros are in the left s-half plane. This is only possible if all the coefficients of the polynomial $P(s)$ are numbers. If one or more ot the coefficients are symbolic quantities, a list of expressions is returned, and the polynomial is stable if all expressions are positive.

```
P = Product[(s + 1 / k + k I) (s + 1 / k - k I), {k, 5}] // Expand;

StabilityQ[P, s]

True

StabilityQ[s P, s]

False

StabilityQ[s^4 + s^3 + k s^2 + 2 s + 1, s]
```

$$\left\{1, \frac{1}{-2+k}, \frac{(-2+k)^2}{-5+2k}, \frac{-5+2k}{-2+k}\right\}$$

If all four expressions in the above list are positive the polynomial is stable, if one or more are zero or negative it is unstable. If the `HurwitzTest[P[s],s]` command is used, a list of Boolean expressions appears. If all elements in the list have the value `True`, the polynomial is stable; otherwise it is unstable.

```
HurwitzTest[P, s]

{True}
```

```
HurwitzTest[s P, s]

{False, True}

HurwitzTest[s^4 + s^3 + k s^2 + 2 s + 1, s]

{True, k > 0, -5 + 2 k > 0}
```

So, if $k > 5/2$ the polynomial is stable.

5.4.2.2 Z-Stability Test

In order to test whether or not a polynomial $P(z)$ has all its zeros strictly inside the unit-circle of the complex z-plane, use `ZStabilityTest[P[s],s]`. If all elements of the returned list of Boolean expressions have the value `True` the polynomial has all its roots strictly located inside the unit-circle.

```
P = Product[z - k / (k + 1), {k, 5}] // Expand;

ZStabilityTest[P, z]

{True, True}

ZStabilityTest[(z - 1) P, z]

{True, False, True}

ZStabilityTest[-k/9 + (1/18 - 2 k/9) z +
   (1/9 - k) z^2 + (1/2 - 2 k) z^3 + z^4, z]
```

$$\left\{\frac{5}{9}+\frac{10\,k}{9}\neq 0,\ 15-30\,k>0,\ 44-28\,k>0,\ 5+10\,k>0,\right.$$
$$\left.52+12\,k>0,\ 28+36\,k>0,\ 42624+42624\,k+18944\,k^2+18944\,k^3>0\right\}$$

If $k = 0.45$, for example, the polynomial is z-stable.

```
% /. k → 0.45

{True, True, True, True, True, True, True}
```

But for $k = 0.55$ it is unstable.

```
%% /. k → 0.55

{True, False, True, True, True, True, True}
```

Example. Suppose a plant is described by the open-loop transfer function

```
G = 8 (s + 1) / (s (s + 2) (s + 4));
```

The system is unstable due to a pole at the origin. We choose a standard controller configuration with a PI-controller, for example,

```
H = k (1 + 2 / s);
```

The closed-loop transfer function then becomes

```
Gc = ClosedLoopTransferFunction[StandardCC[-1, {G}, {H}, s]]
```

$$\frac{8\,s\,(1+s)}{(2+s)\,(8\,k\,(1+s)+s^2\,(4+s))}$$

The closed-loop characteristic function is

```
Cf = ClosedLoopCharacteristicFunction[StandardCC[-1, {G}, {H}, s]]
```

$$(2+s)\,(8\,k\,(1+s)+s^2\,(4+s))$$

The Hurwitz test yields the conditions

```
HurwitzTest[%, s]
```

$$\{\text{True},\ 16\,k>0,\ 24\,k>0,\ 8+8\,k>0,\ 576\,k+576\,k^2>0\}$$

showing that the closed-loop system is stable for any positive value of the parameter k. Let us take $k = 10$, for example. The closed-loop transfer function then becomes

```
Gc1 = Gc /. k → 10 // ExpandDenominator
```

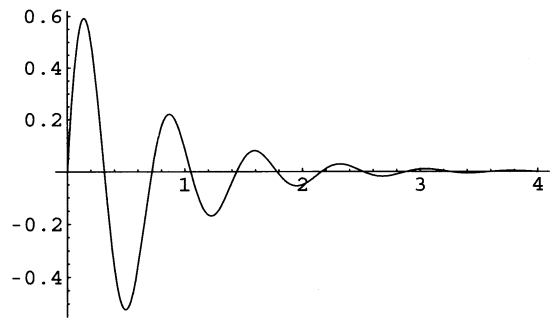
$$\frac{8\,s\,(1+s)}{160+240\,s+88\,s^2+6\,s^3+s^4}$$

The impulse-response of the closed-loop system then is

```
InverseLaplaceTransform[Gc1, s, t] // Chop
```

$$-0.222222\,e^{-2\,t}+0.00462943\,e^{-1.04002\,t}+2\,e^{-1.47999\,t}\,(0.108796\,\text{Cos}[8.64472\,t]+0.455909\,\text{Sin}[8.64472\,t])$$

```
Plot[%, {t, 0, 4}, PlotRange → All];
```

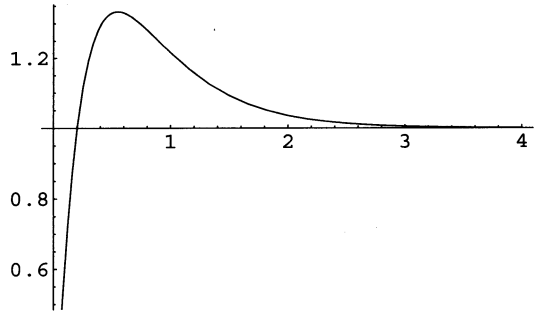


Compare this with the impulse response of the unstable open-loop system.

```
InverseLaplaceTransform[G, s, t]
```

$$1-3\,e^{-4\,t}+2\,e^{-2\,t}$$

```
Plot[%, {t, 0, 4}];
```



While a perturbation in the form of a Dirac-impulse $\delta(t)$ applied to the open-loop system leads to a constant set-off of the response, the closed-loop system returns to the zero position.

5.5 Frequency Analysis

■ 5.5.1 Frequency-Domain Plots

Let us suppose that a system is modeled by its transfer function $G(s)$ and that the system is stable, i.e. all poles of $G(s)$ have strictly negative real parts. In Section 1.3.1.3 we defined the frequency response of the system as

$$\mathcal{L}^{-1}\left[G(s)\,\frac{a}{s - i\,\omega}\right](t)$$

which is the steady-state response of the system to the input

$$x(t) = a\,e^{i\,\omega\,t} \quad (a > 0).$$

If $G(s)$ is a real rational function, as is usually the case in real world applications, the complex exponential function can be replaced by a cosine or sine function. It was shown that the frequency response can be written as

$$y(t) = a\ |\ G(i\,\omega)\ |\ \exp[i\,(\omega\,t\ +\ \arg G(i\,\omega))]$$

which means that the amplitude a of the input is multiplied by the positive factor $|\ G(i\,\omega)\ |$ called the *gain* and that the phase $\omega\,t$ of the input is shifted by the *phase-shift* $\varphi(\omega) := \arg G(i\,\omega)$. Thus the input, being a harmonic oscillation with pulsation ω, in the steady-state again produces a harmonic oscillation with the same pulsation, but with an amplitude and a phase-shift that depends on the pulsation ω. A *Frequency-Response Plot* consists of two graphics. The gain $|\ G(i\,\omega)\ |$ and the phase-shift $\varphi(\omega) := \arg G(i\,\omega)$ are plotted over the angular frequency ω for $0 \le \omega < \infty$. A *Bode Plot* also consists of two plots, but the gain measured in decibels (dB), i.e. $20\log_{10}|\ G(i\,\omega)\ |$, and the phase-shift $\varphi(\omega) := \arg G(i\,\omega)$ are plotted over $\log_{10}\omega$. A *Polar Plot* is the plot of the complex quantity $G(i\,\omega)$ in the complex plane, where the parameter ω ranges from 0 to ∞.

■ 5.5.2 Using the Package

5.5.2.1 Frequency-Response

By using`FrequencyResponse[G[s],s,a Cos[ωt+ϕ],t]` the frequency response of the system, modeled by the transfer function $G(s)$, is returned. It is assumed that the system is stable.

```
G = s / (s^2 + 2 s + 26);

FrequencyResponse[G, s, a Cos[ω t + ϕ], t]
```

$$\left\{a\,\text{Cos}[\phi + t\,\omega],\ a \to a\sqrt{\frac{\omega^2}{676 - 48\,\omega^2 + \omega^4}},\ \phi \to \phi + \text{ArcTan}[2\,\omega^2,\ 26\,\omega - \omega^3]\right\}$$

The cosine function can be replaced by a sine or by a complex exponential function.

```
FrequencyResponse[G, s, a Exp[I (ω t + ϕ)], t]
```

$$\left\{a\,e^{i\,(\phi + t\,\omega)},\ a \to a\sqrt{\frac{\omega^2}{676 - 48\,\omega^2 + \omega^4}},\ \phi \to \phi + \text{ArcTan}[2\,\omega^2,\ 26\,\omega - \omega^3]\right\}$$

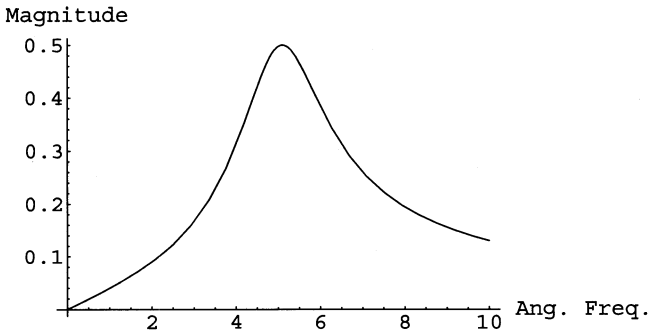
The gain and the phase-shift as functions of the pulsation ω are:

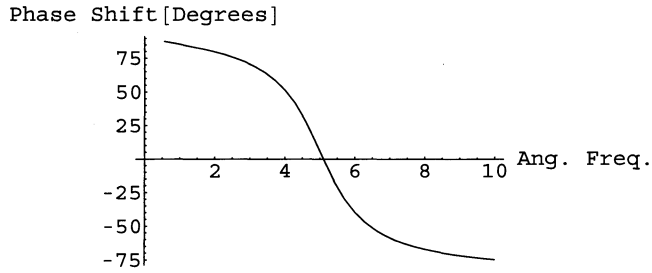
$$\sqrt{\frac{\omega^2}{676 - 48\,\omega^2 + \omega^4}}\ ; \quad \text{ArcTan}[2\,\omega^2,\ 26\,\omega - \omega^3].$$

5.5.2.2 Frequency-Response Plot

By executing `FrequencyResponsePlot[G[s],{s,ω1, ω2}]` the gain and the phase-shift are plotted over the angular frequency ω for $\omega 1 < \omega < \omega 2$.

```
FrequencyResponsePlot[G, {s, 0, 10}];
```

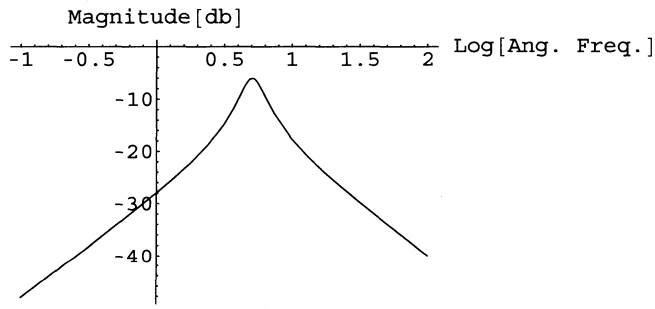


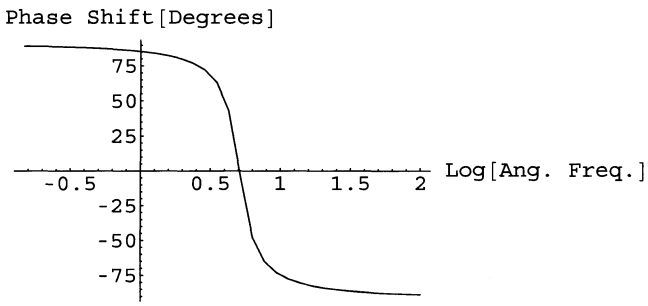


5.5.2.3 Bode Plot

`BodePlot[G[s],{s,ω1, ω2}]` yields the Bode plot. The left boundary of the plot-interval must be positive.

```
BodePlot[G, {s, 10^-1, 10^2}];
```

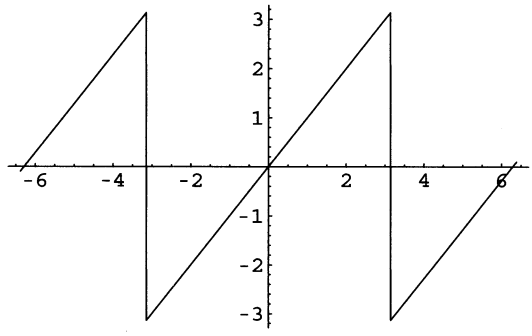




The plot of the argument of a complex-valued function depending continuously on a parameter should have no jumps. Due to the implementation of the ArcTan function most computer

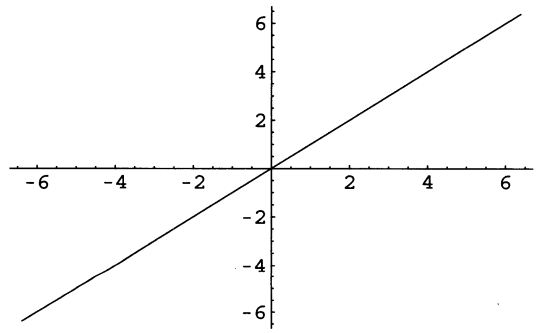
programs yield arguments between $-\pi$ and $+\pi$. This produces a jump in the graph if the parameter passes through $\pm\pi$ or an integral multiple of it.

```
Plot[Arg[Exp[I t]], {t, -2 Pi - 0.1, 2 Pi + 0.1}];
```



The correct plot is

```
Plot[t, {t, -2 Pi - 0.1, 2 Pi + 0.1}];
```



It should be noted that if there is a jump at an integer-multiple of π in the argument plot, this discontinuity probably only appears due to the explained above effect. If you want a continuous plot of the argument of a complex valued function, use the `ContinuousArgPlot[f[t],{t,t1,t2}]` command.

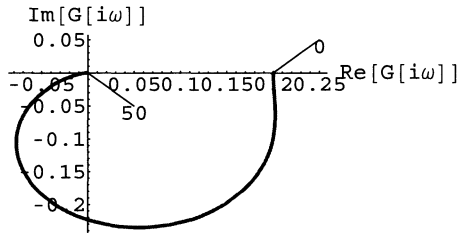
5.5.2.4 Polar Plot

`PolarPlot[G[s], {s, ω1, ω2}]` returns the polar plot over the angular frequency interval `[ω1,ω2]`. For example,

```
G = 1 / (s^2 + 2 s + 5)
```

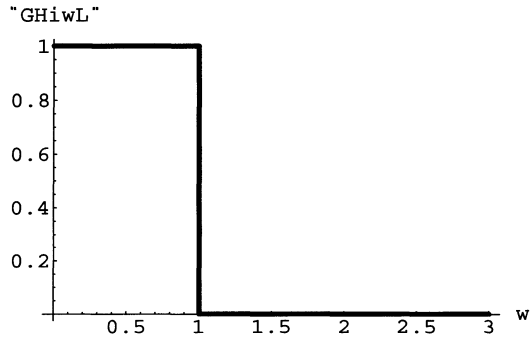
$$\frac{1}{5 + 2\,s + s^2}$$

```
PolarPlot[G, {s, 0, 50}, AxesLabel → {"Re[G[iω]]", "Im[G[iω]]"}];
```



■ 5.5.3 Filters

Laplace (and Fourier) transformations also have many applications in the field of signal processing. A control system can be regarded as a signal processor. An important device in signal processing is a filter. A filter is used to separate the components of a signal with respect to their frequencies. There are mainly three types of filters: low-pass filters, high-pass filters and band-pass filters. A filter is a linear, time invariant system and can therefore be described by its transfer function $G(s)$. A filter must also be a stable system. The frequency-response characteristic of an ideal low-pass filter with cut-off (angular) frequency $\omega_c = 1$ has the following characteristic:



Its phase-shift is a decreasing linear function of the frequency. Unfortunately, an ideal low-pass filter is physically not realizable. There is no rational transfer function having the gain-characteristic above. A simple but, of course, non-ideal filter is given by the second order transfer function

$$G(s) = \frac{a s^2 + b s + c}{s^2 + d s + 1}, \quad (d > 0).$$

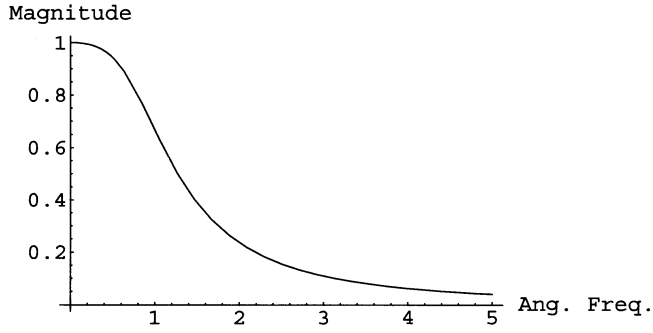
Here we have a

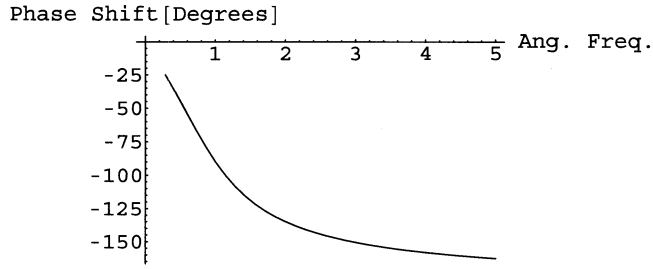
- low-pass filter, if $a = 0,\ b = 0,\ c = 1$;
- high-pass filter, if $a = 1,\ b = 0,\ c = 0$;
- band-pass filter, if $a = 0,\ b = d,\ c = 0$.

The frequency-response for the low-pass filter above looks like

```
G = 1 / (s^2 + 1.5 s + 1);
```

```
FrequencyResponsePlot[G, {s, 0, 5}];
```





5.5.3.1 Butterworth and Chebyshev Filters

Higher order filters have a better cut-off sharpness. A *Butterworth Filter* of order n has a frequency-response gain characteristic given by

$$| G(i\omega) | = \frac{1}{\sqrt{1+\omega^{2n}}}.$$

For a *Chebyshev Filter* of order n the frequency-response gain characteristic is

$$| G(i\omega) | = \frac{1}{\sqrt{1+\epsilon^2 T_n^2(\omega)}},$$

where ϵ is a small positive parameter, and $T_n(\omega)$ is the Chebyshev polynomial of degree n. These are normalized low-pass filters, i.e. the cut-off angular frequency is $\omega_c = 1$. It is possible to find the transfer functions such that the above gain-characteristics are met (see, for example [Arbenz K. and Martin J-C., 1983]). For a Butterworth Filter of order n one obtains

$$G(s) = \frac{1}{s+1} \prod_{k=1}^{(n-1)/2} \frac{1}{s^2 + d_k\, s + 1} \quad (n \text{ odd}),$$

$$G(s) = \prod_{k=1}^{n/2} \frac{1}{s^2 + d_k\, s + 1} \quad (n \text{ even}),$$

with $d_k = 2\sin[(2k-1)\pi/(2n)]$. The transfer function for a Chebyshev Filter of order n is obtained by

$$G(s) = \frac{1}{2^{n-1}\,\epsilon} \frac{1}{s + \sin\eta} \prod_{k=1}^{(n-1)/2} \frac{1}{s^2 + d_k\, s + \Omega_k^2} \quad (n \text{ odd}),$$

$$G(s) = \frac{1}{2^{n-1}\,\epsilon} \prod_{k=1}^{n/2} \frac{1}{s^2 + d_k\, s + \Omega_k^2} \quad (n \text{ even}),$$

with

$$\eta = \frac{\operatorname{argsinh}(1/\epsilon^2)}{n}, \quad d_k = 2\sin\left[\frac{\pi}{n}\left(k - \frac{1}{2}\right)\right]\sinh\eta,$$

$$\Omega_k^2 = \sin^2\left[\frac{\pi}{n}\left(k - \frac{1}{2}\right)\right]\sinh^2\eta + \cos^2\left[\frac{\pi}{n}\left(k - \frac{1}{2}\right)\right]\cosh^2\eta.$$

These transfer functions describe normalized low-pass filters. The transfer function $G_{\omega_0}(s)$ of a low-pass filter with cut-off frequency ω_0 is obtained simply by

$$G_{\omega_0}(s) = G\left(\frac{s}{\omega_0}\right).$$

The transfer function $G_{\omega_0}(s)$ of a high-pass filter with cut-off frequency ω_0 is obtained by

$$G_{\omega_0}(s) = G\left(\frac{\omega_0}{s}\right).$$

A band-pass filter with the two cut-off angular frequencies ω_1 and ω_2 are obtained by connecting in series a high-pass filter with a low-pass filter. The corresponding transfer functions are multiplied:

$$G_{[\omega_1,\omega_2]}(s) = G\left(\frac{\omega_1}{s}\right) G\left(\frac{s}{\omega_2}\right).$$

Another way to obtain a band-pass from a low-pass filter is to replace the variable s by the $(s + 1/s)/2$.

5.5.3.2 Use of the Package

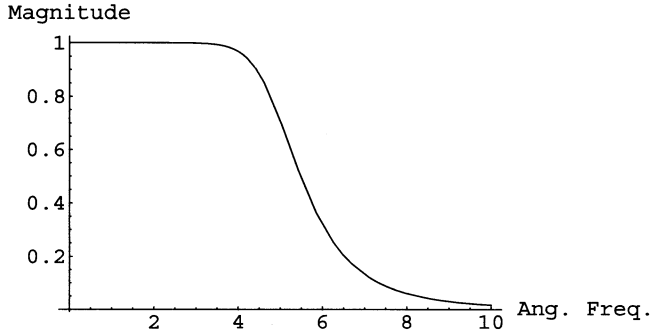
The transfer function of a normalized low-pass Butterworth Filter of order n is obtained by `ButterworthFilter[n,s]`.

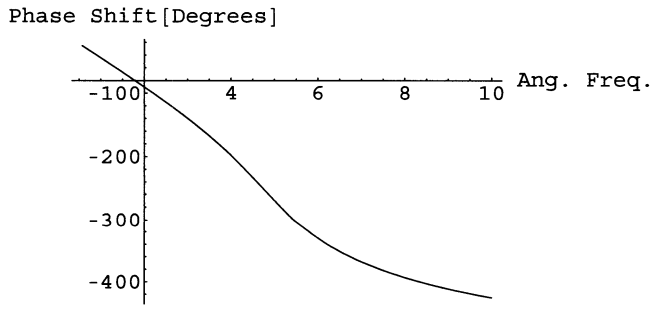
```
G1 = ButterworthFilter[6, s]
```

$$\frac{1}{\left(1 + \sqrt{2}\,s + s^2\right)\left(1 + \frac{\left(-1+\sqrt{3}\right)s}{\sqrt{2}} + s^2\right)\left(1 + \frac{\left(1+\sqrt{3}\right)s}{\sqrt{2}} + s^2\right)}$$

Then the frequency-response characteristics for a low-pass Butterworth filter with cut-off frequency $\omega_0 = 5$ is

```
FrequencyResponsePlot[(G1 /. s → s / 5), {s, 0, 10}, PlotRange → All];
```





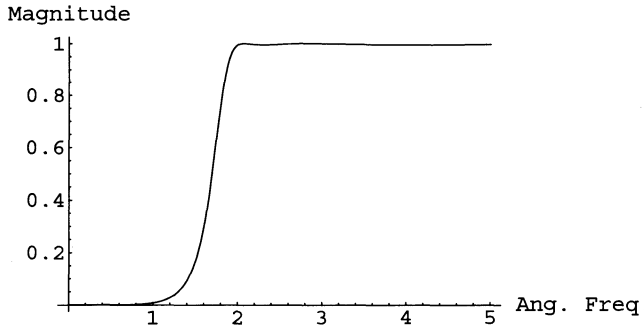
The transfer function of a normalized low-pass Chebyshev Filter of order n is obtained by `ChebyshevFilter[n,ϵ,s]`, where ϵ is a small positive parameter (called the ripple-parameter).

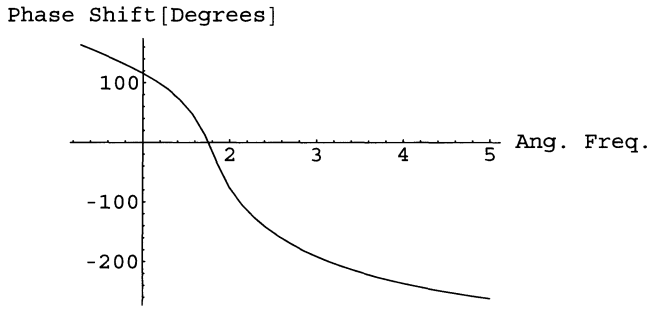
```
G2 = ChebyshevFilter[6, 0.1, s]
```

$$0.3125 / ((1.20421 + 0.269566\,s + s^2)\,(0.771192 + 0.736468\,s + s^2)\,(0.33818 + 1.00603\,s + s^2))$$

The frequency-response plot of a high-pass Chebyshev filter with cut-off frequency $\omega_0 = 2$ is, for example,

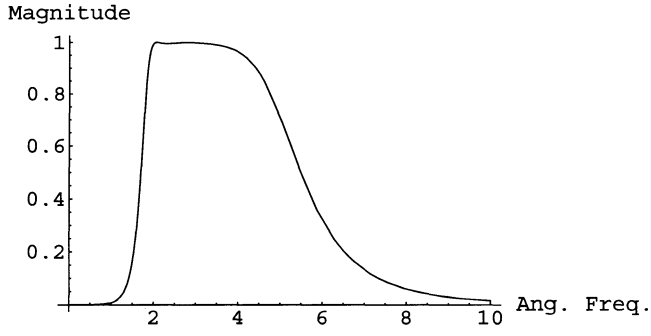
```
FrequencyResponsePlot[
  Together[G2 /. s → 2 / s], {s, 0, 5}, PlotRange → All];
```

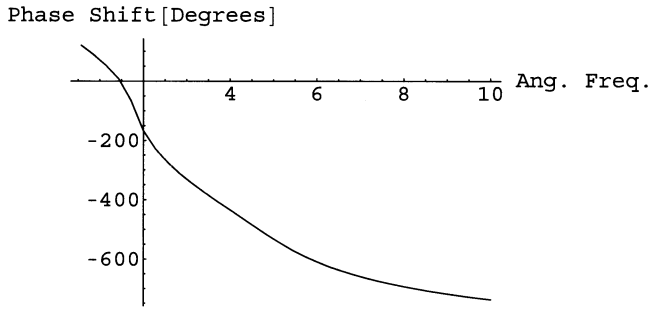




Finally, the transfer function of a band-bass filter with the window [2, 5] is obtained by

```
G3 = Together[G1 /. s → s / 5] Together[G2 /. s → 2 / s];

FrequencyResponsePlot[G3, {s, 0, 10}, PlotRange → All];
```





5.6 Sampled-Data Control Systems

A sampled-data control system consists partly of analog components and partly of digital components. The analog part is generally the plant or the process to be controlled, and the digital component is a digital computer used for the controller. The analog components generate continuous variables (analog signals), while the digital components need discrete signals for the input and generate discrete signals as an output. Discrete signals cannot act directly on an analog device as they lack physical energy. Therefore, a digital to analog converter device (D/A) is needed. Such a device takes as input a discrete signal and yields an analog signal in the form of a modulated impulse train as output. Analog signals also have to be transformed into discrete signals. This is accomplished by the sampling process and a device called the sampler. Therefore sampled-data control systems are systems with three types of signals:

- digitally coded discrete signals (sequences);
- analog signals in the form of modulated impulse trains (sampled signals);
- ordinary analog signals (ordinary time functions).

■ 5.6.1 The Sampling Process

A *discrete signal* is modeled by a sequence of the form $y(kT)$ or $y(kT + \eta)$ where the positive constant T is the sampling period. A discrete signal is a purely mathematical quantity with no physical energy. Discrete signals are put into a digital computer that produces an output in the form of a discrete signal. Discrete signals are also the result of the sampling process, i.e. the periodic sampling of an analog signal by a sampler. Such a sampler is represented by the symbol of Fig. 5.10.

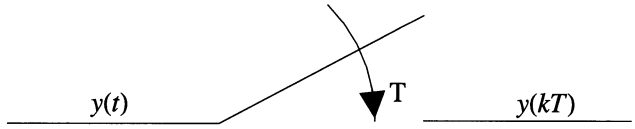
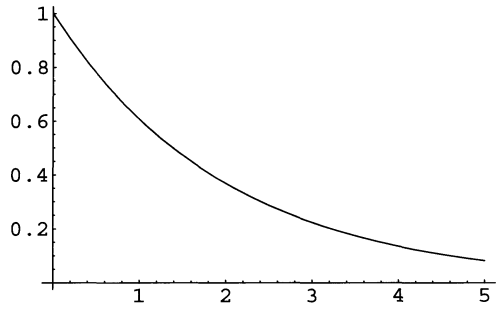


Fig.5.10: Sampler

The input of the sampler is an analog signal $y(t)$ and the output is the sequence $y(kT)$, consisting of the instantaneous values of the analog signal at instants $t = kT$, $(k = 0, 1, 2, \ldots)$.

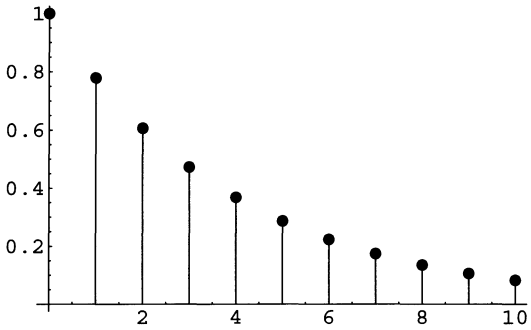
Example. An input signal of the sampler:

```
Plot[Exp[-t / 2], {t, 0, 5}, AxesOrigin → {0, 0}];
```



The output of the sampler with a sampling period of $T = 1/2$ is a discrete signal:

```
Show[DiscreteSignal[Exp[-k / 4], {k, 0, 10}]];
```



As a control signal in the form of a discrete signal has no physical energy and cannot be used to drive a physical plant, we have to use a digital-analog converter (D/A). The output of the D/A is an analog signal in the form of a modulated impulse train. Such a signal now contains physical energy and can act on a plant. It is called the *sampled signal* of $y(t)$. From a given discrete signal $y(kT)$, the sampled signal can be built up by using the unit-impulse train:

$$\delta_T(t) = u(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=0}^{\infty} \delta(t - kT) \circ\!\!-\!\!\bullet\, \Delta_T(s) = \frac{1}{1 - e^{-Ts}}.$$

If we denote by $y^*(t)$ the sampled signal of $y(t)$, we obtain for the modulated impulse train

$$y^*(t) = y(t) \sum_{k=0}^{\infty} \delta(t - kT) = \sum_{k=0}^{\infty} y(kT)\, \delta(t - kT).$$

Remark: If we measure the "energy" of a signal by its area between the graph and the t-axis, and if we want the sampled signal to have approximately the same "energy" as the analog signal given, we multiply the above modulated impulse train by the sampling period T, i.e. we rather set

$$y^*(t) = T\,y(t)\sum_{k=0}^{\infty}\delta(t-kT) = T\sum_{k=0}^{\infty} y(kT)\,\delta(t-kT)$$

(if a hold-device follows the sampler, the effect of the introduced factor T is taken into account by the device). The D/A converter, which symbolizes the transformation of the discrete signal $y(kT)$ into the sampled signal $y^*(t)$, is represented by the block diagram in Fig.5.11.

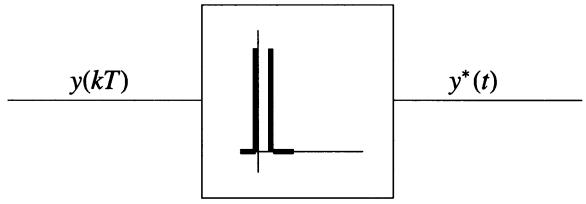


Fig.5.11: D/A converter representing the transformation of a discrete signal into a sampled signal.

The modulation of an analog signal with a unit-impulse train is thus represented by Fig.5.12.

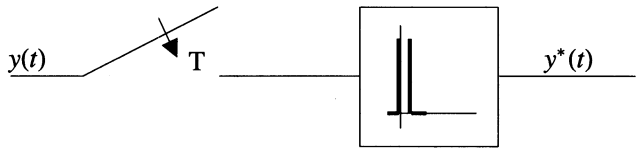


Fig.5.12: Modulation of an analog signal with the unit-impulse train.

The Laplace transform of the sampled signal is

$$Y^*(s) = \sum_{k=0}^{\infty} y(kT)\,e^{-kTs} = \left(\sum_{k=0}^{\infty} y(kT)\,z^{-k}\right)_{z=e^{Ts}} = Y^{\dagger}(e^{Ts})$$

where $Y^{\dagger}(z)$ is the z-transform of the discrete signal $y(kT)$. Therefore, we obtain the following

Proposition. *The Laplace transform of the sampled signal $y^*(t)$ is the z-transform of the discrete signal $y(kT)$, where z is replaced by e^{Ts}.*

If the above sampled signal is directly used to drive the plant or to control a process, we say that we are working with the *impulse-model*. Often the sampler and the D/A converter are followed by a *hold device* which is described by the transfer function $Q(s)$. A *zero-hold* device has the transfer function

$$Q(s) = \frac{1-e^{-Ts}}{s}$$

which is the Laplace transform of $q(t) = u(t) - u(t-T)$, $u(t)$ being the unit-step function. If the sampled signal is the input of the zero-hold, its output has the Laplace transform

$$H(s) = Y^*(s)\;Q(s) = Y^{\dagger}(e^{Ts})\,\frac{1-e^{-Ts}}{s}$$

where $Y^{\dagger}(z)$ is the z-transform of the discrete signal $y(kT)$. Because

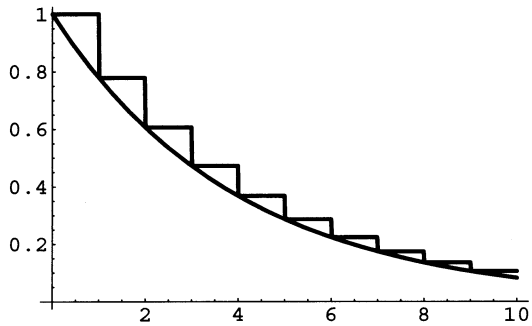
$$H(s) = \frac{1 - e^{-Ts}}{s} \sum_{k=0}^{\infty} y(kT)\, e^{-kTs} = \sum_{k=0}^{\infty} y(kT)\, \frac{e^{-kTs} - e^{-(k+1)Ts}}{s}$$

the inverse Laplace transform is

$$h(t) = \sum_{k=0}^{\infty} y(kT)\, [u(t - kT) - u(t - (k+1)T)]$$

which is the result of the combined sampled-and-zero-hold process in the time domain. This is a step-function with the step equal to the sampling period T, while the height of each step is $y(kT)$. In other words, we get a staircase approximation of the given analog signal $y(t)$. The figure below shows the staircase approximation of the signal of the example above.

```
f[t_] := Sum[If[k <= t < k + 1, Exp[-k / 4], 0], {k, 0, 10}]

Plot[{f[t], Exp[-t / 4]}, {t, 0, 10}, PlotStyle → Thickness[0.0075]];
```



The modulation followed by a zero-order hold is represented in Fig.5.13

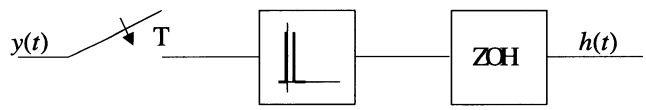


Fig.5.13: Transformation of an analog signal into its staircase approximation.

In order to reduce the number of blocks, we introduce yet another symbol for the combination of a sampler followed by a D/A device and call it a pulse-sampler. It is shown in Fig.5.14.

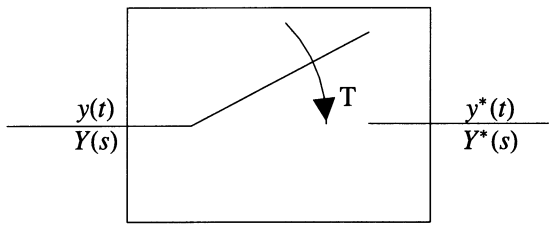


Fig.5.14: Pulse-sampler which is equivalent to the configuration of Fig.5.12.

The sampling process absorbs some of the information. The smaller the sampling period T, the better the approximation of the analog signal by the sampled signal. Often a natural eigenfrequency ω_0 of self-oscillation of the system can be defined. In order to get a sufficiently good approximation by the sampled signal, one should at least have six sample

points per half-period of the system's self-oscillation, i.e. about 12 sample points per period of self-oscillation. This implies the rule

$$12\,T \le \frac{2\pi}{\omega_0} \Longrightarrow \omega_0\,T \lesssim \frac{1}{2}.$$

5.6.1.1 The Pulse Transform

The mapping that maps the Laplace image $Y(s)$ to the Laplace image $Y^*(s)$, where $y^*(t)$ is the sampled signal of the analog signal $y(t)$ is called the *pulse-transformation.* The pulse-transform $Y^*(s)$ is the discrete Laplace transform of the sequence $y(k\,T)$, i.e.,

$$Y^*(s) := \sum_{k=0}^{\infty} y(k\,T)\,e^{-k\,T\,s}.$$

We have to be careful about the meaning of $y(0)$, because $y(0)$ may not be defined. Indeed, in our context $y(0)$ always means $y(0+)$. Ealier, we introduced the generalized discrete Laplace transform

$$Y^*(s,\tau) := \sum_{k=0}^{\infty} y(k\,T+\tau)\,e^{-k\,T\,s} \quad (0 \le \tau < T).$$

In Section 2.1.1.1 we proved the two formulas

$$Y^*(s,0) = \frac{y(0+)}{2} + \mathrm{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} Y\Big(s + i\,\frac{2\pi}{T}\,n\Big),$$

$$Y^*(s,\tau) = \mathrm{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} Y\Big(s + i\,\frac{2\pi}{T}\,n\Big)\exp\Big[\Big(s + i\,\frac{2\pi}{T}\,n\Big)\tau\Big] \quad \text{for } \tau > 0$$

and established the periodicity

$$Y^*\Big(s + i\,\frac{2\pi}{T},\tau\Big) = Y^*(s,\tau).$$

There is a discontinuity at $\tau = 0$ of $Y^*(s,\tau)$ with respect to τ if $y(0+) \neq 0$. In anticipation of the modeling process, we set

$$Y^*(s) := Y^*(s,0+) = \sum_{k=0}^{\infty}\Big[\lim_{\tau\to 0+} y(k\,T+\tau)\Big]\,e^{-k\,T\,s}. \tag{5.11}$$

This implies in the limiting case $\tau \to 0+$ that the second of the above formulas has to be taken. Therefore, the pulse-transform is explicitly defined by

$$Y(s) \mapsto Y^*(s) := \mathrm{pv}\,\frac{1}{T}\sum_{n=-\infty}^{\infty} Y\Big(s + i\,\frac{2\pi}{T}\,n\Big). \tag{5.12}$$

Next we establish the important relation

$$[G(s)\;Y^*(s)]^* = G^*(s)\;Y^*(s) \tag{5.13}$$

which follows from the periodicity of Y^*, for

$$[G(s)\ Y^*(s)]^* = \text{pv}\ \frac{1}{T} \sum_{n=-\infty}^{\infty} G\left(s + i\frac{2\pi}{T} n\right) Y^*\left(s + i\frac{2\pi}{T} n\right)$$

$$= \text{pv}\ \frac{1}{T} \sum_{n=-\infty}^{\infty} G\left(s + i\frac{2\pi}{T} n\right) Y^*(s)$$

$$= Y^*(s)\ \text{pv}\ \frac{1}{T} \sum_{n=-\infty}^{\infty} G\left(s + i\frac{2\pi}{T} n\right) = Y^*(s)\, G^*(s) = G^*(s)\ Y^*(s).$$

5.6.1.2 The z-Transfer Function

Let us examine a configuration consisting of a pulse-sampler, that samples an analog input signal $x(t)$, followed by an analog component with transfer function $G(s)$. If a zero-hold device follows the pulse-sampler, we just absorb its transfer function into the transfer function $G(s)$. The Laplace transform of the output of $G(s)$ is $Y(s) = X^*(s)\, G(s)$, i.e.,

$$Y(s) = \text{pv}\ \frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(s + i\frac{2\pi}{T} n\right) G(s).$$

The image-shift rule and the convolution property allows transforming back this expression, but an infinite series would result in the time domain. This inconvenience may be eliminated by adopting a cruder description of the output $y(t)$. We do this by connecting a fictitious pulse-sampler to the output. This fictitious sampler is assumed to be synchronized with the first sampler. Fig.5.15 shows the configuration.

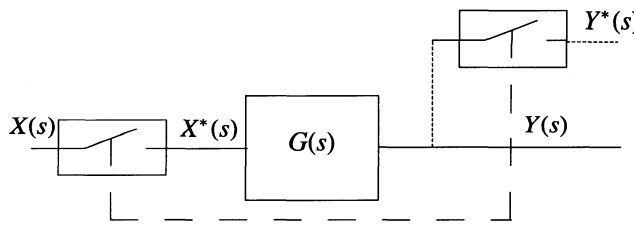


Fig.5.15: Pulse-sampler with analog block followed by a fictitious synchronized pulse-sampler

From (5.13) it follows that

$$Y^*(s) = [G(s)\, X^*(s)]^* = G^*(s)\, X^*(s). \tag{5.14}$$

Let us introduce the z-transforms

$$X^\dagger(z) := Z[\mathcal{L}^{-1}[X(s)]\,(k\,T)]\,(z),$$

$$G^\dagger(z) := Z[\mathcal{L}^{-1}[G(s)]\,(k\,T)]\,(z),$$

$$Y^\dagger(z) := Z[\mathcal{L}^{-1}[Y(s)]\,(k\,T)]\,(z).$$

Writing

$$X^*(s) = X^\dagger(e^{T\,s});\quad G^*(s) = G^\dagger(e^{T\,s});\quad Y^*(s) = Y^\dagger(e^{T\,s})$$

and (5.14) implies $Y^{\dagger}(e^{Ts}) = G^{\dagger}(e^{Ts})\, X^{\dagger}(e^{Ts})$. From this we conclude

$$Y^{\dagger}(z) = G^{\dagger}(z)\, X^{\dagger}(z). \tag{5.15}$$

The fundamental result (5.15) can also be derived without using (5.12). Indeed, we have

$$X^{\dagger}(z) = Z[x(kT)](z); \quad X^{*}(s) = X^{\dagger}(e^{Ts}),$$

$$Y(s) = G(s)\, X^{*}(s) = G(s)\, X^{\dagger}(e^{Ts})$$

$$= G(s) \sum_{k=0}^{\infty} x(kT)\, e^{-kTs} = \sum_{k=0}^{\infty} \{x(kT)\, e^{-kTs}\, G(s)\}.$$

By the original shift rule, one obtains

$$y(t) = \sum_{k=0}^{\infty} x(kT)\, u(t-kT)\, g(t-kT) = \sum_{k=0}^{[t/T]} x(kT)\, g(t-kT)$$

with $g(t) = \mathcal{L}^{-1}[G(s)](t)$ and $[t/T]$ denoting the integral part of t/T. The output $y(t)$ is sampled with the same sampling period and synchronized with the first sampler. Therefore,

$$y(kT) = \sum_{\ell=0}^{k} x(\ell T)\, g(kT - \ell T) = (x * g)(kT)$$

which is the discrete convolution of the two sequences $x(kT)$ and $g(kT)$ (see Section 2.1.2.7). Again, the convolution theorem for the z-transformation implies (5.15).

$G^{\dagger}(z) := Z[\mathcal{L}^{-1}[G(s)](kT)](z)$ is called the *z-transfer function* of the pulse sampled-system modeled by $G(s)$.

5.6.1.3 The z-Transfer Function with Blocks in Cascade

Next, let us consider a sampled-data system with analog blocks in cascade. There are two cases. In the first case each element is followed by a pulse-sampler and the input to the next element is the output of the preceding pulse-sampler. All samplers are synchronized. Fig.5.16 shows the situation.

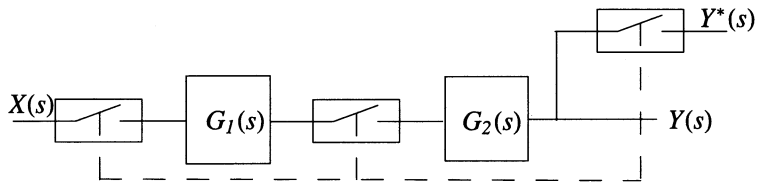


Fig.5.16: Two blocks in cascade, each with synchronized pulse-sampler.

In this first case, we have

$$Y^{*}(s) = G_2^{*}(s)\, G_1^{*}(s)\, X^{*}(s) \Longrightarrow Y^{\dagger}(e^{Ts}) = G_2^{\dagger}(e^{Ts})\, G_1^{\dagger}(e^{Ts})\, X^{\dagger}(e^{Ts})$$

implying

$$Y^{\dagger}(z) = G_2{}^{\dagger}(z)\, G_1{}^{\dagger}(z)\, X^{\dagger}(z). \tag{5.16}$$

The z-transform of two analog blocks in cascade and separated by a pulse-sampler is equal to the product of the z-transform of the two blocks.

In the second case, there is no pulse-sampler between the blocks connected in cascade. We then have

$$Y^{*}(s) = [\, G_2(s)\, G_1(s)]^{*}\, X^{*}(s)\,.$$

For the z-transforms,

$$(G_2\, G_1)^{\dagger}\,(z) := Z[\, \mathcal{L}^{-1}[G_2(s)\, G_1(s)]\,(k\,T)]\,(z)\,,$$

$$Y^{\dagger}(z) = (G_2\, G_1)^{\dagger}\,(z)\; X^{\dagger}(z). \tag{5.17}$$

The composition $Z \circ \mathcal{L}^{-1}$, i.e. the z-transform of a given Laplace image function, will often be used in the following. We call it the *z-transform of a Laplace image,* for short, and use the symbols:

$$Y(s) \;\blacksquare\!\!-\!\!\diamond\; Y^{\dagger}(z) := Z[\mathcal{L}^{-1}[Y(s)]\,(k\,T)]\,(z) = : Z[Y(s)]\,(z).$$

5.6.1.4 Pulse-Sampler followed by a Zero-Hold

If the pulse-sampler is followed by a zero-hold, the transfer function $G(s)$, which follows the pulse-sampler, is simply replaced by $Q(s)\, G(s)$.

$$Q(s) = \frac{1 - e^{-T s}}{s}$$

is the transfer function of the zero-hold. For the configuration of Fig.5.15, but with an added zero-hold between the pulse-sampler and the block $G(s)$, we have

$$Y^{*}(s) = \left[\frac{1 - e^{-T s}}{s}\, G(s) \right]^{*} X^{*}(s).$$

Now

$$\left[G(s)\, \frac{1 - e^{-T s}}{s} \right]^{*} = \mathrm{pv}\, \frac{1}{T} \sum_{n=-\infty}^{\infty} G\!\left(s + i\, \frac{2\pi}{T}\, n\right) \frac{1 - \exp[-T(s + i\, \frac{2\pi}{T}\, n)]}{s + i\, \frac{2\pi}{T}\, n}$$

$$= \mathrm{pv}\, \frac{1 - e^{-T s}}{T} \sum_{n=-\infty}^{\infty} \frac{G(s + i\, \frac{2\pi}{T}\, n)}{s + i\, \frac{2\pi}{T}\, n} = (1 - e^{-T s}) \left\{ \frac{G(s)}{s} \right\}^{*\,\square}\,.$$

If we set

$$\frac{G(s)}{s} \;\blacksquare\!\!-\!\!\diamond\; Z\!\left[\frac{G(s)}{s}\right] = H^{\dagger}(z), \quad \left\{ \frac{G(s)}{s} \right\}^{*} = H^{\dagger}(e^{T s})$$

we obtain

$$Y^{\dagger}(e^{T s}) = Y^{*}(s) = (1 - e^{-T s})\, H^{\dagger}(e^{T s})\, X^{*}(s) = (1 - e^{-T s})\, H^{\dagger}(e^{T s})\, X^{\dagger}(e^{T s}).$$

This finally implies

$$Y^{\dagger}(z) = \left(1 - \frac{1}{z}\right) H^{\dagger}(z)\, X^{\dagger}(z). \tag{5.18}$$

The configuration "pulse-sampler + zero-hold + block with transfer function $G(s)$ + (fictitious) pulse-sampler" has the concise z-transform description

$$\frac{z-1}{z} H^{\dagger}(z)$$

with $G(s)/s \bullet\!\!-\!\!\diamond\ H^{\dagger}(z)$.

■ 5.6.2 Closed-Loop Sampled-Data Systems

There are different possibilities for the configuration of a sampled-data system, depending on how many and where the sampler(s) are inserted. As an example, we consider the configuration of Fig.5.17, which also shows the principal advantage of a sampled-data system by means of a digital computer for the controller.

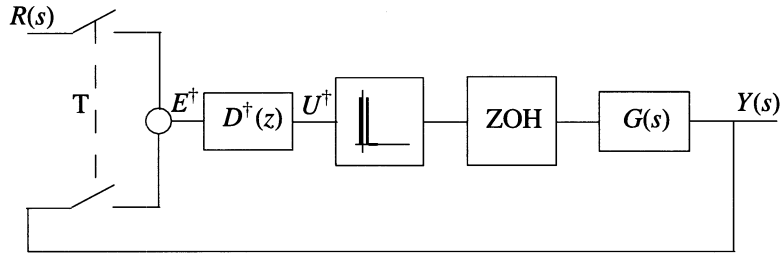


Fig.5.17: A sampled-data control system with a digital controller in cascade.

Two synchronized samplers are sampling the reference analog input $R(s)$ and the analog output $Y(s)$. The controller (a digital computer) is represented by the z-transfer function $D^{\dagger}(z)$. The input of the controller is the sequence given by $e(kT) = r(kT) - y(kT)$, whose z-transform is $E^{\dagger}(z) = R^{\dagger}(z) - Y^{\dagger}(z)$. The output of the controller is the sequence $u(kT) \circ\!\!-\!\!\diamond\ U^{\dagger}(z)$. A D/A device followed by a zero-hold ZOH yields the sampled signal, which is the input of the plant $G(s)$. With

$$G_1{}^{\dagger}(z) := Z\left[\frac{G(s)}{s}\right];\quad r(kT) \circ\!\!-\!\!\diamond\ R^{\dagger}(z);\quad y(kT) \circ\!\!-\!\!\diamond\ Y^{\dagger}(z)$$

we set

$$U^{\dagger}(z) = D^{\dagger}(z)\, E^{\dagger}(z) = D^{\dagger}(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)],$$

$$Y^{\dagger}(z) = \frac{z-1}{z} G_1{}^{\dagger}(z)\, U^{\dagger}(z) = \frac{z-1}{z} G_1{}^{\dagger}(z)\, D^{\dagger}(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)].$$

This gives the closed-loop z-transfer function

$$\frac{Y^{\dagger}(z)}{R^{\dagger}(z)} = \frac{\frac{z-1}{z} G_1{}^{\dagger}(z)\, D^{\dagger}(z)}{1 + \frac{z-1}{z} G_1{}^{\dagger}(z)\, D^{\dagger}(z)} = \frac{(z-1)\, G_1{}^{\dagger}(z)\, D^{\dagger}(z)}{z + (z-1)\, G_1{}^{\dagger}(z)\, D^{\dagger}(z)}$$

and the actuating signal ratio

$$\frac{E^{\dagger}(z)}{R^{\dagger}(z)} = \frac{z}{z + (z-1)\,G_1{}^{\dagger}(z)\,D^{\dagger}(z)}.$$

If no zero-hold is used, just substitute

$$\frac{z-1}{z}\,G_1{}^{\dagger}(z)$$

by $G^{\dagger}(z) := Z[G(s)]$ in the expressions above .

The z-transfer function $D^{\dagger}(z)$ of the controller has to be chosen such that the poles of

$$\frac{(z-1)\,G_1{}^{\dagger}(z)\,D^{\dagger}(z)}{z + (z-1)\,G_1{}^{\dagger}(z)\,D^{\dagger}(z)}$$

are located inside the unit-circle of the complex z-plane. Once the rational function $D^{\dagger}(z)$ is determined, it is easy to obtain the difference equation that corresponds to this transfer function (see Section 2.2.1). The algorithm that solves the difference equation is easily programmed on the computer. Compared with an analog controller this gives many more degrees of freedom.

■ 5.6.3 State Control Sampled-Data Systems

Control systems with one single variable to be controlled are best treated in the transfer function philosophy. If there are multi-inputs and/or multi-outputs, inputs and outputs being vectors, the transfer function approach often is not adequate. This is particularly the case for sampled-data systems with multiple inputs and outputs. In this case the state variable approach is more favorable. First, we shall show how, given an analog state control system, the equivalent discrete control system (5.7) of Section 5.2.4 is obtained.

5.6.3.1 Equivalent Discrete State Control System

Let

$$\frac{d\boldsymbol{x}}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{w}(t),$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

be a given analog state control system (the matrices **D** and **H** in (5.7) must be the zero matrices). The input vector is sampled with a sampling period $T > 0$. A zero-order hold device is used such that the input vector $u(t)$ is replaced by

$$\mathbf{u}(k) := \mathbf{u}(kT + t), \quad (kT \le t < (k+1)T), \quad (k = 0, 1, 2, \ldots).$$

The perturbation vector is synchronically sampled and replaced by

$$\mathbf{w}(k) := \mathbf{w}(kT + t), \quad (kT \le t < (k+1)T), \quad (k = 0, 1, 2, \ldots).$$

We now use the general transition formulas of Section 5.2.3

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t - \tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau\,, \tag{5.19}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{\Phi}(t - t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}\,\mathbf{\Phi}(t - \tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau\,, \tag{5.20}$$

with the state-transition matrix

$$\mathbf{\Phi}(t) := \mathcal{L}^{-1}[\mathbf{R}(s, \mathbf{A})]\,(t) = e^{t\,\mathbf{A}}. \tag{5.21}$$

By setting $t = (k + 1)\,T$, $t_0 = k\,T$, we obtain

$$\mathbf{x}((k + 1)\,T) = \mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{k\,T}^{k\,T+T} \mathbf{\Phi}(k\,T + T - \tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau\,,$$

$$\mathbf{y}((k + 1)\,T) = \mathbf{C}\,\mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{k\,T}^{k\,T+T} \mathbf{C}\,\mathbf{\Phi}(k\,T + T - \tau)\,[\mathbf{B}\,\mathbf{u}(\tau) + \mathbf{E}\,\mathbf{w}(\tau)]\,d\tau\,.$$

By changing the variables in the integrals, one obtains

$$\mathbf{x}((k + 1)\,T) = \mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{0}^{T} \mathbf{\Phi}(T - \tau)\,[\mathbf{B}\,\mathbf{u}(\tau + k\,T) + \mathbf{E}\,\mathbf{w}(\tau + k\,T)]\,d\tau\,,$$

$$\mathbf{y}((k + 1)\,T) = \mathbf{C}\,\mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{0}^{T} \mathbf{C}\,\mathbf{\Phi}(T - \tau)\,[\mathbf{B}\,\mathbf{u}(\tau + k\,T) + \mathbf{E}\,\mathbf{w}(\tau + k\,T)]\,d\tau\,,$$

which, due to the sampling and the zero-hold device, becomes

$$\mathbf{x}((k + 1)\,T) = \mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{0}^{T} \mathbf{\Phi}(T - \tau)\,[\mathbf{B}\,\mathbf{u}(k) + \mathbf{E}\,\mathbf{w}(k)]\,d\tau\,,$$

$$\mathbf{y}((k + 1)\,T) = \mathbf{C}\,\mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{0}^{T} \mathbf{C}\,\mathbf{\Phi}(T - \tau)\,[\mathbf{B}\,\mathbf{u}(k) + \mathbf{E}\,\mathbf{w}(k)]\,d\tau\,.$$

Another change of variables yields

$$\mathbf{x}((k + 1)\,T) = \mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{0}^{T} \mathbf{\Phi}(\tau)\,[\mathbf{B}\,\mathbf{u}(k) + \mathbf{E}\,\mathbf{w}(k)]\,d\tau$$

$$\mathbf{y}((k + 1)\,T) = \mathbf{C}\,\mathbf{\Phi}(T)\,\mathbf{x}(k\,T) + \int_{0}^{T} \mathbf{C}\,\mathbf{\Phi}(\tau)\,[\mathbf{B}\,\mathbf{u}(k) + \mathbf{E}\,\mathbf{w}(k)]\,d\tau\,.$$

Identifying $\mathbf{x}((k + 1)\,T)$ with $\mathbf{x}(k + 1)$ and $\mathbf{x}(k\,T)$ with $\mathbf{x}(k)$ and introducing the matrices

$$\mathbf{A}_s := \mathbf{\Phi}(T) = \mathcal{L}^{-1}[\mathbf{R}(s, \mathbf{A})]\,(T) = e^{T\,\mathbf{A}}, \tag{5.22}$$

$$\mathbf{B}_s := \int_{0}^{T} \mathbf{\Phi}(\tau)\,d\tau\;\mathbf{B} = \int_{0}^{T} e^{\tau\,\mathbf{A}}\,d\tau\;\mathbf{B}\,, \tag{5.23}$$

$$\mathbf{E}_s := \int_{0}^{T} \mathbf{\Phi}(\tau)\,d\tau\;\mathbf{E} = \int_{0}^{T} e^{\tau\,\mathbf{A}}\,d\tau\;\mathbf{E}\,, \tag{5.24}$$

$$\mathbf{C}_s := \mathbf{C}\,\Phi(T)\,, \tag{5.25}$$

$$\mathbf{D}_s := \mathbf{C}\int_0^T \Phi(\tau)\,d\tau\;\mathbf{B}\,, \tag{5.26}$$

$$\mathbf{H}_s := \mathbf{C}\int_0^T \Phi(\tau)\,d\tau\;\mathbf{E}\,, \tag{5.27}$$

we obtain the *equivalent discrete state control system* that describes the sampled-data control system with zero-hold devices

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}_s\,\mathbf{x}(k) + \mathbf{B}_s\,\mathbf{u}(k) + \mathbf{E}_s\,\mathbf{w}(k)\,,\\ \mathbf{y}(k+1) &= \mathbf{C}_s\,\mathbf{x}(k) + \mathbf{D}_s\,\mathbf{u}(k) + \mathbf{H}_s\,\mathbf{w}(k)\,,\end{aligned}$$

or

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}_s\,\mathbf{x}(k) + \mathbf{B}_s\,\mathbf{u}(k) + \mathbf{E}_s\,\mathbf{w}(k)\,,\\ \mathbf{y}(k) &= \mathbf{C}_s\,\mathbf{x}(k-1) + \mathbf{D}_s\,\mathbf{u}(k-1) + \mathbf{H}_s\,\mathbf{w}(k-1)\,.\end{aligned} \tag{5.28}$$

Remark. If the matrix $\mathbf{A}$ is nonsingular, we have

$$\int_0^T e^{\tau\mathbf{A}}\,d\tau = \mathbf{A}^{-1}[\Phi(T) - \mathbf{I}].$$

If a full state feedback is introduced, i.e.,

$$\mathbf{u}(k) = \mathbf{r}(k) - \mathbf{K}\,\mathbf{x}(k), \quad \mathbf{U}(z) = \mathbf{R}(z) - \mathbf{K}\,\mathbf{X}(z)$$

where $\mathbf{K}$ is a $p \times n$-matrix, and $\mathbf{r}(k)$ is the reference input, we can write

$$\mathbf{Y}(z) = \mathbf{G}_r(z)\,\mathbf{R}(z) + \mathbf{G}_w(z)\,\mathbf{W}(z)\,,$$

where the transfer matrix between the reference input and the output is

$$\mathbf{G}_r(z) = \mathbf{G}(z)\,\mathbf{B}_s + \mathbf{D}_s\,,$$

the transfer matrix between the perturbation and the output is

$$\mathbf{G}_w(z) = \mathbf{G}(z)\,\mathbf{E}_s + \mathbf{H}_s\,,$$

and

$$\mathbf{G}(z) = (\mathbf{C}_s - \mathbf{D}_s\,\mathbf{K})\,\mathbf{R}(z,\,\mathbf{A}_s + \mathbf{B}_s\,\mathbf{K})\,\mathbf{E}_s + \mathbf{H}_s.$$

The common denominator polynomial of the entries of the matrix $\mathbf{G}(z)$ is the characteristic equation of the closed-loop sampled-data system. For a stable system, all its roots must be located inside the unit-circle.

■ 5.6.4 Use of the Package

5.6.4.1 Discrete Laplace Transforms

In order to compute the discrete Laplace transform of a given sequence $y(k)$, use `DiscreteLaplaceTransform[y[k],k,T,s]`.

```
DiscreteLaplaceTransform[k Cos[k], k, T, s]
```

$$\frac{e^{sT}\,T\,(-2\,e^{sT}+\text{Cos}[T]+e^{2\,sT}\,\text{Cos}[T])}{(1+e^{2\,sT}-2\,e^{sT}\,\text{Cos}[T])^2}$$

5.6.4.2 The s-Image to z-Image Transform

In order to compute the z-transform of a given Laplace image function, i.e. $Z[Y(s)](z)$, use `LaplaceToZImage[Y[s],s,T,z]`.

```
LaplaceToZImage[(s + 1) / (s^2 + 2 s + 5), s, T, z]
```

$$\frac{e^{2\,T}\,z^2-e^{T}\,z\,\text{Cos}[2\,T]}{1+e^{2\,T}\,z^2-2\,e^{T}\,z\,\text{Cos}[2\,T]}$$

If the advanced z-transform of the given Laplace image is wanted, add the parameter τ.

```
LaplaceToZImage[(s + 1) / (s^2 + 2 s + 5), s, T, τ, z]
```

$$\frac{e^{-\tau}\,(-e^{T}\,z\,\text{Cos}[2\,(T-\tau)]+e^{2\,T}\,z^2\,\text{Cos}[2\,\tau])}{1+e^{2\,T}\,z^2-2\,e^{T}\,z\,\text{Cos}[2\,T]}$$

Example. Consider a system with the transfer function

```
G = 1 / (s^2 + 2 s + 10);
```

The eigen-frequency of the self-oscillation of the system is $\omega_0 = 3$. Thus we choose $T = 0.1$ for the sampling period in order to satisfy $\omega_0 T \lesssim 1/2$. We shall examine the unit-step response. First, we take the simple impulse-model (no hold-device). The z-transform of the sampled-data signal of the discrete unit-step is

```
T = 0.1; F* = z / (z - 1);

G* = LaplaceToZImage[G, s, T, z] /. T → 0.1
```

$$\frac{0.108867\,z}{1-2.11162\,z+1.2214\,z^2}$$

The z-transform of the output sequence is

```
InverseZTransform[G* F*, z, k] /. ExpandGenFac // Chop
```

```
0.108867
 (9.10892 1.^k + 2 0.904837^k (-4.55446 Cos[0.3 k] - 0.778275 Sin[0.3 k]))
```

In order to compare the discrete signal output with the analog output, we have to multiply the former by T (see the Remark in Section 5.6.1 concerning the sampling process).

```
yd = %T; yd1 = DiscreteSignal[yd, {k, 0, 30}];
```

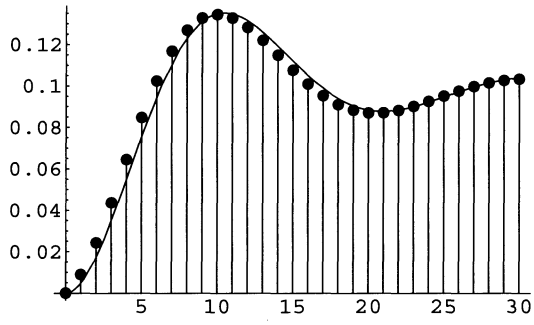
The analog output is

```
ya = InverseLaplaceTransform[G / s, s, t]
```

$$\frac{1}{10} + 2\,e^{-t}\left(-\frac{1}{20}\,\text{Cos}[3\,t] - \frac{1}{60}\,\text{Sin}[3\,t]\right)$$

Comparing the discrete signal output with the analog output on the interval [0, 5] yields

```
yap = Plot[Evaluate[ya /. t → T t], {t, 0, 30}];

Show[{yd1, yap}];
```



A zero-hold device between the pulse-sampler and the analog block is now inserted. We then have

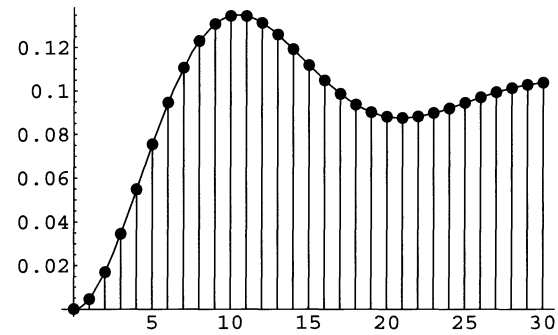
```
Gh† = (z - 1) / z LaplaceToZImage[G / s, s, T, z] // Together /. T → 0.1
```

$$\frac{0.0530567\,z + 0.0567259\,z^2}{10\,z\,(1 - 2.11162\,z + 1.2214\,z^2)}$$

```
ydh = InverseZTransform[Gh† F†, z, k] /. ExpandGenFac // Chop
```

$$\frac{1}{10}\,(1.\,1.^{k} + 2\,0.904837^{k}\,(-0.5\,\text{Cos}[0.3\,k] - 0.166667\,\text{Sin}[0.3\,k]))$$

```
yd2 = DiscreteSignal[ydh, {k, 0, 30}];

Show[yd2, yap];
```



The two graphics show the improved approximation of the discrete signal output when a hold device is used. It also shows that the simple impulse-model without hold device in general is sufficient, provided the sampling period is not chosen to be too large.

5.6.4.3 Closed-Loop Systems

Example. We consider the sampled-data control system of Fig.5.17, where the plant has the transfer function

$$G(s) = \frac{1}{s(s+1)}.$$

Let us compute $G_1^{\dagger}(z)$.

```
T =.

G1 = LaplaceToZImage[1 / (s^2 (s + 1)), s, T, z] // Together
```

$$\frac{z\,(-1 + e^T - T + z - e^T z + e^T T z)}{(-1+z)^2\,(-1+e^T z)}$$

```
(z - 1) / z G1;
```

We choose a sample period of $T = 0.1$.

```
H = % /. T → 0.1;
```

We try a controller with the z-transfer function

```
D1 = (z - a) / (z^2 + b z + c);
```

The closed-loop transfer function for the cascade configuration is obtained by

```
M = ClosedLoopTransferFunction[CascadeCC[-1, {D1}, {H}, z]]

((0.00517092 + 0.00534617 z) (-a + z)) /
 (-0.00517092 a + c + (0.00517092 - 0.00534617 a + b - 2.10517 c) z +
   (1.00535 - 2.10517 b + 1.10517 c) z^2 + (-2.10517 + 1.10517 b) z^3 + 1.10517 z^4)
```

The closed-loop characteristic function is given by

```
P = Collect[
  ClosedLoopCharacteristicFunction[CascadeCC[-1, {D1}, {H}, z]], z]

-0.00517092 a + 1. c + (0.00517092 - 0.00534617 a + 1. b - 2.10517 c) z +
 (1.00535 - 2.10517 b + 1.10517 c) z^2 + (-2.10517 + 1.10517 b) z^3 + 1.10517 z^4

test = ZStabilityTest[P, z]

{4.21052 + 0.000175256 a - 4.21034 b + 4.21034 c ≠ 0,
 0.0105171 - 0.0105171 a > 0,
 8.64137 + 0.00999132 a - 0.210342 b - 8.21034 c > 0,
 0.2 + 0.031376 a + 0.210342 b + 0.210342 c > 0,
 4.62033 - 0.0310255 a + 4.21034 b + 3.78966 c > 0,
 4.21052 + 0.000175256 a - 4.21034 b + 4.21034 c > 0,
 7.03143 + 1.93901 a - 0.00935783 a^2 - 8.84877 × 10^-6 a^3 + 15.3327 b +
   1.03316 a b + 0.00555782 a^2 b + 7.43884 b^2 + 0.0384656 a b^2 + 0. b^3 +
   8.33044 c - 1.74995 a c + 0.00324254 a^2 c - 1.1245 b c - 1.00155 a b c -
   7.43884 b^2 c - 8.56334 c^2 - 0.261328 a c^2 - 14.1698 b c^2 - 6.73094 c^3 > 0}
```

These are the conditions for the two parameters a and b in order to get a stable system.

```
test /. {a → -0.5, b → -1, c → 1}
```

```
{True, True, True, True, True, True, True}
```

A controller that yields a stable system is, for example,

```
D11 = (z - a) / (z^2 + b z + c) /. {a → -0.5, b → -1, c → 1}
```

$$\frac{0.5 + z}{1 - z + z^2}$$

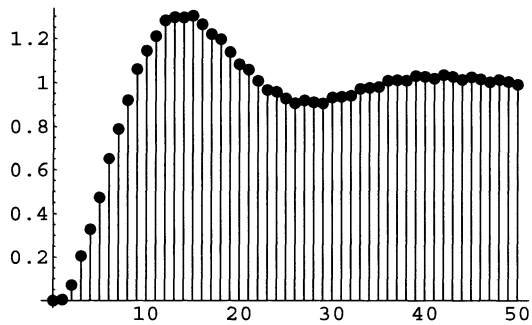
The closed-loop z-transfer function then becomes

```
M1 = ClosedLoopTransferFunction[CascadeCC[-1, {D11}, {H}, z]]
```

$$\frac{0.00483742\,(0.5 + z)\,(0.967218 + z)}{0.907177 - 2.80258\,z + 3.81451\,z^2 - 2.90484\,z^3 + z^4}$$

Let us compute the response to a unit-step input:

```
InverseZTransform[M1 z / (z - 1), z, k] // Chop;

% /. ExpandGenFac // Chop

0.00483742 (206.722 1.^k +
   2 0.956901^k (-104.723 Cos[0.114829 k] - 43.833 Sin[0.114829 k]) +
   2 0.995357^k (1.36192 Cos[1.04239 k] + 0.361648 Sin[1.04239 k]))

Show[DiscreteSignal[(% /. k → 2 k), {k, 0, 50}]];
```



Another example: Let

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{w}(t),$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

be an analog state control system. If the input and the perturbation are sampled with a sampling period T, i.e.,

$$\mathbf{u}(k) := \mathbf{u}(k\,T + t), \quad (k\,T \le t < (k+1)\,T), \quad (k = 0, 1, 2, \ldots),$$
$$\mathbf{w}(k) := \mathbf{w}(k\,T + t), \quad (k\,T \le t < (k+1)\,T), \quad (k = 0, 1, 2, \ldots),$$

the equivalent discrete state control system is

$$\mathbf{x}(k+1) = \mathbf{A}_s\,\mathbf{x}(k) + \mathbf{B}_s\,\mathbf{u}(k) + \mathbf{E}_s\,\mathbf{w}(k),$$
$$\mathbf{y}(k+1) = \mathbf{C}_s\,\mathbf{x}(k) + \mathbf{D}_s\,\mathbf{u}(k) + \mathbf{H}_s\,\mathbf{w}(k),$$

with $\mathbf{x}(k)$, $\mathbf{y}(k)$ identified with $\mathbf{x}(kT)$, $\mathbf{y}(kT)$ respectively. The six matrices $\mathbf{A_s}$,..., $\mathbf{H}_s$ are obtained by `ToDiscreteStateControlSystem[A,B,C,E,T]`. The result is returned in the form of {`StateControlSystem[`A_s, B_s, C_s, D_s`]` E_s, H_s}. **Example.**

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -5 & -2\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + u(t)\begin{pmatrix}0\\ 1\end{pmatrix},$$

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}.$$

```
Am = {{0, 1}, {-5, -2}};
Bm = {{0}, {1}};
Cm = IdentityMatrix[2];
Em = NullMatrix[2, 1];
```

We choose the sampling period $T = 0.5$.

```
ToDiscreteStateControlSystem[Am, Bm, Cm, Em, 0.5]
```

```
{StateControlSystem[{{0.582899, 0.255189}, {-1.27594, 0.0725209}},
  {{0.0834202}, {0.255189}},
  {{0.582899, 0.255189}, {-1.27594, 0.0725209}},
  {{0.0834202}, {0.255189}}], {{0.}, {0.}}, {{0.}, {0.}}}
```

The z-transform of the response $\mathbf{y}[k+1]$ of a unit-step then is

```
DiscreteDynamicSystemResponse[%[[1]], %[[2]], %[[3]]][
  {UnitStep[k]}, {0}, k, z] // Chop
```

$$\left\{\frac{0.0834202\,z^2\,(0.708122+z)}{(-1.+1.\,z)\,(0.367879-0.65542\,z+1.\,z^2)},\ \frac{0.255189\,(-1.+z)\,z^2}{(-1.+1.\,z)\,(0.367879-0.65542\,z+1.\,z^2)}\right\}$$

Thus the z-transform of $\mathbf{y}[k]$ is, by the shift-rule,

```
% / z
```

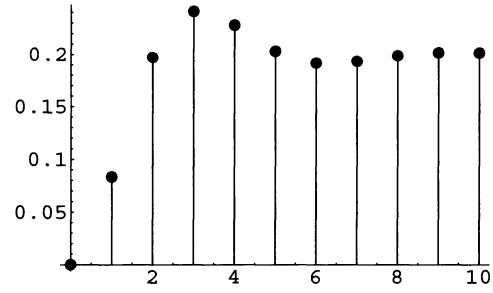
$$\left\{\frac{0.0834202\,z\,(0.708122+z)}{(-1.+1.\,z)\,(0.367879-0.65542\,z+1.\,z^2)},\ \frac{0.255189\,(-1.+z)\,z}{(-1.+1.\,z)\,(0.367879-0.65542\,z+1.\,z^2)}\right\}$$

which gives the values for $y(t)$ at discrete instants $t = kT$:

```
InverseZTransform[%[[1]], z, k] /. ExpandGenFac // Chop
```

```
0.0834202 (2.3975 1.^k + 2 0.606531^k (-1.19875 Cos[1. k] - 0.599375 Sin[1. k]))
```

```
DiscreteSignal[%, {k, 0, 10}] // Show;
```



Let us compare this with the result obtained by applying the transfer function philosophy. The given analog state control system is equivalent with the analog input-output system described by the transfer function

```
G = 1 / (s^2 + 2 s + 5);
```

Now we use a sampler, followed by a D/A converter, which in turns is followed by a zero-order hold device. In the z-domain the system is described by the z-transfer function

```
Gz = (z - 1) / z * LaplaceToZImage[G / s, s, 0.5, z] // Together
```

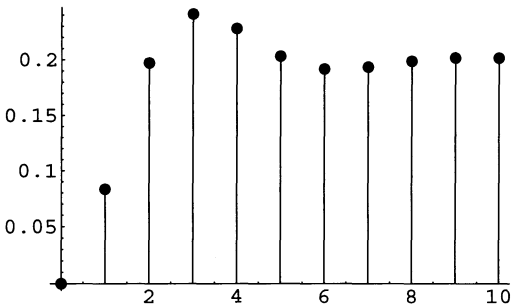
$$\frac{0.802868\,z + 1.1338\,z^2}{5\,z\,(1 - 1.78162\,z + 2.71828\,z^2)}$$

The response of a discrete unit-step is

```
InverseZTransform[z / (z - 1) Gz, z, k] /. ExpandGenFac // Chop
```

$$\frac{1}{5}\,(1.\,1.^k + 2\,0.606531^k\,(-0.5\,\mathrm{Cos}[1.\,k] - 0.25\,\mathrm{Sin}[1.\,k]))$$

```
DiscreteSignal[%, {k, 0, 10}] // Show;
```



in agreement with the above result.

Chapter 6
Laplace Transformation: Further Topics

6.1 The Complex Inversion Formula

Remember that if the Laplace integral

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt$$

converges for some finite real value $s = x_0$, it converges for all complex s with $\Re s > x_0$ and thus defines a holomorphic or analytic function on the right-half plane $\Re s > x_0$. The smallest of all such x_0's is called the *abscissa of simple convergence* and the corresponding right half-plane is called the *half-plane of simple convergence.* Fortunately, the situation in most applications is more convenient with the Laplace integral converging absolutely for a finite value of $s = x_1 \in \mathbb{R}$, i.e.,

$$\int_0^\infty e^{-x_1 t} \mid f(t) \mid dt < \infty.$$

It can then be shown that the Laplace integral absolutely converges for all s in the right half-plane $\Re s > x_1$, which is called the *half-plane of absolute convergence.* The smallest of all such x_1's is called the *abscissa of absolute convergence.* We always have $x_0 \le x_1$, i.e. the half-plane of absolute convergence is contained in the half-plane of simple convergence (absolute convergence implies simple convergence).

Definition. A function $f(t)$ defined on the interval $[a, b]$ is said to be of *bounded variation,* if for *any* subdivision $a = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = b$,

$$\sum_{k=0}^{m-1} \mid f(t_{k+1}) - f(t_k) \mid \; \le M < \infty,$$

for some bound M, that does not depend on the subdivision chosen.

For example, a function $f(t)$ is of bounded variation on $[a, b]$, if $f'(t)$ is piecewise continuous and if

$$\int_a^b \mid f'(t) \mid dt \le M.$$

Proof. For any subdivision it holds that

$$\sum_{k=0}^{m-1} | f(t_{k+1}) - f(t_k) | = \sum_{k=0}^{m-1} \left| \int_{t_k}^{t_{k+1}} f'(t)\, dt \right| \leq \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} | f'(t) |\, dt$$

$$= \int_a^b | f'(t) |\, dt \leq M \ . \ \square$$

Because

$$\int_a^b | f'(t) |\, dt < \int_a^b \sqrt{1 + [f'(t)]^2}\, dt$$

we can say that a finite arc-length of the graph of $f(t)$ over $[a, b]$ implies bounded variation. Conversely, it is readily seen that a function of bounded variation on $[a, b]$ implies a finite arc-length of its graph. Thus the class of functions of bounded variation on $[a, b]$ coincides with the class of functions with finite arc-length of their graphs. A continuous function is not necessarily of bounded variation and a function of bounded variation is not necessarily continuous! However, it can be proved that if a function is of bounded variation on $[a, b]$, then for each point $t \in (a, b)$ the left limit $f(t-)$ and the right limit $f(t+)$ exist. This implies that a function $f(t)$ of bounded variation can be normalized or redefined by

$$f(t) := \frac{f(t-) + f(t+)}{2}.$$

At points where f is continuous nothing is changed, but if f has a discontinuity at t, which must then be a discontinuity of the first kind, the value in the middle of the jump is the redefined function value at t.

We now state the basic

Theorem. *If a given original function $f(t)$ satisfies the two conditions:*

(i) $f(t)$ has a right half-plane of absolute convergence $\mathrm{R}s > x_1$,

(ii) $f(t)$ is locally of bounded variation on the real axis,

then the complex inversion formula (Bromwich's Formula)

$$\frac{f(t-) + f(t+)}{2} = \mathrm{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds \tag{6.1}$$

holds. The integration contour (a vertical straight line) must be located inside the half-plane of absolute convergence, i.e. $c > x_1$.

Saying that $f(t)$ is locally of bounded variation means that for any t there is a finite interval $[a, b]$ containing t where $f(t)$ is of bounded variation. Remember that an original vanishes on the negative part of the real axis. Thus, if the original $f(t)$ is continuous and locally of bounded variation on $[0, \infty)$ and satisfies the above condition (i), then

$$\frac{f(0+)}{2} = \mathrm{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, ds,$$

$$f(t) = \mathrm{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds, \quad t > 0.$$

For a proof of this fundamental theorem, which in fact is a result from the theory of Fourier transforms, see for example [Doetsch G. and Nader W. (1974)].

If the original $f(t)$ only has a right half-plane of simple convergence $\Re s > x_0$ and is not necessarily of bounded variation, the complex inversion theorem takes a weaker form.

Theorem. *Assume that a given original function $f(t)$ has only a right half-plane of simple convergence $\Re s > x_0$, then*

$$u(t) \int_0^t f(x)\,dx = \operatorname{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} \frac{F(s)}{s}\,ds, \tag{6.2}$$

where $c > \max\{x_0, 0\}$.

This can be proved by observing that

$$\varphi(t) := u(t) \int_0^t f(x)\,dx$$

satisfies the above conditions (i) and (ii), see [Doetsch G. and Nader W. (1974)]. The complex inversion formula is rarely applied as it stands, since an evaluation by direct integration is not readily made. However, the formula (6.1) is the starting point for other fruitful methods.

■ 6.1.1 Evaluation of the Inverse Laplace Transform by the Calculus of Residues

Assume that the image function has a right half-plane of absolute convergence where it represents a holomorphic function. Image functions of solutions of partial differential equations often have the property that their analytic extension to the left is a meromorphic function. This means that all singularities are poles which are located to the left of the abscissa of convergence x_1, and there are no other singularities (no essential singularities, no branch points). A special case is a rational image function where we have a finite number of poles. Often, we have an infinite number of poles which are evenly spaced along a line parallel to the imaginary axis.

We now assume a meromorphic image function $F(s)$ where the poles a_ν ($\nu = 0, 1, 2, \ldots$) are arranged such that

$|a_0| \le |a_1| \le |a_2| \le \cdots$. We then consider the sequence of closed integration contours C_n satisfying the following two conditions (Fig.6.1):

1. Each C_n consists of the vertical integration path $z(t) = c + it$, $-R_n \le t \le R_n$ and of a left circular arc

   $A_n : z(t) = \rho_n \exp[i\varphi]$ with $\rho_n = \sqrt{R_n^2 + c^2}$ and

   $\pi/2 - \arctan(c/R_n) \le \varphi \le 3\pi/2 + \arctan(c/R_n)$.

2. Each ρ_n is chosen such that the circular arc passes between two consecutive poles a_n and a_{n+1}.

We then consider the contour integral

$$\frac{1}{2\pi i} \int_{C_n} e^{ts} F(s)\,ds = \frac{1}{2\pi i} \int_{c-iR_n}^{c+iR_n} e^{ts} F(s)\,ds + \frac{1}{2\pi i} \int_{A_n} e^{ts} F(s)\,ds.$$

The theorem of residues implies that

$$\frac{1}{2\pi i}\int_{C_n} e^{ts}\,F(s)\,ds = \sum_{\nu=0}^{n} \operatorname*{res}_{s=a_\nu} \{e^{ts}\,F(s)\}.$$

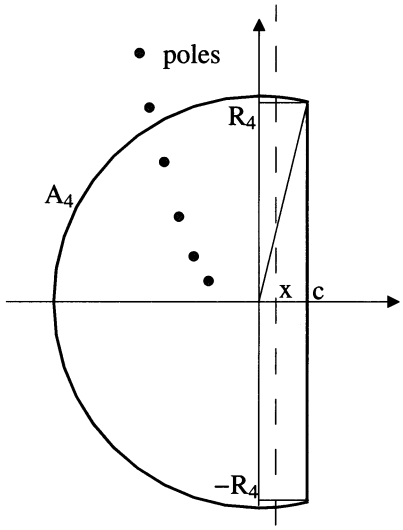


Fig.6.1: Integration contour C_n

In addition to the two conditions above we add a third one, which we name the *Closing Contour Condition.*

3. The contour integrals over the left circular arcs tend to zero as $\rho_n \to \infty$:

$$\lim_{\rho_n\to\infty} \frac{1}{2\pi i}\int_{A_n} e^{ts}\,F(s)\,ds = 0. \tag{6.3}$$

The following proposition holds now.

Proposition. *If the three conditions 1, 2, 3 are satisfied, then*

$$f(t) = \operatorname{pv}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\,F(s)\,ds = \sum_{\nu=0}^{\infty} f_\nu(t) \tag{6.4}$$

with

$$f_\nu(t) = \operatorname*{Res}_{s=a_\nu} \{e^{ts}\,F(s)\}.$$

If between two consecutive circular arcs there is more than one pole, then

$$f_\nu(t) = \sum_{\rho_\nu < |a_i| < \rho_{\nu+1}} \operatorname*{Res}_{a_i} \{e^{ts}\,F(s)\}\,.$$

The assumed property that the arcs closing the integration contour to the left are circular is inessential. The circular arcs may be replaced by rectangles or parabolic arcs or any other curve, as long as Condition 3 is satisfied. Also, these left arcs are often terminated at the

imaginary axes and the gap between the imaginary axis and the vertical integration contour is completed by horizontal straight line segments. Clearly, the crucial hypothesis for the validity of the above proposition is the Closing Contour Condition (6.3):

$$\lim_{\rho_n \to \infty} \frac{1}{2\pi i} \int_{A_n} e^{ts} F(s)\, ds = 0.$$

A tool to test Condition 3 is

Jordan's Lemma. *Let us denote the left semi-circle with center c_0 and radius ρ by*

$$A: z(\varphi) = c_0 + \rho e^{i\varphi}, \quad \frac{\pi}{2} < \varphi < \frac{3\pi}{2}$$

and the corresponding right semi-circle with center c_0 and radius ρ by

$$\tilde{A}: z(\varphi) = c_0 + \rho e^{i\varphi}, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}.$$

Let us also assume that as the radius ρ of the semi-circle tends to infinity the image function $F(s)$ tends to zero uniformly with respect to the argument of s on A, and $\tilde{A}$ respectively. This is, for example, the case if

$$\left| F(s) \right| \leq \frac{M}{|s|^{\alpha}}, \quad \alpha > 0$$

on A, and $\tilde{A}$ respectively. Then

(i) $$\lim_{\rho \to \infty} \frac{1}{2\pi i} \int_{A} e^{ts} F(s)\, ds = 0, \quad \text{for } t > 0,$$

(ii) $$\lim_{\rho \to \infty} \frac{1}{2\pi i} \int_{\tilde{A}} e^{ts} F(s)\, ds = 0, \quad \text{for } t < 0.$$

A proof of Jordan's lemma is found, for example, in [Doetsch G. and Nader W. (1974)] or in [Proudnikov A. and Ditkine V. (1979)].

Clearly, the statement of Jordan's Lemma remains true if the semi-circles are replaced by only parts of them. We now prove that in the case of $t > 0$ and $c > 0$ the left semi-circles can be extended until they reach the vertical integration contour. This is helpful if the abscissa of convergence of $f(t)$ is nonnegative and the vertical integration contour is on the right of the imaginary axis. With c denoting the abscissa of the vertical integration contour let

$$\gamma: z(\varphi) = \rho e^{i\varphi}, \quad \frac{\pi}{2} - \epsilon \leq \varphi \leq \frac{\pi}{2}, \quad \epsilon = \arcsin \frac{c}{\rho}$$

$$\left| \int_{\gamma} e^{ts} F(s)\, ds \right| \leq \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} |e^{t\rho e^{i\varphi}}| \, |F(\rho e^{i\varphi})| \, |i\rho e^{i\varphi}| \, d\varphi$$

$$= \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} e^{t\rho\cos\varphi} \left| F(\rho e^{i\varphi}) \right| \rho \, d\varphi$$

$$= \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} e^{tc} \left| F(\rho e^{i\varphi}) \right| \rho \, d\varphi \leq M_{\rho} e^{tc} \rho \epsilon$$

where $M_\rho \to 0$ for $\rho \to \infty$. But

$$\rho\epsilon = \rho \arcsin\frac{c}{\rho} \to c \text{ for } \rho \to \infty$$

which proves that

$$\int_\gamma e^{ts} F(s)\, ds \to 0 \text{ for } \rho \to \infty.$$

The same is true for the contribution to the integral along the lower arc

$$\tilde{\gamma}: z(\varphi) = \rho e^{i\varphi}, \quad \frac{3\pi}{2} \le \varphi \le \frac{3\pi}{2} + \epsilon, \quad \epsilon = \arcsin\frac{c}{\rho}.$$

6.1.1.1 Expansion Theorems of the Second Kind

In Section 1.1.2.12 we constructed the so-called expansion theorem of the first kind. The calculus of residues now yields another type of expansion theorems, sometimes called expansion theorems of the second kind. The series representation from Section 6.1.1,

$$f(t) = \sum_{\nu=0}^{\infty} \operatorname*{Res}_{s=a_\nu} \{e^{ts} F(s)\},$$

valid under the Closing Contour Condition (6.3), i.e., for example,

$$\left|F(s)\right| \le \frac{M}{|s|^\alpha} \quad (\alpha > 0) \text{ for } |s| \to \infty,$$

now yields several forms of expansion theorems. First, the residues are computed. The residue at a_ν is determined by the principal part of the Laurent expansion of $F(s)$ at a_ν. If $F(s)$ has a pole of order n, we have

$$F(s) = \frac{c_{\nu n}}{(s-a_\nu)^n} + \frac{c_{\nu n-1}}{(s-a_\nu)^{n-1}} + \cdots + \frac{c_{\nu 1}}{s-a_\nu} + \varphi(s)$$

where $\varphi(s)$ is holomorphic at a_ν. From this, we obtain

$$\operatorname*{Res}_{s=a_\nu} \{e^{ts} F(s)\} = \sum_{k=1}^{n} c_{\nu k} \frac{1}{2\pi i} \int_{\gamma_\nu} \frac{e^{ts}}{(s-a_\nu)^k}\, ds,$$

where the integration contour γ_ν is a small circle around a_ν traversed counter-clockwise. Cauchy's integral theorem yields

$$\frac{1}{2\pi i} \int_{\gamma_\nu} \frac{e^{ts}}{(s-a_\nu)^k}\, ds = \frac{1}{(k-1)!} \left(\frac{d^{k-1}}{ds^{k-1}} \{e^{ts}\}\right)_{s=a_\nu} = \frac{t^{k-1} e^{ta_\nu}}{(k-1)!}.$$

Therefore, if $F(s)$ has a pole of order n at a_ν,

$$f_\nu(t) = \sum_{k=1}^{n} \frac{c_{\nu k}}{(k-1)!} t^{k-1} e^{ta_\nu}. \tag{6.5}$$

The coefficients $c_{\nu k}$ are determined as follows:

$$(s-a_\nu)^n F(s) = c_{\nu n} + c_{\nu n-1}(s-a_\nu) + \cdots + c_{\nu 1}(s-a_\nu)^{n-1} + \varphi(s)(s-a_\nu)^n .$$

From this we conclude

$$c_{\nu n} = \{(s-a_\nu)^n F(s)\}_{s=a_\nu}, \quad c_{\nu n-1} = \frac{d}{ds}\{(s-a_\nu)^n F(s)\}_{s=a_\nu},$$

$$c_{\nu n-2} = \frac{1}{2!}\frac{d^2}{ds^2}\{(s-a_\nu)^n F(s)\}_{s=a_\nu}, \ \ldots,$$

$$c_{\nu 1} = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\{(s-a_\nu)^n F(s)\}_{s=a_\nu}.$$

Now assume that $F(s) = Q(s)/P(s)$ where $Q(s)$ and $P(s)$ are entire functions and the denominator function $P(s)$ has only simple zeros. Then the meromorphic function $F(s)$ has only poles of the first order. Hence

$$c_{\nu 1} = \{(s-a_\nu)F(s)\}_{s=a_\nu} = \left\{(s-a_\nu)\frac{Q(s)}{P(s)}\right\}_{s=a_\nu}$$

$$= \left\{\frac{Q(s)}{\frac{P(s)-P(a_\nu)}{s-a_\nu}}\right\}_{s=a_\nu} = \lim_{s\to a_\nu}\frac{Q(s)}{\frac{P(s)-P(a_\nu)}{s-a_\nu}} = \frac{Q(a_\nu)}{P'(a_\nu)}.$$

This gives the first expansion theorem.

Expansion Theorem I. *Let $F(s) = Q(s)/P(s)$ be the quotient of two entire functions. If the denominator has only simple zeros a_ν ($\nu = 0, 1, 2, \ldots$), and if the Closing Contour Condition (6.3) for $F(s)$ is satisfied, for example, if*

$$\left|\frac{Q(s)}{P(s)}\right| \leq \frac{M}{|s|^\alpha} \qquad (\alpha > 0)$$

as $|s| \to \infty$, then

$$f(t) = \sum_{\nu=0}^{\infty}\frac{Q(a_\nu)}{P'(a_\nu)}e^{ta_\nu}. \tag{6.6}$$

Next assume that $F(s) = Q(s)/(s^m P(s))$ where $Q(s)$ and $P(s)$ are entire functions, and the denominator function $P(s)$ has only simple zeros which are all different from 0. The meromorphic function $F(s)$ then has only poles of the first order at $a_\nu \neq 0$ and a pole of order m at $a_0 = 0$. The contribution from $a_\nu \neq 0$ ($\nu \geq 1$) to the sum of residues is

$$\frac{Q(a_\nu)/a_\nu^m}{P'(a_\nu)}e^{ta_\nu},$$

while the contribution from the zero of order m at 0 is

$$f_0(t) = \sum_{k=1}^{n}\frac{c_{0k}}{(k-1)!}t^{k-1}$$

with

$$c_{0n} = \left\{\frac{Q(s)}{P(s)}\right\}_{s=0} = \frac{Q(0)}{P(0)},$$

$$c_{\nu n-1} = \frac{d}{ds}\left\{\frac{Q(s)}{P(s)}\right\}_{s=0} = \frac{Q'(0)P(0) - Q(0)P'(0)}{P(0)^2},$$

$$c_{\nu n-2} = \frac{1}{2!}\frac{d^2}{ds^2}\left\{\frac{Q(s)}{P(s)}\right\}_{s=0}, \ \cdots,$$

$$c_{\nu 1} = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left\{\frac{Q(s)}{P(s)}\right\}_{s=0}.$$

For $m = 1$, we have

Expansion Theorem II.

$$f(t) = \frac{Q(0)}{P(0)} + \sum_{\nu=1}^{\infty} \frac{Q(a_\nu)}{a_\nu P'(a_\nu)} e^{t a_\nu} \tag{6.7}$$

provided, the Closing Contour Condition (6.3) is met, with, for example,

$$\left|\frac{Q(s)}{P(s)}\right| \leq M |s|^{\alpha} \qquad (\alpha < 1) \text{ for } |s| \to \infty$$

or, less restrictively, if $Q(s)/(sP(s))$ tends to zero on the left semi-circles, uniformly with respect to the argument, if the radii of the semi-circles tend to infinity.

For $m = 2$,

$$f(t) = \frac{Q(0)}{P(0)} t + \frac{Q'(0)P(0) - Q(0)P'(0)}{P(0)^2} + \sum_{\nu=1}^{\infty} \frac{Q(a_\nu)}{a_\nu^2 P'(a_\nu)} e^{t a_\nu} \tag{6.8}$$

provided, for example,

$$\left|\frac{Q(s)}{P(s)}\right| \leq M |s|^{\alpha} \qquad (\alpha < 2) \text{ for } |s| \to \infty.$$

The expansions (6.6), (6.7) and (6.8) are in a convenient form if the zeros are real. Often, the meromorphic image function has a finite number of real zeros and an infinite number of pairs of complex conjugate zeros. Moreover in a real world problem, the given image function is real in the sense that $F(\bar{s}) = \overline{F(s)}$. This implies that the coefficients of the principal part corresponding to the pole $\overline{a}_\nu$ are the conjugates of the coefficients corresponding to the pole a_ν. The contribution of such a pair of poles in (6.5) is

$$\frac{c_{\nu k}}{(k-1)!} t^{k-1} e^{t a_\nu} + \frac{\overline{c_{\nu k}}}{(k-1)!} t^{k-1} e^{t \overline{a_\nu}} = \frac{t^{k-1}}{(k-1)!}\{c_{\nu k} e^{t a_\nu} + \overline{c_{\nu k}} e^{t \overline{a_\nu}}\}$$

$$= \frac{2\, t^{k-1}}{(k-1)!} \Re\,[c_{\nu k} e^{t a_\nu}] = \frac{2\, t^{k-1}}{(k-1)!} e^{t \Re a_\nu}\{\Re\, c_{\nu k} \cos(a_\nu t) - \mathrm{I} c_{\nu k} \sin(a_\nu t)\}.$$

Therefore, in the case where the real meromorphic image function $F(s)$ has a pair of complex conjugate poles $(a_\nu, \overline{a}_\nu)$ of order n, (6.5) is to be replaced by

$$f_\nu(t) = 2 \sum_{k=1}^{n} \frac{t^{k-1}\, e^{t\,\mathrm{R}\, a_\nu}}{(k-1)!} \{\mathrm{R}\, c_{\nu k} \cos(a_\nu t) - \mathrm{Ic}_{\nu k} \sin(a_\nu t)\}. \tag{6.9}$$

Example. Assume that $F(s) = Q(s)/(s^m P(s))$ where $Q(s)$ and $P(s)$ are entire functions and the denominator $P(s)$ has only simple zeros that are different from 0 and evenly spaced along the positive part and the negative part of the imaginary axis, i.e.,

$$a_\nu = i\nu\omega \quad (\nu = \pm 1, \pm 2, \ldots).$$

The meromorphic function $F(s)$ then has a pair of complex conjugate poles a_ν, $\overline{a}_\nu$ of the first order and a real pole of order m at $a_0 = 0$. The expansion for $m = 1$ becomes

$$f(t) = \frac{Q(0)}{P(0)} + \frac{2}{\omega} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \{B_\nu(\omega) \cos[\nu\omega t] + A_\nu(\omega) \sin[\nu\omega t]\}$$

where

$$\frac{Q(i\nu\omega)}{P'(i\nu\omega)} =: A_\nu(\omega) + iB_\nu(\omega).$$

For $m = 2$,

$$f(t) = \frac{Q(0)}{P(0)} t + \frac{Q'(0)P(0) - Q(0)P'(0)}{P(0)^2} - \frac{2}{\omega^2} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \{A_\nu(\omega) \cos[\nu\omega t] - B_\nu(\omega) \sin[\nu\omega t]\},$$

provided that the Closing Contour Condition (6.3)

$$\sup_{s \in A_\nu} \frac{Q(s)}{s^2 P(s)} \to 0$$

holds, as the radius of the left semi-circle A_ν tends to infinity.

An example is given by

$$F(s) = \frac{\sinh x s}{s^2 \sinh a s}, \quad (0 \le x \le a).$$

There are a pole of order 2 at $s = 0$ and pairs of simple complex conjugate poles $a_\nu = i\nu\pi/a$ ($\nu = \pm 1, \pm 2, \ldots$). The condition $0 \le x \le a$ ensures that $\sinh x s / \sinh a s$ remains bounded on the left semi-circles, if their radii tend to infinity. This implies that the Closing Contour Condition (6.3) is met.

$$\frac{Q(0)}{P(0)} = \lim_{s \to 0} \frac{\sinh x s}{\sinh a s} = \frac{x}{a}; \quad \frac{Q'(0)P(0) - Q(0)P'(0)}{P(0)^2} = 0,$$

$$\frac{Q(i\nu\frac{\pi}{a})}{P'(i\nu\frac{\pi}{a})} =: A_\nu\left(\frac{\pi}{a}\right) + iB_\nu\left(\frac{\pi}{a}\right) = i\frac{(-1)^\nu}{a} \sin\left(\nu \frac{\pi}{a} x\right)$$

$$\Longrightarrow A_\nu\left(\frac{\pi}{a}\right) = 0, \quad B_\nu\left(\frac{\pi}{a}\right) = \frac{(-1)^\nu}{a} \sin\left(\nu \frac{\pi}{a} x\right).$$

Finally,

$$f(t) = \mathcal{L}^{-1}\left[\frac{\sinh x\,s}{s^2 \sinh a\,s}\right](t)$$

$$= \frac{x}{a}\,t + \frac{2a}{\pi^2}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu^2}\sin\left(\nu\,\frac{\pi}{a}\,x\right)\sin\left(\nu\,\frac{\pi}{a}\,t\right).$$

Example.

$$F(s) = \frac{\cosh x\sqrt{s}}{\sqrt{s}\sinh a\sqrt{s}}, \quad (0 \le x \le a).$$

At first glance, this image function seems to have a branch point at $s = 0$ due to the square root and therefore is not a meromorphic function. But the Taylor expansions of the numerator and the denominator show that this is not the case and we again have a quotient of two entire functions with a simple zero at $s = 0$.

$$\frac{\cosh x\sqrt{s}}{\sqrt{s}\sinh a\sqrt{s}} = \frac{\cosh x\sqrt{s}}{s\left(\sinh a\sqrt{s}\,/\sqrt{s}\right)}$$

$$\Longrightarrow Q(s) = \cosh x\sqrt{s} = 1 + \frac{x^2}{2}\,s + \cdots$$

$$\Longrightarrow P(s) = \frac{\sinh a\sqrt{s}}{\sqrt{s}} = a + \frac{a^2}{6}\,s + \cdots.$$

The other simple poles are located along the negative real axis

$$a_\nu = -\frac{\pi^2}{a^2}\,\nu^2 \quad (\nu = 1, 2, \ldots).$$

We then have

$$f(t) = \frac{Q(0)}{P(0)} + \sum_{\nu=1}^{\infty}\frac{Q(a_\nu)}{a_\nu\,P'(a_\nu)}\,e^{t\,a_\nu}$$

$$= \frac{1}{a} - \frac{a^2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{Q\left(-\frac{\pi^2}{a^2}\nu^2\right)}{\nu^2\,P'\left(-\frac{\pi^2}{a^2}\nu^2\right)}\exp\left[-\frac{\pi^2}{a^2}\,\nu^2\,t\right],$$

$$Q\left(-\frac{\pi^2}{a^2}\,\nu^2\right) = \cosh\left(x\,\frac{\pi}{a}\,i\,\nu\right) = \cos\left(x\,\frac{\pi}{a}\,\nu\right),$$

$$P'(s) = \frac{a\cosh\left(a\sqrt{s}\right) - \sinh\left(a\sqrt{s}\right)/\sqrt{s}}{2\,s}$$

$$\Longrightarrow P'\left(-\frac{\pi^2}{a^2}\,\nu^2\right) = -\frac{a^3}{2\,\pi^2\,\nu^2}\,(-1)^\nu.$$

Finally, since the Closing Contour Condition (6.3) is also satisfied,

$$f(t) = \frac{1}{a} + \frac{2}{a}\sum_{\nu=1}^{\infty}(-1)^{\nu}\exp\left[-\frac{\pi^2}{a^2}\nu^2 t\right]\cos\left(x\frac{\pi}{a}\nu\right).$$

■ 6.1.2 Use of the Package

In Section 3.5.2 the inverse Laplace transform of meromorphic image functions was introduced. Now we explore several commands that are helpful in computing inverse Laplace transforms by use of the calculus of residues.

6.1.2.1 Direct Computation of Residues

The `ResidueLaplace[F[s],{a1,a2,...},s,t]` command computes the residues of the meromorphic function $F(s)\exp(s\,t)$ at the specified poles a1, a2, ... and adds them up. The next example is an instructive one.

```
F = Exp[-a s] / ((s - 2) (s^2 + 4));
```

Of course, we can directly invert this expression by use of the NamedRule `ImageRules` and obtain

```
InverseLaplaceTransform[F, s, t] /. ImageRules
```

$$\left(\frac{1}{8}e^{2(-a+t)} + 2\left(-\frac{1}{16}\text{Cos}[2(-a+t)] - \frac{1}{16}\text{Sin}[2(-a+t)]\right)\right)\text{UnitStep}[-a+t]$$

The meromorphic function given has simple poles at 2, $2\,i$, $-2\,i$. We compute the sum of the residues at these poles and obtain:

```
ResidueLaplace[F, {2 I, -2 I, 2}, s, t] // ComplexExpand
```

$$\frac{1}{8}e^{-2a+2t} - \frac{1}{8}\text{Cos}[2a-2t] + \frac{1}{8}\text{Sin}[2a-2t]$$

How come we do not get the same result? The reason is that the Closing Contour Condition (6.3) is not satisfied! However, we observe that for $t \geq a$, the two computations give the same result. This is not a coincidence, as we will see later when talking about asymptotics. In fact, the second expression obtained by adding up the contributions of the residues yields the correct asymptotic expansion for large values of t of the original (see Section 6.1.3.2). A proper rational image function always meets the Closing Contour Condition (6.3) and the inverse Laplace transform can be computed by `ResidueLaplace`.

```
F = (s + 3) / ((s + 1)^3 (s^2 + 2 s + 17));
```

The poles are -1, $-1+4\,i$, $-1-4\,I$.

```
ResidueLaplace[F, {-1, -1 + 4 I, -1 - 4 I}, s, t] // ComplexExpand
```

$$-\frac{e^{-t}}{128} + \frac{e^{-t}t}{16} + \frac{1}{16}e^{-t}t^2 + \frac{1}{128}e^{-t}\text{Cos}[4t] - \frac{1}{64}e^{-t}\text{Sin}[4t]$$

You may check yourself that the same result is obtained by a direct evaluation via `InverseLaplaceTransform[F,s,t]`.

6.1.2.2 Evaluation by an Expansion Theorem

Let $F(s) = Q(s)/(s^m P(s))$ be a meromorphic function, where $P(s)$ has only simple zeros all different from 0. Furthermore, let the Closing Contour Condition (6.3) be satisfied so that the inverse Laplace transform can be evaluated by the calculus of residues. This is done by the following command: `ExpansionTheorem[Q[s],s^m P[s],{a1,a2,...},{χ1+iω1,χ2+iω2,...},s,t]`. Here a1, a2, ... are the real poles and $\chi 1+i\omega 1, \chi 2+i\omega 2, \ldots$ are the nonreal poles with positive imaginary parts. It is assumed that the nonreal poles occur in pairs of complex conjugates. If there are infinitely many zeros of $P(s)$ defined by a formula, the corresponding list can be given in the form `Hold[Table[[a[k],{k,k0, Infinity}]]]`. Use {} for empty lists. Let us test these commands by the way of the last example of Section 6.1.1.1:

$$F(s) = \frac{\cosh x\sqrt{s}}{\sqrt{s}\sinh a\sqrt{s}} = \frac{\cosh x\sqrt{s}}{s\,\frac{\sinh a\sqrt{s}}{\sqrt{s}}}, \quad (0 \le x \le a).$$

The zeros of $P(s) = \sinh a\sqrt{s}\,/\sqrt{s}$ are real and there are no nonreal zeros: $a_\nu = -\pi^2/(a^2\nu^2)$ $(\nu = 1, 2, \ldots)$. Observe that the inserted `Unevaluated` prevents evaluation until the pattern matching is done.

```
ExpansionTheorem[Cosh[x Sqrt[s]],
  Unevaluated[s Sinh[a Sqrt[s]] / Sqrt[s]],
  Hold[Table[-Pi^2 k^2 / a^2, {k, 1, Infinity}]], {}, s, t] //
 PowerExpand /. {Cos[n5 Pi] → (-1)^n5, Sin[n5 Pi] → 0}
```

$$\frac{1}{a} + \text{Summation}\left[-\frac{a^2\, e^{-\frac{n1^2\pi^2 t}{a^2}}\,\text{Cos}\left[\frac{n1\pi x}{a}\right]}{n1^2\pi^2\left(-\frac{a^3\,\text{Cos}[n1\pi]}{2\,n1^2\pi^2}+\frac{a^3\,\text{Sin}[n1\pi]}{2\,n1^3\pi^3}\right)}, \{n1, 1, \infty\}\right]$$

Another example is

$$F(s) = \frac{\text{Cosh}[x\,s]}{s^3\,\text{Cosh}[a\,s]}, \quad (0 \le x \le a)$$

```
ExpansionTheorem[Cosh[x s], s^3 Cosh[a s], {},
 Hold[Table[I (1 / 2 + k) Pi / a, {k, 0, Infinity}]], s, t]
```

$$\frac{1}{2}\left(-a^2 + t^2 + x^2\right) +$$

$$2\,\text{Summation}\left[\frac{8\,(-1)^k\,a^2\,\text{Cos}\left[\frac{(\pi+2k\pi)\,t}{2a}\right]\text{Cos}\left[\frac{\left(\frac{1}{2}+k\right)\pi x}{a}\right]}{(\pi+2k\pi)^3}, \{k, 0, \infty\}\right]$$

The `ExpansionTheorem` procedure is not necessary for the two examples. You may check that you obtain the same result by directly using

```
InverseLaplaceTransform[Cosh[x s] / (s^3 Cosh[a s]), s, t];
```

However, the `ExpansionTheorem` procedure is more flexible, for example:

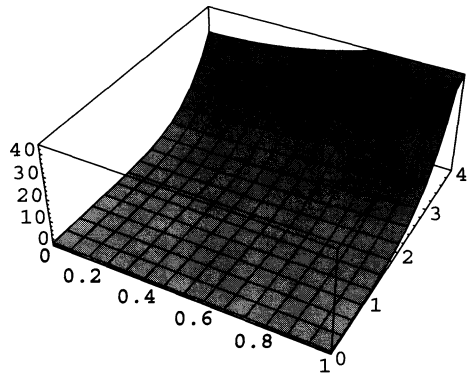
$$F(s) = \frac{(s+2)\,\text{Cosh}[x\,s]}{s^3\,(s^2-1)\,\text{Cosh}[a\,s]}, \quad (0 \le x \le a)$$

where, in addition to the pole of order 3 at the origin, there are two real simple poles 1,–1 and infinitely many complex conjugate poles at $i\,(1/2 + k)\,\pi$, $k \in \mathbb{Z}$.

```
ExpansionTheorem[(s + 2) Cosh[x s], s^3 (s^2 - 1) Cosh[a s],
  {1, -1}, Hold[Table[I (1 / 2 + k) Pi / a, {k, 0, Infinity}]], s, t];
```

Let us take the first ten terms of the series and plot the resulting approximation as a function of two variables t and x, assuming, for example, $a = 1$.

```
f = % /. a → 1 /. TruncateSummation[9];

Plot3D[f, {x, 0, 1}, {t, 0, 4}];
```



Again, you can use the `ExpansionTheorem` command to invert a rational image function of the type assumed at the beginning of this section. Try yourself the following example:

```
F = (s^2 + s + 1) / (s^4 (s - a) (s^2 + 2 s + 1 + ω^2))
```

$$\frac{1 + s + s^2}{s^4\,(-a + s)\,(1 + 2\,s + s^2 + \omega^2)}$$

```
ExpansionTheorem[s^2 + s + 1,
  s^4 (s - a) (s^2 + 2 s + 1 + ω^2) , {a}, {-1 + I ω}, s, t];
```

You may check that the large output (not displayed here) is in a nicer form than the one obtained by the `InverseLaplaceTransform[F,s,t]` command.

The principal use of the latter form of the `ExpansionTheorem` procedure is in computing approximate solutions of certain partial differential equations. There the solution appears as an infinite sum of residues at poles, which first have to be evaluated numerically. Examples are given in Chapter 10 that deal with problems of heat conduction.

A word of caution: It is up to the user to control whether or not the Closing Contour Condition (6.3) is met! We shall see in Section 6.1.3.2 what the computed series represents if the Closing Contour Condition fails to hold.

■ 6.1.3 Translation and Deformation of the Vertical Integration Contour

The vertical integration contour $\Im s = c$ of the complex inversion formula may be any parallel to the imaginary axis, as long as this straight line remains inside the right half-plane of absolute convergence $\Re s > x_1$. We then have

$$f(t) = \mathrm{pv}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\,F(s)\,ds \qquad (t > 0)$$

provided that $f(t)$ is of bounded variation and continuous in a neighborhood of t.

6.1.3.1 Left Translation of the Integration Contour

First suppose that $F(s)$ has a finite number m of poles a_i on the left of the abscissa of absolute convergence x_1, followed by non-isolated singularities farther to the left. In order to settle the situation, we assume that

$$\Re a_m < \Re a_{m-1} < \cdots < \Re a_2 < \Re a_1 < x_1 < c.$$

We consider the following closed rectangular integration contour C_1: $c - Ri \to c + Ri \to \beta_1 + Ri \to \beta_1 - Ri \to c - Ri$ where $\Re a_2 < \beta_1 < \Re a_1$. By Cauchy's theorem of residues

$$\frac{1}{2\pi i} \int_{C_1} e^{ts} F(s)\, ds = f_1(t) := \operatorname*{Res}_{a_1} \{e^{ts} F(s)\}.$$

In Section 6.1.1.1 we showed that, if

$$\frac{c_{1n}}{(s-a_1)^n} + \frac{c_{1n-1}}{(s-a_1)^{n-1}} + \cdots + \frac{c_{11}}{s-a_1}$$

is the principal part of the Laurent expansion of $F(s)$ at $s = a_1$, we can write

$$f_1(t) = \sum_{k=1}^{n} \frac{c_{1k}}{(k-1)!}\, t^{k-1} e^{ta_1}.$$

Now let us make the following assumption: On the horizontal parts of the integration contour, $F(s) \to 0$ uniformly with respect to $\beta_1 < \Re s < c$ as $R \to \infty$. This is for example the case if

$$\left| F(x + iy) \right| \le \frac{M_1}{(|y|)^{\sigma_1}}, \quad \sigma_1 > 0,\ \beta_1 < x < x_1,\ |y| > Y_1.$$

Then the contributions to the contour integral on the two horizontal contours tend to zero as $R \to \infty$, and

$$\operatorname{pv} \frac{1}{2\pi i} \int_{\beta_1 + i\infty}^{\beta_1 - i\infty} e^{ts} F(s)\, ds + \operatorname{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds = f_1(t).$$

Since the second integral equals $f(t)$,

$$f(t) = f_1(t) + \operatorname{pv} \frac{1}{2\pi i} \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} e^{ts} F(s)\, ds.$$

By iterating this procedure, we finally arrive at the formula

$$f(t) = \sum_{\nu=1}^{m} f_\nu(t) + \operatorname{pv} \frac{1}{2\pi i} \int_{\beta_m - i\infty}^{\beta_m + i\infty} e^{ts} F(s)\, ds \tag{6.10}$$

where it is supposed that the above assumption on the behavior of $F(s)$, as $R \to \infty$, is extended to each strip $\beta_{\nu+1} < \Re s < c$ with $\Re a_{\nu+1} < \beta_\nu < \Re a_\nu$ ($\nu = 1, \ldots, m$, $\Re a_{m+1} = -\infty$). β_m is now an abscissa on the left of the left-most pole a_m of $F(s)$. Also,

$$f_\nu(t) = \sum_{k=1}^{n} \frac{c_{\nu k}}{(k-1)!}\, t^{k-1}\, e^{t a_\nu}$$

where

$$\frac{c_{\nu n}}{(s-a_\nu)^n} + \frac{c_{\nu n-1}}{(s-a_\nu)^{n-1}} + \cdots + \frac{c_{\nu 1}}{s-a_\nu}$$

is the principal part of the Laurent expansion of $F(s)$ at $s = a_\nu$.

We call the procedure resulting in (6.10) the *"left translation of the integration contour by pole skipping"*.

Before we go on, let us examine the integral in (6.10). We can write it as

$$\begin{aligned} &\mathrm{pv}\,\frac{1}{2\pi i}\int_{\beta_m - i\infty}^{\beta_m + i\infty} e^{ts}\, F(s)\, ds \\ &= e^{\beta_m t}\,\mathrm{pv}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ity}\, F(\beta_m + iy)\, dy. \end{aligned}$$

For any $R > 0$, we have

$$\mathrm{pv}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ity}\, F(\beta_m + iy)\, dy = \int_{-\infty}^{-R} \ldots dy + \int_{-R}^{R} \ldots dy + \int_{R}^{\infty} \ldots dy.$$

If we assume that the improper integral

$$\int_{-\infty}^{\infty} e^{ity}\, F(\beta_m + iy)\, dy$$

converges uniformly with respect to $t > T$, we can make the first and the last integral above as small as we wish, by choosing R sufficiently large. For the integral in the middle, we have

$$\int_{-R}^{R} e^{ity}\, F(\beta_m + iy)\, dy \to 0 \text{ as } t \to \infty$$

due to the lemma of Riemann–Lebesgue. Therefore, if the integral converges uniformly with respect to t, we can write

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} e^{ity}\, F(\beta_m + iy)\, dy = 0.$$

This means that

$$\mathrm{pv}\,\frac{1}{2\pi i}\int_{\beta_m - i\infty}^{\beta_m + i\infty} e^{ts}\, F(s)\, ds = o(e^{\beta_m t}) \qquad (6.11)$$

as $t \to \infty$.

Uniform convergence with respect to t is certainly fulfilled if we have

$$\int_{-\infty}^{\infty} |\, F(\beta_m + iy)\,|\, dy < \infty,$$

but also if, for example, only

$$F(\beta_m + i\,y) = \frac{1}{(\beta_m + i\,y)^\sigma} \quad (\sigma > 0)$$

holds. This is because integration by parts yields

$$\begin{aligned}
&\int_R^\infty \frac{e^{ity}}{(\beta_m + i\,y)^\sigma}\,dy \\
&= \left(\frac{e^{ity}}{it}\,\frac{1}{(\beta_m + i\,y)^\sigma}\right)_{y=R}^{y\to\infty} + \frac{\sigma}{t}\int_R^\infty \frac{e^{ity}}{(\beta_m + i\,y)^{\sigma+1}}\,dy \\
&= \frac{e^{itR}}{it}\,\frac{1}{(\beta_m + iR)^\sigma} + \frac{\sigma}{t}\int_R^\infty \frac{e^{ity}}{(\beta_m + i\,y)^{\sigma+1}}\,dy.
\end{aligned}$$

For the last integral, we have the estimate

$$\begin{aligned}
&\left|\int_R^\infty \frac{e^{ity}}{(\beta_m + i\,y)^{\sigma+1}}\,dy\right| \le \int_R^\infty \frac{1}{|\,\beta_m + i\,y\,|^{\sigma+1}}\,dy \le \int_R^\infty \frac{1}{y^{\sigma+1}}\,dy \\
&= \frac{1}{\sigma R^\sigma}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\int_R^\infty \frac{e^{ity}}{(\beta_m + i\,y)^\sigma}\,dy\right| \le \frac{1}{T}\left\{\frac{1}{|\,\beta_m + iR\,|^\sigma} + \frac{1}{R^\sigma}\right\} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

uniformly with respect to $t \ge T > 0$.

More generally and by the use of Abel's theorem, it can be shown that (6.11) is satisfied if the real and imaginary part of $F(\beta_m + i\,y)$ are strictly positive / negative functions of y which are decreasing / increasing towards 0 as $y \to \infty$.

6.1.3.2 A Theorem on Asymptotic Expansion

We will deal with asymptotics in more detail in Section 6.2.2. But the result of the last section enables us to formulate a theorem concerning the asymptotic expansion of the original as $t \to \infty$ if all singularities of the image function are poles.

In Section 6.2.2 we shall define that a sequence of functions $r_n(t)$ $(n = 0, 1, 2, \ldots)$ is called an *asymptotic sequence or a scale as* $t \to \infty$, if for all n,

$$r_{n+1}(t) = o[r_n(t)] \quad (t \to \infty),$$

i.e. $\lim_{t\to\infty} r_{n+1}(t)/r_n(t) = 0$.

Also, a function $f(t)$ has the *asymptotic expansion with respect to the scale* $r_n(t)$ *as* $t \to \infty$, denoted by

$$f(t) \sim \sum_{k=0}^{\infty} c_k\, r_k(t) \qquad (t \to \infty),$$

if for each nonnegative integer n,

$$f(t) = \sum_{k=0}^{n} c_n\, r_n(t) + o[r_n(t)] \qquad (t \to \infty).$$

Now, if $\beta_n < a_n$, then $f(t) = o[e^{\beta_n t}] \Longrightarrow f(t) = o[p_n(t)\ e^{a_n t}]$ for any polynomial $p_n(t)$, since

$$\frac{f(t)}{p_n(t)\ e^{a_n t}} = \frac{e^{-(a_n-\beta_n)t}}{p_n(t)} \frac{f(t)}{e^{\beta_n t}} \to 0 \quad (t \to \infty).$$

Under the assumptions of Section 6.1.3.1 and by the use of (6.10) and (6.11) we can write for an integer n,

$$f(t) = \sum_{\nu=1}^{n} f_\nu(t) + o(e^{\beta_n t}).$$

Here

$$f_\nu(t) = \sum_{k=1}^{m} \frac{c_{\nu k}}{(k-1)!}\, t^{k-1}\, e^{t a_\nu} \tag{6.12}$$

where

$$\frac{c_{\nu m}}{(s-a_\nu)^m} + \frac{c_{\nu m-1}}{(s-a_\nu)^{m-1}} + \cdots + \frac{c_{\nu 1}}{s-a_\nu}$$

is the principal part of the Laurent expansion of $F(s)$ at a_ν. Therefore,

$$f(t) = \sum_{\nu=1}^{n} \left\{ \frac{c_{\nu m}}{(k-1)!}\, t^{m-1} + \frac{c_{\nu m-1}}{(k-2)!}\, t^{m-2} + \cdots + c_{\nu 1} \right\} e^{t a_\nu} + o[f_n(t)]$$

as $(t \to \infty)$.

Since the foregoing considerations remain valid if the condition

$$\cdots < a_{n+1} < a_n < a_{n-1} < \cdots < a_2 < a_1 < a_0 < x_1 < c$$

is replaced by

$$\cdots < \mathrm{R}\, a_{n+1} < \mathrm{Ra}_n < \cdots < \mathrm{Ra}_2 < \mathrm{Ra}_1 < \mathrm{Ra}_0 < x_1 < c,$$

we have proved the following

Proposition. *Suppose that the only singularities of the image function $F(s)$ are poles a_ν with*

$$\cdots < \mathrm{R}\, a_{n+1} < \mathrm{Ra}_n < \cdots < \mathrm{Ra}_2 < \mathrm{Ra}_1 < \mathrm{Ra}_0 < x_1 < c$$

where

$$\frac{c_{\nu m}}{(s-a_\nu)^m} + \frac{c_{\nu m-1}}{(s-a_\nu)^{m-1}} + \cdots + \frac{c_{\nu 1}}{s-a_\nu}$$

is the principal part of the Laurent expansion of $F(s)$ about a_ν. If x_1 is the abscissa of absolute convergence of $f(t)$, then

$$f(t) = \operatorname{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds.$$

Furthermore, let us assume that for every β_ν with $\Re a_{\nu+1} < \beta_\nu < \Re a_\nu$ the following two conditions are satisfied:

1. *In each vertical strip $\beta_\nu < \Re s < c$ as $|\Im s| \to \infty$, $F(s) \to 0$ uniformly with respect to $\Re s$.*
2. *The integral*

$$\int_{-\infty}^{\infty} e^{ity} F(\beta_\nu + iy)\, dy$$

converges uniformly with respect to t $(t > T)$.

Under these assumptions the asymptotic expansion

$$f(t) \sim \sum_{\nu=0}^{\infty} \left\{ \frac{c_{\nu m}}{(k-1)!} t^{m-1} + \frac{c_{\nu m-1}}{(k-2)!} t^{m-2} + \cdots + c_{\nu 1} \right\} e^{ta_\nu} \tag{6.13}$$

for $t \to \infty$, with the scale (6.12), holds.

The above assumptions 1 and 2 are satisfied, for example, if for each ν there is an estimate of the form

$$\left| F(x+iy) \right| \le \frac{M(\beta_\nu)}{(|y|)^\sigma} \quad (\sigma > 0),\ \beta_\nu \le x \le c.$$

The theorem shows that if the Closing Contour Condition 3 of Section 6.1.1 is replaced by the above weaker conditions 1 and 2, we no longer necessarily obtain a convergent series representation, but only an asymptotic expansion of $f(t)$ for $t \to \infty$.

The example at the end of Section 6.1.2.1 may serve as an illustration of this proposition. Another example is given by

```
F = (1 - Exp[-a s]) ^ 3 / (s ^ 2 + ω ^ 2) / s
```

$$\frac{(1 - e^{-a\,s})^3}{s\,(s^2 + \omega^2)}$$

By first expanding the numerator and then applying the original shift rule, the exact inverse Laplace transform can be computed and a lengthy expression will result. But if you are mainly interested in the behavior of the original for large values of time, the above theorem can be applied which amounts to adding up the contributions of the three residues at the poles at 0, $i\omega$, $-i\omega$.

```
ResidueLaplace[F, {0, I ω, -I ω}, s, t] // ComplexExpand // Simplify
```

$$\frac{8\,\text{Sin}[\frac{a\omega}{2}]^3\,\text{Sin}[\frac{3a\omega}{2} - t\omega]}{\omega^2}$$

According to the above theorem, this is the asymptotic expression for $t \to \infty$ of the original. In fact, it is the exact expression for $t \geq 3\,a$.

One more example for the above proposition is

$$F(s) = e^{s^2}\,\frac{\mathrm{Erfc}(s)}{s-a}\,.$$

The exact inverse Laplace transform is

```
InverseLaplaceTransform[Exp[s^2] Erfc[s] / (s - a), s, t] // Simplify
```

$$e^{a\,(a+t)}\left(-\mathrm{Erf}[a] + \mathrm{Erf}\left[a + \frac{t}{2}\right]\right)$$

The residue at $s = a$ of $e^{st}\,F(s)$ is

$$e^{at+a^2}\,\mathrm{Erfc}(a) = e^{at+a^2}\,[1 - \mathrm{Erf}(a)] = e^{at+a^2}\,[\mathrm{Erf}(a + t/2) - \mathrm{Erf}(a)] + r(t)$$

with

$$r(t) = e^{at+a^2}\,\frac{2}{\sqrt{\pi}}\int_{a+t/2}^{\infty} e^{-t^2}\,dt \to 0 \qquad (t \to \infty)$$

which by the above theorem, describes the asymptotic behavior of the inverse Laplace transform for large values of t. Finally a last example: Find the asymptotic expansion for large values of t for the inverse Laplace transform of

$$F(s) = e^{s^2}\,\frac{\mathrm{Erfc}(s)}{s^2\,(s^2 - a^2)}\,.$$

```
F = Exp[s^2] Erfc[s] / (s^2 - a^2) / s^2; ResidueLaplace[F, {0, a, -a}, s, t]
```

$$\frac{2 - \sqrt{\pi}\,t}{a^2\,\sqrt{\pi}} - \frac{e^{a^2+a\,t}\,(-1 + \mathrm{Erf}[a])}{2\,a^3} - \frac{e^{a^2-a\,t}\,(1 + \mathrm{Erf}[a])}{2\,a^3}$$

6.1.3.3 Bending the Vertical Integration Contour to the Left

Suppose that the image function $F(s)$ has a finite number m of poles a_i located to the left of the abscissa of absolute convergence x_1 and followed to the left by a branch point b, i.e. we assume

$$\Re b < \Re a_m < \Re a_{m-1} < \cdots < \Re a_2 < \Re a_1 < x_1.$$

If in the vertical strip $\beta_m < \Re s < x_1$, as $|\,\mathrm{Im}\,s\,| \to \infty$, $F(s) \to 0$ uniformly with respect to $\Re s$, we know from Section 6.1.3.1 that

$$f(t) = \sum_{\nu=1}^{m} f_\nu(t) + \mathrm{pv}\,\frac{1}{2\pi i}\int_{\beta_m - i\infty}^{\beta_m + i\infty} e^{ts}\,F(s)\,ds$$

where β_m is a real number such that $\Re b < \beta_m < \Re a_m$. If, in addition, the assumption of the uniform convergence of $F(s) \to 0$ as $|\,\mathrm{Im}\,s\,| \to \infty$ is extended to the adjacent strip $b \leq \Re s \leq \beta_m$, then

$$f(t) = \sum_{\nu=1}^{m} f_\nu(t) + \frac{1}{2\pi i} \int_{C_b} e^{ts} F(s)\, \boldsymbol{d}s.$$

The integration contour C_b is the vertical straight line $\Re s = b$ that has an indentation to the right of the branch point $s = b$. If, moreover, on the left semi-circles

$$C_R : z(\varphi) = b + R e^{i\varphi}, \quad \frac{\pi}{2} \le \varphi \le \frac{3\pi}{2},$$

$F(s) \to 0$ as $R \to \infty$, uniformly with respect to the argument of s, then, by Jordan's lemma,

$$f(t) = \sum_{\nu=1}^{m} f_\nu(t) + \frac{1}{2\pi i} \int_{A(b,\psi)} e^{ts} F(s)\, \boldsymbol{d}s, \tag{6.14}$$

where the integration contour $A(b, \psi)$ with $\pi/2 < \psi \le \pi$ traces an angle, that starts at $\infty\, e^{-i\psi}$ approaching $s = b$ along a straight line under the angle $-\psi$; bypassing the branch point $s = b$ to the right along a small circular arc and going back to $\infty\, e^{i\psi}$ along a straight line inclined at angle ψ. Fig.6.2 illustrates the situation. Let us call the complex inversion formula, where the vertical integration contour is replaced by an angle-contour $A(b, \psi)$, an angle-contour integral.

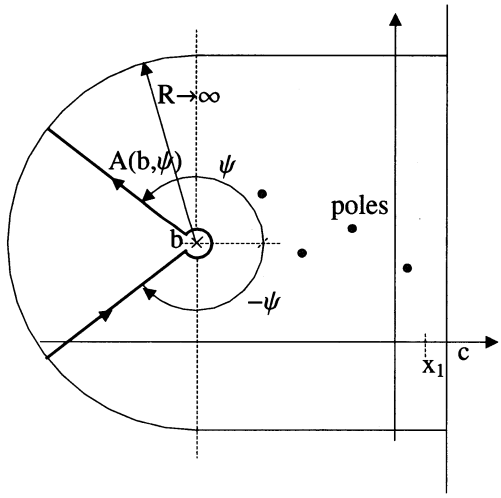


Fig.6.2: The integration contour of the angle-contour integral.

The above assumptions concerning the behavior of the image function are satisfied, for example, if

$$\left| F(s) \right| \le \frac{M}{|s|^{\sigma}} \text{ for } |s| \to \infty \qquad (\sigma > 0).$$

In the limit case where $\psi = \pi$, the angle-contour integral is also called a *Hankel's loop integral* or an integral of the Hankel type and the following notation is commonly used:

$$\frac{1}{2\pi i} \int_{-\infty}^{(b+)} e^{ts} F(s)\, \boldsymbol{d}s.$$

For a Hankel type integral with $b = 0$, the integration contour starts at $-\infty$ runs along and below the negative real axis towards $s = 0$, winds in the counter-clock wise around 0, and goes back above the negative real axis to -∞. A famous example for a loop integral of Hankel type is

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^s s^{-a}\, ds.$$

It can be used to define the gamma function for complex a. By replacing s by ts $(t > 0)$, we obtain an important formula

$$\frac{t^{a-1}}{\Gamma(a)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{ts} s^{-a}\, ds \tag{6.15}$$

or, by setting $a = -\gamma$,

$$\frac{1}{\Gamma(-\gamma)\, t^{\gamma+1}} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{ts} s^{\gamma}\, ds. \tag{6.16}$$

If γ is a nonnegative integer, the left-hand side of the last formula is to be replaced by 0. Otherwise γ can be any arbitrary complex number.

Example. Find the inverse Laplace transform of

$$F(s) = \frac{\log(s+b)}{s+a} \quad (b > a).$$

The given image function has a pole at $s = -a$ and a branch point at $s = -b$. In order to apply the above procedure we assume $-b < -a$, i.e. $b > a$. By the angle-contour integral with $\psi = \pi$,

$$f(t) = f_0(t) + \frac{1}{2\pi i}\int_{-\infty}^{(-b+)} e^{ts}\,\frac{\log(s+b)}{s+a}\, ds$$

where

$$f_0(t) = \operatorname*{Res}_{s=-a}\left\{e^{ts}\,\frac{\log(s+b)}{s+a}\right\} = e^{-ta}\log(-a+b).$$

Furthermore,

$$\frac{\log(s+b)}{s+a} = \frac{\log(s+b)}{s+b}\,\frac{s+b}{s+a}$$
$$= \frac{\log s + \log(1+b/s)}{s}\,\frac{1}{1+b/s}\,\frac{s+b}{s+a} \le M\,\frac{\log s}{s}$$

shows that the required uniform convergence behavior of the image function is satisfied. On the path from $-\infty$ to $-b$ and below the real axis, we have the contribution

$$\frac{1}{2\pi i}\int_{-\infty}^{-b} e^{tx}\,\frac{\log|x+b| - \pi i}{x+a}\, dx.$$

Back from $-b$ to $-\infty$ above the real axis the contribution is

$$\frac{1}{2\pi i}\int_{-b}^{-\infty} e^{tx}\,\frac{\log|x+b|+\pi i}{x+a}\,dx.$$

The contribution on the small circle with radius ρ around $s=-b$ tends to zero as $\rho\to 0$. Therefore

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{(-b+)} e^{ts}\,\frac{\log(s+b)}{s+a}\,dx\\ &=\frac{1}{2\pi i}\int_{-b}^{-\infty}\frac{e^{tx}}{x+a}\{-\log|x+b|+\pi i+\log|x+b|+\pi i\}\,dx\\ &=\int_{-b}^{-\infty}\frac{e^{tx}}{x+a}\,dx.\end{aligned}$$

By substituting $t(x+a)=-u$

$$\begin{aligned}&\int_{-b}^{-\infty}\frac{e^{tx}}{x+a}\,dx=e^{-at}\int_{-b}^{-\infty}\frac{e^{t(x+a)}}{x+a}\,dx\\ &=e^{-at}\int_{(b-a)t}^{\infty}\frac{e^{-u}}{-u/t}\,du/-t=-e^{-at}\,Ei[-(b-a)t]\end{aligned}$$

where

$$-Ei(-x):=\int_{x}^{\infty}\frac{e^{-u}}{u}\,du\quad(x>0)$$

is the exponential integral. Finally, we get $(b>a)$:

$$\frac{\log(s+b)}{s+a}\;\blacksquare\!-\!\square\;e^{-at}\{\log(b-a)-Ei[-(b-a)t]\}\,.$$

The assumption that the branch point has to lie on the left of the poles is not always necessary for applying the above method. This is illustrated in the next example.

Example. Solve the following initial value problem:

$$\begin{aligned}&y''+2\kappa y'+(1+\kappa^2)\,y(t)=\frac{1}{\sqrt{t}}\,e^{bt},\qquad (b\in\mathbb{R})\\ &y(0+)=y'(0)=0.\end{aligned}$$

It is readily seen that the Laplace transform of the solution is given by

$$\begin{aligned}Y(s)&=\frac{\sqrt{\pi}}{(s^2+2\kappa s+1+\kappa^2)\sqrt{s-b}}\\ &=\frac{\sqrt{\pi}}{[s-(-\kappa+i)]\,[s-(-\kappa-i)]\,\sqrt{s-b}}.\end{aligned}$$

We have a pair of complex conjugate poles $-\kappa\pm i$ and a branch point at b. We use the contour (Fig.6.2a).

Since the Closing Contour Condition (6.3) is fulfilled

$$y(t) = \frac{1}{2\pi i}\int_{-\infty}^{(b+)} e^{st}\,Y(s)\,ds + \operatorname*{Res}_{-\kappa\pm i}\{e^{st}\,Y(s)\}.$$

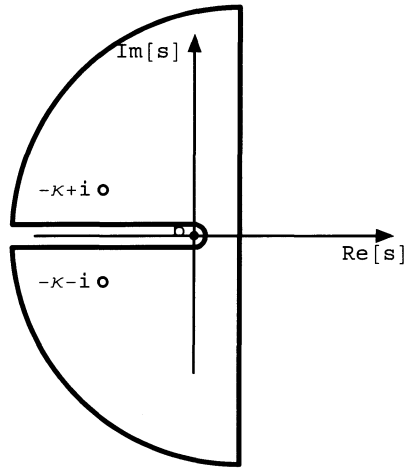


Fig.6.2a: Integration contour in the presence of a branch point b.

The computation of the two residues gives

$$\operatorname*{Res}_{-\kappa\pm i}\{e^{st}\,Y(s)\} = \sqrt{\pi}\,e^{-\kappa t}\,\mathrm{I}\left\{\frac{\cos t + i\sin t}{\sqrt{-A+i}}\right\}$$

$$= \sqrt{\pi}\,\frac{e^{-\kappa t}}{\sqrt{A^2+1}}\left\{-\sqrt{\frac{\sqrt{A^2+1}+A}{2}}\cos t + \sqrt{\frac{\sqrt{A^2+1}-A}{2}}\sin t\right\}$$

with $A = b + \kappa$. After a change of variable in the Hankel-type integral,

$$y(t) = \sqrt{\pi}\,\frac{e^{bt}}{2\pi i}\int_{-\infty}^{(0+)} \frac{e^{st}}{(s^2+2As+1+A^2-b^2)\sqrt{s}}\,ds$$

$$+\sqrt{\pi}\,\frac{e^{-\kappa t}}{\sqrt{A^2+1}}\left\{-\sqrt{\frac{\sqrt{A^2+1}+A}{2}}\cos t + \sqrt{\frac{\sqrt{A^2+1}-A}{2}}\sin t\right\}.$$

The Hankel-type integral can be transformed into a Laplace integral

$$\frac{e^{bt}}{2\pi i}\int_{-\infty}^{(0+)} \frac{e^{st}}{(s^2+2As+1+A^2-b^2)\sqrt{s}}\,ds$$

$$= \frac{e^{bt}}{\pi}\int_0^{\infty} \frac{e^{-xt}}{(x^2-2Ax+1+A^2-b^2)\sqrt{x}}\,dx.$$

The Hankel–typeintegral or its version as a Laplace integral can be used to find an asymptotic expression of the solution for large values of t (see below).

Example. Let us consider the linear flow of heat in a semi-infinite solid $x > 0$, if a periodic temperature, for example $A \cos \omega t$, is applied at the left boundary $x = 0$. The Laplace transform of the distribution of temperature $y(x, t)$ inside the solid is given by

$$Y(x, s) = A \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}/\kappa},$$

see Section 10.1.1.1. It is readily verified that the Closing Contour Condition (6.3) is satisfied (consider the modulus of $e^{-x\sqrt{s}/\kappa}$). If we use the same contour as in Fig.6.2.a, with the branch point $b = 0$ and the two poles at $\pm i\omega$ inside the contour, the theorem of residues yields

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}\, Y(x, t)\, ds - \int_{-\infty}^{(0+)} e^{st}\, Y(x, s)\, ds$$
$$= \operatorname*{Res}_{\{i\omega, -i\omega\}} \{e^{st}\, Y(x, s)\} .$$

Because the contributions to the contour integral on the two left quarter-circles tend to zero if their radii go to infinity. The first expression on the left-hand side is $y(x, t)$. The sum of the two residues is

```
ResidueLaplace[A s Exp[-x Sqrt[s] / κ] / (s^2 + ω^2), {-I ω, I ω}, s, t] //
   ComplexExpand // PowerExpand;

% /. {Arg[-I ω] -> -π / 2, Arg[I ω] -> π / 2}
```

$$A\, e^{-\frac{x\sqrt{\omega}}{\sqrt{2}\,\kappa}} \operatorname{Cos}\left[\frac{x\sqrt{\omega}}{\sqrt{2}\,\kappa} - t\,\omega\right]$$

Therefore, we have

$$y(x, t) = A\, e^{-x\sqrt{\omega}/\sqrt{2}\,\kappa} \cos\left[t\omega - \frac{x\sqrt{\omega}}{\sqrt{2}\,\kappa}\right]$$
$$+ A \int_{-\infty}^{(0+)} e^{st} \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}/\kappa}\, ds .$$

For the last term on the right-hand side an asymptotic expansion for large values of t can be found.

6.1.3.4 The Angle-Contour Integral

Passing from the complex inversion formula with a vertical integration path to the inversion integral with an angle-contour is an essential feature in the theory of Laplace transforms. It is worth it to hold on an instant and gather some striking properties implied by this step. The complex inversion formula is essentially a Fourier transform integral:

$$f(t) = \operatorname{pv} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts}\, F(s)\, ds = e^{ct} \operatorname{pv} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ity}\, F(c + iy)\, dy.$$

A Fourier integral has the following properties:

- it can represent rather general functions, for example functions with a jump-discontinuity,
- it cannot be differentiated under the integral sign.

In contrast, the angle-contour integral with $\pi/2 < \psi \leq \pi$ behaves quite differently, because it is essentially a Laplace integral and not a Fourier integral. This is seen as follows: On the upper ray $\overline{A}(b, \psi)$: $s(r) := b + r\,e^{i\psi}$ $(\rho < r < \infty)$, we may write

$$\int_{\overline{A}(b,\psi)} e^{t\,s}\,F(s)\,\boldsymbol{d}s = \int_{\rho}^{\infty} \exp[t\,(b + r\,e^{i\psi})]\;F(b + r\,e^{i\psi})\;e^{i\psi}\,\boldsymbol{d}r$$

$$= e^{t\,b+i\psi} \int_{\rho}^{\infty} \exp[-t\;r\,e^{i(\psi-\pi)}]\;F(b + r\,e^{i\psi})\;\boldsymbol{d}r.$$

This is essentially the Laplace transform of the function

$$f(r) := u(r-\rho)\;F(b + r\,e^{i\psi})$$

evaluated at $p = t\,e^{i(\psi-\pi)}$. Now if a Laplace integral converges for $p_0 = t_0\,e^{i(\psi-\pi)}$, it will also converge for all p with $\Re p = \Re\,[t\,e^{i(\psi-\pi)}] > \Re\,[t_0\,e^{i(\psi-\pi)}] = :\sigma_0$. But this defines a half-plane with the interior unit-normal $e^{i(\pi-\psi)}$. On the lower ray, we have a similar situation, only ψ has to be replaced by $-\psi$. If the Laplace integral converges for some $p_1 = t_1\,e^{i(-\psi-\pi)}$, it will also converge for all p with $\Re p = \Re\,[t\,e^{i(-\psi-\pi)}] > \Re\,[t_1\,e^{i(-\psi-\pi)}] = :\sigma_1$. But this is a half-plane with the interior unit-normal $e^{i(\psi-\pi)}$. The intersection of these two half-planes is a sector with a vertex v_0 and such that

$$\pi - \psi - \pi/2 = \pi/2 - \psi < \arg(t - v_0) < \psi - \pi + \pi/2 = \psi - \pi/2.$$

If $0 < \phi := \psi - \pi/2 < \pi/2$, then $-\phi < \arg(t - v_0) < \phi$ (Fig.6.3).

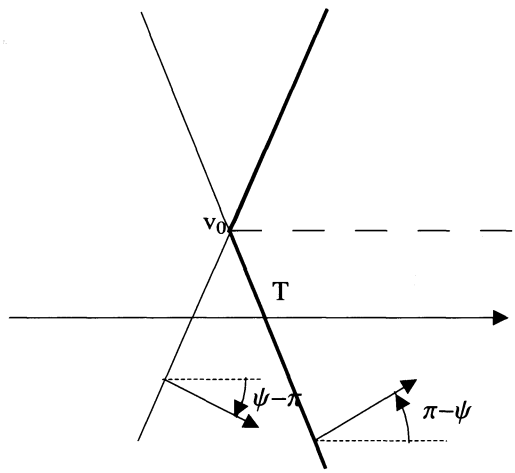


Fig.6.3: Sector of convergence with vertex v_0.

Conclusion: If the angle-contour integral converges for some complex v_0, it will converge for all complex t in the sector with vertex v_0 and $-\phi < \arg(t - v_0) < \phi$. The positive angle ϕ is given by $\phi := \psi - \pi/2 \leq \pi/2$. Moreover, the angle-contour integral represents an analytic function of t in that sector. Since a Laplace integral can be differentiated under the integral sign, this is also true for the the angle-contour integral. The integral over the small circular arc of finite length around b represents an analytic function and can be differentiated under the integral sign. Let us summarize these striking properties concerning the angle-contour integral.

The angle-contour integral with $\pi/2 < \psi \leq \pi$ has the following properties:

- It is a Laplace integral and represents a holomorphic function of t in a sector with vertex v_0 and $-\phi \leq \arg(t - v_0) < \phi$ where $\phi := \psi - \pi/2$.
- The integral can be differentiated under the integral sign.
- If the image function is real in the sense that $F(\bar{s}) = \overline{F(s)}$, then the vertex of the sector of analyticity lies on the real axis which bisects the sector angle and $f(t)$ is real for a real t.

The sector of analyticity will contain the real axis from some $T > 0$ onwards. This means that an angle-contour integral can only represent functions of the real variable t which are real analytic functions of t for $t > T$! Also observe that if $\psi = \pi$, then

$$\frac{1}{2\pi i}\int_{A(b,\pi)} e^{ts}\,F(s)\,ds = \frac{1}{2\pi i}\int_{-\infty}^{(b+)} e^{ts}\,F(s)\,ds$$

represents an analytic function of t in some right half-plane. This is the case for

$$\frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{ts}\,s^{\gamma}\,ds \qquad (\Re t > 0),$$

which holds for any γ that is different from a nonnegative integer. More interesting details about angle-contour integrals can be found in [Doetsch G. (1976)] or [Doetsch G. and Nader W. (1974)].

6.1.3.5 A Generalized Correspondence and an Expansion Theorem

If $\gamma < 0$, we have the correspondence

$$s^{\gamma} \blacksquare\!-\!\square \frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}},$$

valid in the classical or conventional sense. By the complex inversion formula, we can write

$$\frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}} = \mathrm{pv}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\,s^{\gamma}\,ds \quad (\gamma < 0).$$

By referring to the closed integration contour of Fig.6.4, still assuming $\gamma < 0$, the Closing Contour Condition (6.3) is satisfied by Jordan's lemma. Thus, Cauchy's theorem yields

$$\frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}} = \mathrm{pv}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\,s^{\gamma}\,ds = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{ts}\,s^{\gamma}\,ds$$

i.e.,

$$s^{\gamma} \blacksquare\!-\!\square \frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}} \quad (\gamma < 0).$$

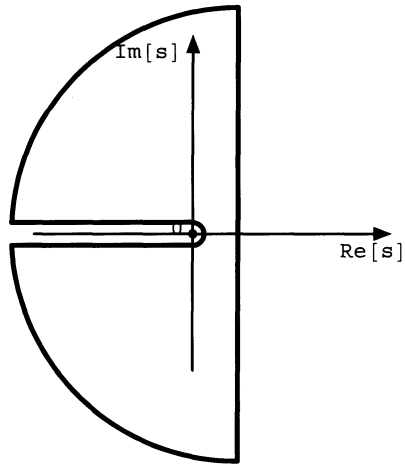


Fig.6.4: Integration contour if 0 is a branch point

For a non-integral $\gamma \geq 0$, this correspondence does no longer hold, since the Closing Contour Condition (6.3) is not satisfied and, therefore,

$$\mathrm{pv}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\,s^{\gamma}\,ds \;\neq\; \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{ts}\,s^{\gamma}\,ds\,.$$

In addition, s^{γ} cannot be an image function in the classical sense. However, the relation

$$\frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{ts}\,s^{\gamma}\,ds$$

still holds for any γ which differs from a nonnegative integer. Thus one may be tempted to consider the generalized correspondence

$$s^{\gamma} \;\blacksquare\!\!-\!\!\circ\; \frac{1}{\Gamma(-\gamma)\,t^{\gamma+1}} \qquad \frac{1}{t^{\gamma+1}} \qquad (\gamma > 0 \text{ non-integral})\,. \tag{6.17}$$

Indeed, if the Laplace transformation is extended to generalized functions, this correspondence would come up with a precise meaning. It is beyond the scope of this book to develop a consistent theory of the Laplace transform of generalized functions. But remember that we have already introduced heuristically (Chapter 1) the generalized correspondences for $\gamma = n$, a positive integer:

$$s^{n} \;\blacksquare\!\!-\!\!\circ\; \delta^{(n)}(t) \qquad (n \in \mathbb{N}).$$

An application of what was just said is the following

Expansion Theorem III. *Suppose that*

1. *the image function $F(s)$ is holomorphic in the right-half plane $\Re s > x_1 > 0$,*
2. *the only singularity of the image function $F(s)$ in the finite complex plane is a branch point of finite order at $s = 0$,*
3. *in the left half-plane $\Re s < c$ $(c > x_1)$, as $s \to \infty$, $F(s) \to 0$ uniformly with respect to the argument of s (the hypothesis of Jordan's lemma is satisfied),*

then the fractional power expansion of $F(s)$ at the origin

$$F(s) = s^{\alpha} \sum_{k=0}^{\infty} c_k \, s^{k\,n/m} \qquad (n,\, m \in \mathbb{N})$$

implies the following expansion of the (generalized) original:

$$f(t) = \frac{1}{t^{\alpha+1}} \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(-\alpha - k\,n/m)} \frac{1}{t^{k\,n/m}} \,. \tag{6.18}$$

Here the terms where $\alpha + k\,n/m$ are positive integers are omitted in the sum.

In order to get the principal idea in the proof, consider again the closed integration contour of Fig.6.4. Cauchy's theorem yields

$$\frac{1}{2\pi i} \int_C e^{s\,t} F(s)\, \boldsymbol{d}s \;=\; 0.$$

By Jordan's lemma the Closing Contour Condition is satisfied, i.e. the contribution to the integral from the two quarter-circles tend to zero as the radius tends to infinity. Therefore,

$$f(t) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t\,s} F(s)\, \boldsymbol{d}s \,.$$

If we substitute the expansion of $F(s)$ into the integral and assume that term by term integration is permitted, we get

$$f(t) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t\,s} F(s)\, \boldsymbol{d}s = \sum_{k=0}^{\infty} c_k \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t\,s}\, s^{k\,n/m+\alpha}\, \boldsymbol{d}s$$

$$= \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(-\alpha - k\,n/m)} \frac{1}{t^{k\,n/m+\alpha+1}} \,. \;\square$$

Example.

$$F(s) = \frac{1}{\sqrt{s}}\, e^{(i-1)\sqrt{s}} \,.$$

Let $s = R\,e^{i\varphi}$. With $(i-1) = \sqrt{2}\,\exp[i\,3\pi/4]$ we write $(i-1)\sqrt{s} = \sqrt{2\,s}\,\exp[i\,3\pi/4]$, then

$$(i-1)\sqrt{s} = \sqrt{2R}\,\exp\Big[i\Big(\frac{\varphi}{2} + 3\pi/4\Big)\Big],$$

$$\big|\, e^{(i-1)\sqrt{s}} \,\big| \;=\; \exp\Big[\,\sqrt{2R}\,\cos\Big(\frac{\varphi}{2} + 3\pi/4\Big)\Big].$$

For $\pi/2 \le \varphi \le 3\pi/2$, we have $\cos(\frac{\varphi}{2} + 3\pi/4) \le 0$. On the horizontal segments $0 \le \Re s \le c$ and for large $|\,\Im s\,|$, we have $(i-1)\sqrt{s} \sim -\sqrt{2R}$ or $(i-1)\sqrt{s} \sim -\,i\,\sqrt{2R}$, respectively. This shows that as $R \to \infty$, $F(s)$ tends to zero uniformly with respect to the argument of s. The hypotheses of the above expansion theorem are satisfied and

$$F(s) = \frac{1}{\sqrt{s}} e^{(i-1)\sqrt{s}} = \sum_{k=0}^{\infty} \frac{2^{k/2} \exp[i\,3\pi k/4]}{k!} s^{(k-1)/2}.$$

If $(k-1)/2$ is a positive integer, there is no contribution in the sum of (6.18). Therefore, witl $k = 2n$,

$$f(t) = \sum_{n=0}^{\infty} \frac{2^n (-i)^n}{(2n)!\,\Gamma(1/2-n)} \frac{1}{t^{n+1/2}}.$$

By the formulas

$$\frac{1}{\Gamma(1/2-n)} = \frac{1\,3\,5\ldots(2n-1)}{2^n} \frac{(-1)^n}{\sqrt{\pi}}$$
$$(2n)! = 1\,3\,5\ldots(2n-1)\,2^n\,n!$$

we have

$$f(t) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{i^n}{2^n\,n!} \frac{1}{t^{n+1/2}} = \frac{1}{\sqrt{\pi t}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2t}\right)^n = \frac{e^{i/2t}}{\sqrt{\pi t}}.$$

Hence

$$\frac{1}{\sqrt{s}} e^{(i-1)\sqrt{s}} = \frac{e^{-\sqrt{s}}}{\sqrt{s}} \{\cos\sqrt{s} + i\cos\sqrt{s}\} \blacksquare\!-\!\square \frac{e^{i/2t}}{\sqrt{\pi t}}$$
$$= \frac{\cos\frac{1}{2t} + \sin\frac{1}{2t}}{\sqrt{\pi t}}$$

or (in a generalized sense)

$$\frac{e^{-\sqrt{s}}\cos\sqrt{s}}{\sqrt{s}} \blacksquare\!-\!\square \frac{\cos(\frac{1}{2t})}{\sqrt{\pi t}}, \quad \frac{e^{-\sqrt{s}}\sin\sqrt{s}}{\sqrt{s}} \blacksquare\!-\!\square \frac{\sin(\frac{1}{2t})}{\sqrt{\pi t}}.$$

6.2 Laplace Transforms and Asymptotics

■ 6.2.1 Why Asymptotic Analysis

In Section 6.1.3.2 we introduced briefly the notion of asymptotic expansion and we stated a theorem concerning the asymptotic expansion of the original function for large values of t. In this section we will go further into this matter. In mathematical education, convergent series expansions are generally emphasized and it is often believed that an infinite series expansion has to be convergent, otherwise it will be useless. This point of view is wrong.

Besides a convergent series expansion of a given function there is another type of expansion (asymptotic expansion) which will often be superior, especially if the expansion is used for numerical evaluation of the given function. For an asymptotic series expansion there is no notion of convergence. The symbolic notation for an asymptotic expansion is almost the same

as for a convergent series representation, but, regarded as an infinite series, it will generally be a divergent series. In fact, the symbol ∞, which is also used in an asymptotic series representation, is misleading. Asymptotic expansions are essentially finite expansions.

Another point connected with the development of operational calculus is of interest. Oliver Heaviside 1850–1925, an English electrical engineer is considered to be the father of the operational calculus. He invented purely manipulative methods for solving ordinary and partial differential equations, which very often and efficiently yielded correct results, but were rejected by the mathematical community due to their lack of rigor. Carson, Bromwich, Doetsch, Wiener and others justified in the sequel the purely formal methods of Heaviside by use of Laplace transformation. Operational calculus became an affair of mathematical analysis, especially of complex variables. About 1950 there was another turning point. The Polnish mathematician J.G. Mikusinski presented a rigorous foundation for the operational calculus without using Laplace transformation. It was essentially an algebraic approach that used the convolution product of two originals to start with. No integral transforms, no complex variables were used. It was very general in scope, since from the beginning, generalized functions were dealt with. The enthusiasm of the mathematical community about Mikusinski's operational calculus went high. Voices of pure mathematicians were heard, announcing the rapid death of the Laplace transformation. It is true, from a logical point of view, that the discovery of Mikusinski is a beautiful flower flourishing forever. However 50 years later, the Laplace transformation is still here, it was not and will not be replaced by Mikusinski's approach. Why? There are several reasons which we cannot all discuss here, we shall mention only one: asymptotics! There are natural asymptotic relations between an original function and its Laplace transform, and vice versa. We have, for example, the fact that a convergent series expansion of an image function unfortunately does not always imply a corresponding convergent series expansion of the original. On the level of asymptotic expansions, however, it does. The algebraic approach of Mikusinski lacks the possibility of asymptotic analysis. This is the principal reason why modern treatises of operational calculus always seek the relation to the Laplace transformation, even if the groundwork is based on Mikusinski's idea.

Next, we present a simple example intended to persuade you of the usefulness of an apparent divergent expansion.

6.2.1.1 An Example

Suppose you want to compute the Laplace transform of $1/(1+t)$ and you do not know that

$$\frac{1}{1+t} \circ\!-\!\bullet\; -e^{s}\,\mathrm{Ei}(-s) \tag{6.19}$$

where

$$-\mathrm{Ei}(-s) := \int_{s}^{\infty} \frac{e^{-x}}{x}\,dx$$

is the exponential integral. You remember the geometric series expansion

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

and you proceed as follows

$$\frac{1}{1+t} \square - \blacksquare \int_0^\infty \frac{e^{-st}}{1+t}\,dt = \int_0^\infty e^{-st}\{1 - t + t^2 - t^3 + \cdots\}\,dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_0^\infty e^{-st} t^k\,dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{k!}{s^{k+1}} = \frac{1}{s} - \frac{1!}{s^2} + \frac{2!}{s^3} - \frac{3!}{s^4} + - \cdots .$$

You have obtained a series which is divergent for all finite values of s and thus appears to be meaningless. Your math teacher would perhaps say: this is the punishment you deserve because you forgot that the geometric series expansion converges only for $|t| < 1$, while your integration domain is the entire real axis. You are not so rapidly discouraged and you take the first four terms of the computed series in order to compute the image function at $s = 10$, say.

```
N[1 / 10 - 1 / 100 + 2 / 1000 - 6 / 10000 ]
```

```
0.0914
```

Your math teacher insists and says: I will show you how wrong your result is. Let us compute the exact value with *Mathematica*:

```
N[-Exp[10] ExpIntegralEi[-10]]
```

```
0.0915633
```

Curious thing, the error is only $1.63\ \ 10^{-4}$!!! Your divergent nonsense series is perhaps not so meaningless.

Let us now have a closer look. Instead of the infinite geometric series, you take a finite one with a remainder

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t} .$$

This is correct for all $t \neq -1$. We insert this finite expansion in the Laplace integral and get

$$\int_0^\infty \frac{e^{-st}}{1+t}\,dt = \int_0^\infty e^{-st}\left\{1 - t + t^2 - t^3 + \ldots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}\right\} dt$$

$$= \sum_{k=0}^{n-1} (-1)^k \frac{k!}{s^{k+1}} + \epsilon_n(s) ,$$

where the truncation error is given by

$$\epsilon_n(s) = (-1)^n \int_0^\infty e^{-st} \frac{t^n}{1+t}\,dt .$$

It can readily be estimated (we suppose s is real):

$$\left|\epsilon_n(s)\right| \leq \int_0^\infty e^{-st} \frac{t^n}{1}\,dt = \frac{n!}{s^{n+1}}$$

i.e. the truncation error is smaller in absolute value than the modulus of the next neglected term of the series expansion. For the example above,

$$\left| \epsilon_4(s) \right| \leq \frac{24}{100\,000} = 2.4\,10^{-4}.$$

The larger s, the smaller the truncation error. This is done for the same number of terms. For small s, the expansion becomes useless. This all is summarized in the notation

$$\int_0^\infty \frac{e^{-st}}{1+t}\,dt \sim \sum_{k=0}^{\infty} (-1)^k \frac{k!}{s^{k+1}} \quad (s \to \infty)$$

and we say that the function

$$-e^s \operatorname{Ei}(-s) = \int_0^\infty \frac{e^{-st}}{1+t}\,dt$$

has the asymptotic expansion

$$\sum_{k=0}^{\infty} (-1)^k \frac{k!}{s^{k+1}} \quad (s \to \infty)$$

valid for large s. Note that we can write for every positive integer n,

$$\int_0^\infty \frac{e^{-st}}{1+t}\,dt = \sum_{k=0}^{n-1} c_k\, r_k(s) + O[r_n(s)]$$

where

$$c_k = (-1)^k\, k!, \quad r_k(s) = \frac{1}{s^{k+1}}.$$

The Landau symbol "uppercase O" means that, for sufficiently large s, there is a constant A, independent of s, such that

$$| O[r_n(s)] | \leq A \, | r_n(s) | .$$

■ 6.2.2 Definition of an Asymptotic Expansion

Definition. A sequence of functions $r_n(x)$ $(n = 0, 1, 2, \ldots)$ is said to be an *asymptotic sequence or a scale as* $x \to \infty$, if for all n,

$$r_{n+1}(x) = o[r_n(x)] \quad (x \to \infty),$$

where the Landau symbol "lowercase o" means that

$$\lim_{x\to\infty} \frac{r_{n+1}(x)}{r_n(x)} = 0.$$

Example. $r_n(x) = x^{-n}$ $(n = 0, 1, 2, \ldots)$ is an asymptotic sequence as $x \to \infty$.

Example. $r_n(x) = x^{m_n}\, e^{a_n x}$ $(n = 1, 2, 3, \ldots)$ is an asymptotic sequence as $x \to \infty$ if

$$\cdots < a_{n+1} < a_n < a_{n-1} < \cdots < a_2 < a_1 < a_0,$$

because

$$\frac{x^{m_{n+1}}\, e^{a_{n+1}\, x}}{x^{m_n}\, e^{a_n\, x}} = x^{m_{n+1}-m_n}\, e^{-(a_n - a_{n+1})\, x} \to 0 \quad (x \to \infty).$$

Example. More generally, the sequence $r_n(x) = p_n(x)\, e^{a_n\, x}$, $(n = 0, 1, 2, 3, \ldots)$ where $p_n(x)$ is a polynomial in x, is a scale, if we again assume that

$$\cdots < a_{n+1} < a_n < a_{n-1} < \cdots < a_2 < a_1 < a_0.$$

Definition. A function $f(x)$ has the *asymptotic expansion with respect to the scale* $r_n(x)$ *as* $x \to \infty$,

$$f(x) \sim \sum_{k=0}^{\infty} c_k\, r_k(x) \qquad (x \to \infty),$$

if for each nonnegative integer n it holds that

$$f(x) = \sum_{k=0}^{n} c_k\, r_k(x) + o[r_n(x)] \qquad (x \to \infty).$$

We can write

$$\sum_{k=0}^{n} c_k\, r_k(x) + o[r_k(x)] = \sum_{k=0}^{n-1} c_k\, r_k(x) + c_n\, r_n(x) + o[r_n(x)]$$

but

$$\left| \frac{c_n\, r_n(x) + o[r_n(x)]}{r_n(x)} \right| \leq |\, c_n\, | + 1 = A \quad (x \to \infty).$$

So the representation

$$f(x) = \textstyle\sum_{k=0}^{n} c_k\, r_k(x) + o[r_n(x)] \quad (x \to \infty)$$

implies

$$f(x) = \textstyle\sum_{k=0}^{n-1} c_k\, r_k(x) + O[r_n(x)] \quad (x \to \infty).$$

Conversely, if one assumes

$$f(x) = \sum_{k=0}^{n} c_k\, r_k(x) + O[r_{n+1}(x)] \qquad (x \to \infty)$$

we can write

$$|\, O[r_{n+1}(x)]\, | \leq A\, |\, r_{n+1}(x)\, | = A\, |\, r_n(x)\, |\, \frac{|\, r_{n+1}(x)\, |}{|\, r_n(x)\, |}.$$

Thus, by taking into account that $r_n(x)$ is a scale,

$$\frac{|\ O[r_{n+1}(x)]\ |}{|\ r_n(x)\ |} \le A \frac{|\ r_{n+1}(x)\ |}{|\ r_n(x)\ |} \to 0 \quad (x \to \infty) \implies O[r_{n+1}(x)] = o[r_n(x)].$$

We proved that the expressions

$$f(x) = \sum_{k=0}^{n} c_k\, r_k(x) + o[r_n(x)] \quad (x \to \infty),$$

$$f(x) = \sum_{k=0}^{n-1} c_k\, r_k(x) + O[r_n(x)] \quad (x \to \infty)$$

are equivalent.

Definition. A sequence of functions $r_n(x)$ $(n = 0, 1, 2, \ldots)$ is said to be an *asymptotic sequence or a scale as* $x \to 0$, if for all n,

$$r_{n+1}(x) = o[r_n(x)] \quad (x \to 0).$$

A function $f(x)$ has the *asymptotic expansion with respect to the scale* $r_n(x)$ *as* $x \to 0$,

$$f(x) \sim \sum_{k=0}^{\infty} c_k\, r_k(x) \qquad (x \to 0),$$

if, for each nonnegative integer n, one of the following two equivalent expressions holds

$$f(x) = \sum_{k=0}^{n} c_k\, r_k(x) + o[r_n(x)] \qquad (x \to 0),$$

$$f(x) = \sum_{k=0}^{n-1} c_k\, r_k(x) + O[r_n(x)] \qquad (x \to 0).$$

It is important to distinguish between a convergent series and an asymptotic expansion. In order to be specific, let us take the case of an expansion about $x = 0$.

- Let $s(x) = \sum_{k=0}^{\infty} c_k\, r_k(x)$ be a convergent series with the partial sum $s_n(x) := \sum_{k=0}^{n} c_k\, r_k(x)$. By choosing n sufficiently large, the difference between $s(x)$ and $s_n(x)$ can be made as small as you want, for all x inside the interval of convergence $|\ x\ | < \rho$.
- Let $s(x) \sim \sum_{k=0}^{\infty} c_k\, r_k(x)$ $(x \to 0)$ be an asymptotic expansion with the partial sum $s_n(x) := \sum_{k=0}^{n} c_k\, r_k(x)$, where n is fixed now. By choosing x sufficiently close to $x = 0$, the difference between $s(x)$ and $s_n(x)$ can be made small in the sense that the quotient of this difference and the last retained term $r_n(x)$ can be made as small as you want.
- In the case of a convergent series expansion, the number of terms of the partial sum can become very large in order to get $s_n(x)$ sufficiently near $s(x)$. But the error can be made as small as you want by taking more and more terms.
- In the case of an asymptotic expansion, only a few terms in general are needed in order that the partial sum $s_n(x)$ is sufficiently near $s(x)$. But the error is always non-zero for an $x \neq 0$. By taking more terms, the error is, in general, not reduced. For a given $x \neq 0$, there is, in general, an optimal number n of terms for which a minimum error results.

The most important case is the one with the asymptotic sequence of increasing or decreasing powers. It is called a *Poincaré expansion*, i.e. for $x \to 0+$,

$$\cdots,\ r_n(x) = x^{\lambda_n},\ r_{n+1}(x) = x^{\lambda_{n+1}},\ \cdots$$

where $\cdots < \lambda_n < \lambda_{n+1} < \cdots$, or for $x \to \infty$

$$\cdots,\ r_n(x) = x^{-\lambda_n},\ r_{n+1}(x) = x^{-\lambda_{n+1}},\ \cdots$$

where $\ldots < \lambda_n < \lambda_{n+1} < \ldots$, respectively. A convergent power series

$$s(x) = \sum_{n=0}^{\infty} c_k\, x^{\lambda_n} \quad (|x| < \rho)$$

is always an asymptotic series

$$s(x) \sim \sum_{n=0}^{\infty} c_n\, x^{\lambda_n} \quad (x \to 0).$$

In short: convergent expansion implies asymptotic expansion. This is seen as follows (for the case $x \to 0$):

$$\left| s(x) - \sum_{k=0}^{n} c_k\, x^{\lambda_n} \right| = \left| \sum_{k=n+1}^{\infty} c_k\, x^{\lambda_k} \right|$$

$$= \left| x^{\lambda_{n+1}} \right| \left| \sum_{k=n+1}^{\infty} c_k\, x^{\lambda_k - \lambda_{n+1}} \right| \le \left| x^{\lambda_{n+1}} \right| \sum_{k=n+1}^{\infty} |c_k| \left| x^{\lambda_k - \lambda_{n+1}} \right|$$

$$\le \left| x^{\lambda_{n+1}} \right| \sum_{k=n+1}^{\infty} |c_k|\, \rho_1^{\lambda_k - \lambda_{n+1}}$$

$$= C_n\, |x^{\lambda_{n+1}}| \quad (x \le \rho_1 < \rho).$$

Therefore,

$$\left| \frac{s(x) - \sum_{k=0}^{n} c_k\, x^{\lambda_k}}{x^{\lambda_n}} \right| \le C_n \left| x^{\lambda_{n+1} - \lambda_n} \right| \to 0 \quad (x \to 0)$$

since $(\lambda_{n+1} - \lambda_n) > 0$. This proves that

$$s(x) - \sum_{k=0}^{n} c_k\, x^{\lambda_k} = o[x^{\lambda_n}] \quad (x \to 0).$$

In the above asymptotic expansions we assumed that $x \to 0$ or $x \to \infty$ for real values of x. But often a 2-dimensional approach in the complex plane is needed. In such cases the approach is in general restricted to a sector. For example,

$$f(z) \sim \sum_{n=1}^{\infty} \frac{c_n}{z^n} \quad \left(z \to \infty,\ |\arg z| < \frac{\pi}{2}\right)$$

is an asymptotic expansion of $f(z)$ for $z \to \infty$ where the complex variable z is restricted to the right half-plane.

More basic materials about asymptotic expansions can be found, for example, in [Olver F.W.J. (1974)], [Bleistein N. and Handelsman A. (1986)].

■ 6.2.3 Asymptotic Relations between Original and Image Functions

6.2.3.1 Watson's Lemma

Watson's lemma yields the asymptotic expansion of the image function for $s \to \infty$, provided the asymptotic expansion of the original for $t \to 0$ is known.

Watson's Lemma. *Suppose that the Laplace integral*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

converges for a finite value x_0 and that $f(t)$ is piecewise continuous on the positive part of the real axis. Then the asymptotic expansion of the original

$$f(t) \sim \sum_{k=0}^{\infty} a_k\, t^{(k+\lambda-\mu)/\mu} \qquad (t \to 0+,\ t \in \mathbb{R}) \tag{6.20}$$

implies the following asymptotic expansion of the image

$$F(s) \sim \sum_{k=0}^{\infty} \Gamma\Big(\frac{k+\lambda}{\mu}\Big) \frac{a_k}{s^{(k+\lambda)/\mu}} \qquad \Big(s \to \infty,\ \big|\arg s\big| \le \frac{\pi}{2} - \delta\Big). \tag{6.21}$$

Here μ is a positive constant and λ is a complex constant with $\Re\lambda > 0$. The power $s^{(k+\lambda)/\mu}$ has its principal value and δ is an arbitrary small positive constant with $\delta < \pi/2$.

A proof of this fundamental lemma can be found in any textbook on asymptotics, for example [Olver F.W.J. (1974)].

Consider the following special case, which is often used: Assume that the original has the convergent expansion

$$f(t) = \frac{1}{t} \sum_{k=0}^{\infty} a_k\, t^{(k+\lambda)/\mu} \ .$$

Then

$$F(s) \sim \sum_{k=0}^{\infty} \Gamma\Big(\frac{k+\lambda}{\mu}\Big) \frac{a_k}{s^{(k+\lambda)/\mu}} \qquad \Big(s \to \infty,\ \big|\arg s\big| \le \frac{\pi}{2} - \delta\Big)$$

because a convergent expansion is an asymptotic expansion, too.

Example.

$$f(t) = \exp(-at^2) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} t^{2k} \qquad \left(\lambda = \mu = \frac{1}{2},\ a > 0\right),$$

$$F(s) \sim \sum_{k=0}^{\infty} \Gamma(2k+1) \frac{(-a)^k}{k!} \frac{1}{s^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(-a)^k (2k)!}{k!} \frac{1}{s^{2k+1}} \qquad \left(s \to \infty,\ |\arg s| \le \frac{\pi}{2} - \delta\right).$$

Note the following:

1. We obtained the image expansion simply by taking the images of the original expansion term by term.
2. From the point of view of convergent series, this example shows that, in general, you cannot transcribe the series of an original term by term, because the resulting series of the image often diverges.
3. The image expansion may be a divergent series which, however, is not useless. It gives the asymptotic expansion of the image function for $s \to \infty$, where s is restricted to a proper sector of the right half-plane.

Since we know the correspondence

$$f(t) = \exp(-at^2) \circ\!-\!\bullet \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(\frac{s^2}{4a}\right) \operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right) = F(s),$$

we get the asymptotic expansion of the complementary error function:

$$\operatorname{erfc}(s) \sim \frac{1}{\sqrt{\pi}\, s} e^{-s^2} \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k}\, k!} \frac{1}{s^{2k}}$$

$$\left(s \to \infty,\ |\arg s| \le \frac{\pi}{2} - \delta\right).$$

Example. We use the following correspondence, derived in Section 1.1.3.3,

$$\frac{u(2-t)\ [t(2-t)]^{n-1/2}}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})\, 2^n} \circ\!-\!\bullet\ s^{-n} e^{-s} I_n(s)$$

to find an asymptotic expansion of the modified Bessel function of order n. For $t < 2$,

$$[t(2-t)]^{n-\frac{1}{2}} = (2t)^{n-1/2} (1 - t/2)^{n-1/2} = (2t)^{n-1/2} \sum_{k=0}^{\infty} \binom{n-1/2}{k} \left(-\frac{t}{2}\right)^k,$$

$$f(t) = \frac{(2t)^{n-1/2}}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})\, 2^n} \sum_{k=0}^{\infty} \binom{n-1/2}{k} \left(-\frac{t}{2}\right)^k$$

$$= \frac{1}{\sqrt{2\pi}\ \Gamma(n+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \binom{n-1/2}{k} t^{n+k-1/2}.$$

We have $\lambda = n + 1/2$, $\mu = 1$, hence for

$$\left(s \to \infty, \quad \left|\arg s\right| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}\right)$$

we can write

$$F(s) \sim \frac{1}{\sqrt{2\pi}\ \Gamma(n+\frac{1}{2})} \times \sum_{k=0}^{\infty} \Gamma(k+n+1/2)\ \frac{(-1)^k}{2^k} \binom{n-1/2}{k} \frac{1}{s^{k+n+1/2}}.$$

The formula

$$A_k(n) := \binom{n-1/2}{k} \frac{\Gamma(k+n+1/2)}{\Gamma(n+1/2)\ 2^k}$$
$$= \frac{(4n^2-1^2)(4n^2-3^2)\ldots(4n^2-(2k-1)^2)}{k!\,8^k}$$

yields

$$F(s) \sim \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{A_k(n)}{s^{k+n+1/2}}.$$

Finally, the asymptotic expansion of the modified Bessel function is

$$I_n(s) \sim \frac{e^s}{\sqrt{2\pi s}} \sum_{k=0}^{\infty} (-1)^k \frac{A_k(n)}{s^k}, \quad \left(s \to \infty, \quad \left|\arg s\right| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}\right).$$

Example. Now we use the other correspondence derived in Section 1.1.3.3, i.e.,

$$\frac{u(2-t)\,[t(2-t)]^{n-1/2}}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})\,2^n} \quad \square - \blacksquare \quad s^{-n}\, e^{-s}\, e^{-n i \pi/2}\, J_n(s\, e^{i\pi/2}).$$

As in the last example,

$$F(s) \sim \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{A_k(n)}{s^{k+n+1/2}}$$

but now

$$F(s) = s^{-n}\, e^{-s}\, e^{-n i \pi/2}\, J_n(s\, e^{i\pi/2}).$$

Therefore, by replacing s by $s\, e^{-i\pi/2}$, we have for $|\arg s\, e^{-i\pi/2}| < \pi/2 - \delta < \pi/2$, hence for $0 < \delta < \arg s < \pi - \delta$, and $s \to \infty$,

$$J_n(s) \sim \frac{e^{-is}}{\sqrt{2\pi s}} \sum_{k=0}^{\infty} (-1)^k \frac{e^{i\pi/2\,(k+n+1/2)}\ A_k(n)}{s^k}.$$

This is the asymptotic expansion of the Bessel function for $s \to \infty$ in the upper half-plane. The asymptotic expansion of $J_n(x)$ for real x and $x \to \infty$ cannot be found by this method.

Remark. Observe that in order to apply Watson's lemma, it is not necessary that $f(t)$ be an analytic function of t on the real axis. $f(t)$ only needs the assumed asymptotic expansion for $t \to 0+$. This is used in the last two examples, where the function

$$u(2-t)\ [t(2-t)]^{n-1/2},$$

which vanishes identically on the interval $(2, \infty)$, is not analytic on the real axis.

Watson's Lemma can be generalized to allow limiting processes towards zero other than just $t \to 0+$ along the real axis.

Generalized Watson's Lemma. *Assume that:*

1. *$f(t)$ is holomorphic within the sector $S: \alpha_1 < \arg t < \alpha_2$, where $\alpha_1 < 0$ and $\alpha_2 > 0$, such that the real axis lies inside S.*
2. *$f(t)$ is within the sector S of bounded exponential growth, i.e. $f(t) = O[e^{\beta|t|}]$ as $t \to \infty$ in the subsector $S_\delta: \alpha_1 + \delta < \arg t < \alpha_2 - \delta$.*
3. *For $t \to 0$ within the subsector S_δ, the asymptotic expansion*

$$f(t) \sim \sum_{k=0}^{\infty} a_k\, t^{(k+\lambda-\mu)/\mu}$$

is assumed to hold.

Then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

or its analytic continuation has the asymptotic expansion

$$F(s) \sim \sum_{k=0}^{\infty} \Gamma\left(\frac{k+\lambda}{\mu}\right) \frac{a_k}{s^{(k+\lambda)/\mu}} \quad (s \to \infty), \tag{6.22}$$

which holds in the sector

$$-\frac{\pi}{2} + \delta - \alpha_2 \le \arg s \le \frac{\pi}{2} - \delta - \alpha_1.$$

The positive number δ is restricted to $\delta < (\alpha_2 - \alpha_1)/2$.

For a proof, see, for example, [Olver F.W.J. (1974)]. Note that now $f(t)$ has to be an analytic function in the approaching sector. This is more restrictive than in the real case of Watson's lemma, but the spread of the sector of validity of the asymptotic expansion of $F(s)$ is widened. Hypothesis 2 is needed to ensure the absolute convergence of the Laplace integral. Also observe that if the approaching sector for t is rotated counterclockwise (the bisectrix of S forms an angle $\gamma := (\alpha_2 + \alpha_1)/2$ with the positive x-axis), then the sector within $s \to \infty$ rotates clockwise (angle of its bisectrix is $-\gamma$).

Example. Find the asymptotic expansion of

$$F(s) = \int_0^\infty e^{-s^2 t} \log\left(1 + \sqrt{t}\right) dt \quad (s \to \infty).$$

$f(t)$ is holomorphic in the sector $\alpha_1 = -\pi < \arg t < \pi = \alpha_2$.

$$f(t) = \log\bigl(1+\sqrt{t}\bigr) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, t^{k/2} \quad (t \to 0)$$

$\mu = 2$, $\lambda = 2$. Therefore, in the sector $-3\pi/2 + \delta \le \arg s^2 \le 3\pi/2 - \delta$ we have the asymptotic expansion

$$F(s) \sim \sum_{k=1}^{\infty} \Gamma\Bigl(\frac{k+2}{2}\Bigr) \frac{(-1)^{k-1}}{k} \frac{1}{s^{2(k+2)/2}} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k/2)}{s^{k+2}}.$$

Finally, for $s \to \infty$, $|\arg s| \le 3\pi/4 - \delta'$,

$$F(s) \sim \frac{1}{2s^2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(\frac{k}{2})}{s^k}.$$

6.2.3.2 Watson's Lemma for Loop Integrals

The generalization of Watson's lemma for loop integrals, or more generally, for angle-contour integrals of the type considered in Section 6.1.3.4, is of great importance for the asymptotics of Laplace transforms. Let

$$f(t) = \frac{1}{2\pi i} \int_{A(b,\psi)} e^{ts} F(s)\, ds$$

be an angle-contour integral with $\pi/2 < \psi \le \pi$. In the case of $\psi = \pi$ it also is denoted by

$$f(t) = \frac{1}{2\pi i} \int_{-\infty}^{(b+)} e^{ts} F(s)\, ds.$$

The angle formed by the lower and upper ray of the angle-contour and the positive x–axismust be greater than $\pi/2$ in absolute value, such that the real part of the integration variable s goes to $-\infty$.

Watson's Lemma for Angle-Contour Integrals. *Assume that*

1. *$F(s)$ is piecewise continuous on the integration path and holomorphic for $0 < |s-b| < \rho$. $F(s)$ has a branch point at $s = b$.*
2. *The asymptotic expansion*

$$F(s) \sim \sum_{k=0}^{\infty} a_k (s-b)^{(k+\lambda-\mu)/\mu} \qquad (s \to b)$$

holds in the entire sector $|\arg(s-b)| \le \pi$. The constant μ is a positive and λ is an unrestricted complex constant.

Then the asymptotic expansion

$$f(t) \sim e^{bt} \sum_{k=0}^{\infty} \frac{a_k}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right)} \frac{1}{t^{(k+\lambda)/\mu}} \tag{6.23}$$

holds as $t \to \infty$ *in the sector* $|\arg t| \leq \pi/2 - \delta < \pi/2$.

A proof of this theorem can be found in [Doetsch (1976)] or in [Olver (1974)].

The main difference between Watson's lemma and Watson's lemma for loop integrals is that in the former $\Re\lambda > 0$, while in the latter the complex constant λ is unrestricted.

Remark. If $F(s)$ is a single-valued function in a neighborhood of b, then $\mu = 1$ and λ is an integer. For a nonnegative λ , $F(s)$ is a holomorphic function in the neighborhood of b and for a negative integer λ, $F(s)$ has a pole at $s = b$. At any rate, $1 - \lambda - k$ is a negative integer or 0 for $k \geq \max\{1 - \lambda, 0\}$, hence $1/\Gamma(1 - \lambda - k) = 0$. This means that (6.23) does not yield a ful asymptotic series if $F(s)$ is holomorphic or has an isolated singularity at $s = b$. Therefore, (6.23) yields the full asymptotic expansion only if b is a branch point of $F(s)$!

6.2.3.3 The Asymptotic Behavior of the Original for Large Values of t

Assume that the original $f(t)$ has an abscissa of absolute convergence x_1 and is locally of bounded variation. Then the image function $F(s)$ is a holomorphic function for $\Re s > x_1$. Assume further that $F(s)$ has a finite number m of poles a_i which are between x_1 and a branch point b such that

$$\Re b < \Re a_m < \Re a_{m-1} < \cdots < \Re a_2 < \Re a_1 < x_1.$$

Under the conditions of Section 6.1.3.3,

$$f(t) = \sum_{\nu=1}^{m} f_\nu(t) + \frac{1}{2\pi i} \int_{A(b,\psi)} e^{ts} F(s)\, ds$$

with

$$f_\nu(t) = \left\{ \frac{c_{\nu m}}{(k-1)!} t^{m-1} + \frac{c_{\nu m-1}}{(k-2)!} t^{m-2} + \cdots + c_{\nu 1} \right\} e^{t a_\nu},$$

if

$$\frac{c_{\nu m}}{(s-a_\nu)^m} + \frac{c_{\nu m-1}}{(s-a_\nu)^{m-1}} + \cdots + \frac{c_{\nu 1}}{s-a_\nu}$$

is the principal part of the Laurent expansion of $F(s)$ at the pole a_ν.

Then Watson's lemma for loop integrals implies that the asymptotic expansion

$$f(t) \sim \sum_{\nu=1}^{m} f_\nu(t) + e^{bt} \sum_{k=0}^{\infty} \frac{a_k}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right)} \frac{1}{t^{(k+\lambda)/\mu}} \quad (t \to \infty), \tag{6.24}$$

holds for large values of t. The coefficients a_k are given by the asymptotic expansion

$$F(s) \sim \sum_{k=0}^{\infty} a_k \, (s-b)^{(k+\lambda-\mu)/\mu} \qquad (s \to b)$$

of the image function at the branch point b. If b is the only branch point of $F(s)$ which, in a real world problem, then is real, we have all that is needed.

Example. Find the asymptotic behavior for large values of t of the original whose Laplace transform is given by

$$F(s) = \frac{1}{s^2 \, (s-a) \sqrt{s+b}} \qquad (a > 0, \; b > 0) .$$

The image function has a simple pole at $s = a$, a double pole at $s = 0$, and a branch point at $s = -b$. The principal part of the Laurent expansion at $s = a$ is

$$\frac{1}{a^2 \sqrt{a+b}} \, \frac{1}{s-a},$$

the one at $s = 0$ is

$$-\frac{1}{a\sqrt{b}} \, \frac{1}{s^2} + \frac{a-2b}{2a^2 b\sqrt{b}} \, \frac{1}{s} .$$

Therefore,

$$f(t) = -\frac{1}{a\sqrt{b}} \, t + \frac{a-2b}{2a^2 b\sqrt{b}} + \frac{e^{at}}{a^2\sqrt{a+b}} + \frac{1}{2\pi i} \int_{-\infty}^{(-b+)} \frac{e^{ts}}{s^2 \, (s-a)\sqrt{s+b}} \, ds .$$

Let us find the asymptotic expansion of the integrand at $s = -b$. We take its convergent Taylor expansion and use *Mathematica* to compute the first three terms.

```
F = 1 / (s^2 (s - a) Sqrt[s + b]);

Series[F, {s, -b, 2}]
```

$$\frac{1}{(-a-b)\, b^2 \sqrt{s+b}} + \left(\frac{2}{(-a-b)\, b^3} - \frac{1}{(-a-b)^2 \, b^2} \right) \sqrt{s+b} + \left(\frac{3}{(-a-b)\, b^4} - \frac{2}{(-a-b)^2 \, b^3} + \frac{1}{(-a-b)^3 \, b^2} \right) (s+b)^{3/2} + O[s+b]^{5/2}$$

We have

$$a_0 = -\frac{1}{(a+b)\, b^2 \sqrt{s+b}}, \quad a_1 = -\frac{2}{(a+b)\, b^3} - \frac{1}{(a+b)^2 \, b^2},$$

$$a_2 = -\frac{3}{(a+b)\, b^4} - \frac{2}{(a+b)^2 \, b^3} - \frac{1}{(a+b)^3 \, b^2}, \; \ldots$$

and $\mu = 1$, $\lambda = 1/2$. Finally, for large values of t, the orginal has the asymptotic expansion

$$f(t) = -\frac{1}{a\sqrt{b}} \, t + \frac{a-2b}{2a^2 b\sqrt{b}} + \frac{e^{at}}{a^2\sqrt{a+b}}$$

$$+\frac{e^{-bt}}{\sqrt{\pi t}}\left\{-\frac{1}{(a+b)\,b^2\sqrt{s+b}}+\left\{\frac{2}{(a+b)\,b^3}+\frac{1}{(a+b)^2\,b^2}\right\}\frac{1}{2t}\right.$$
$$\left.+\left\{-\frac{3}{(a+b)\,b^4}-\frac{2}{(a+b)^2\,b^3}-\frac{1}{(a+b)^3\,b^2}\right\}\frac{3}{4}\,\frac{1}{t^2}+\cdots\right\}.$$

We shall show in Section 6.2.5 how these computations can be performed by the Package.

Suppose that further left of the poles there are more than one branch point. If there is one real branch point with a greater real part than the others, we take this right-most one and proceed as above. We may do so because the asymptotic contributions of branch points further left may be neglected. This is seen as follows: The asymptotic expansion corresponding to the branch point b which has the greatest real part is

$$f(t) \sim \sum_{k=0}^{\infty} \frac{a_k}{t^{\lambda_k}}\, e^{bt} \qquad (t \to \infty),\quad \lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

This means that for any nonnegative integer n we have

$$f(t) = \sum_{k=0}^{n} \frac{a_k}{t^{\lambda_k}}\, e^{bt} + o\left[\frac{e^{bt}}{t^{\lambda_n}}\right].$$

From this we conclude

$$\frac{t^{\lambda_n}}{e^{bt}}\left\{f(t) - \sum_{k=0}^{n-1} \frac{a_k}{t^{\lambda_k}}\right\} = a_n + o\left[\frac{e^{bt}}{t^{\lambda_n}}\right] \Big/ \left(\frac{e^{bt}}{t^{\lambda_n}}\right).$$

Therefore, for all n we have

$$a_n = \lim_{t\to\infty} \frac{t^{\lambda_n}}{e^{bt}}\left\{f(t) - \sum_{k=0}^{n-1} \frac{a_k}{t^{\lambda_k}}\right\}.$$

If there were a second contribution from a branch point d with $\Re d < \Re b$, i.e.,

$$f(t) \sim \sum_{k=0}^{\infty} \frac{a_k}{t^{\lambda_k}}\, e^{bt} + \sum_{k=0}^{m} \frac{c_k}{t^{\mu_k}}\, e^{dt} \quad (t \to \infty),$$

where m is any positive integer, we could again write

$$\lim_{t\to\infty} \frac{t^{\lambda_n}}{e^{bt}}\left\{\left[f(t) - \sum_{k=0}^{m} \frac{c_k}{t^{\mu_k}}\, e^{dt}\right] - \sum_{k=0}^{n-1} \frac{a_k}{t^{\lambda_k}}\right\}$$
$$= \lim_{t\to\infty} \frac{t^{\lambda_n}}{e^{bt}}\left\{f(t) - \sum_{k=0}^{n-1} \frac{a_k}{t^{\lambda_k}}\right\} - \lim_{t\to\infty} \sum_{k=0}^{m} \frac{c_k}{t^{\mu_k-\lambda_n}}\, e^{-(b-d)t} = a_n.$$

The last summand is zero because $\Re b > \Re d$. This means that the asymptotic expansions of $f(t)$ and of

$$f(t) + \sum_{k=0}^{m} \frac{c_k}{t^{\mu_k}}\, e^{dt}$$

with respect to the scale e^{bt} / t^{λ_n} are the same.

The remaining case is where there are several branch points with equal real parts to the left of the poles. In a typical application problem this may happen with pairs of complex conjugate branch points. Thus let us assume that by scanning through the poles and taking into account their contributions we finally encounter a pair of branch points b, $\overline{b}$. We shall show that the resulting asymptotic expansion is the superposition of the two asymptotic expansions at b and $\overline{b}$. We use the integration contour in (Fig.6.5).

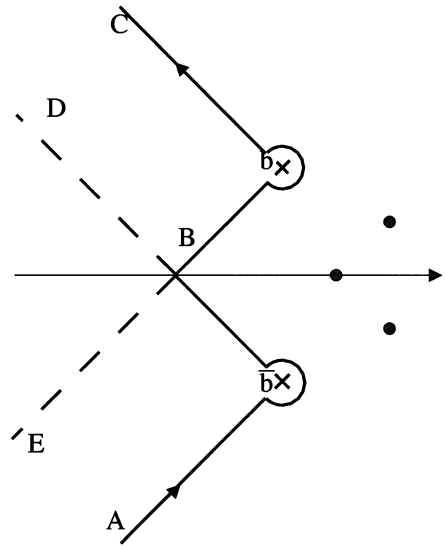


Fig.6.5: Case of a pair of complex conjugate branch points

Assume that the Closing Contour Condition (6.3) of Section 6.1.3.3 is satisfied and that the vertical integration contour is replaced by the contour ABC. By defining $F(s)$ to equal zero along the two segments BD and EB, we can replace contour ABC by the two contours ABD and EBC. We may do so because Watson's lemma for loop integrals does not require $F(s)$ to be analytic. It is sufficient that the image is piecewise continuous on the integration contours. Instead of the single asymptotic expansion we now have to consider the superposition of the two asymptotic expansions

$$e^{bt} \sum_{k=0}^{\infty} \frac{a_k}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right)} \frac{1}{t^{(k+\lambda)/\mu}}, \qquad e^{\overline{b}t} \sum_{k=0}^{\infty} \frac{c_k}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right)} \frac{1}{t^{(k+\lambda)/\mu}},$$

where the coefficients come from the two assumed expansions

$$F(s) \sim \sum_{k=0}^{\infty} a_k (s-b)^{(k+\lambda-\mu)/\mu} \qquad (s \to b),$$

$$F(s) \sim \sum_{k=0}^{\infty} c_k (s-\overline{b})^{(k+\lambda-\mu)/\mu} \qquad (s \to \overline{b}).$$

The superposition will only be valid for real t. We assumed that both expansions have the same constants λ, μ. If the image function is real in the sense that $F(\bar{s}) = \overline{F(s)}$, and if λ is also real (remember that μ is assumed to be positive), then we will have $c_k = \overline{a_k}$ and the superposition of the two expansions can be combined into one compound asymptotic expansion

$$2\, e^{\Re b\, t} \sum_{k=0}^{\infty} \frac{\Re a_k \cos(\Im b\, t) - \Im a_k \cos(\Im b\, t)}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right) t^{(k+\lambda)/\mu}}.$$

Let us summarize this.

Proposition. *Assume that to the left of the poles, there is a pair of complex conjugate branch points $(b, \bar{b})$ with a greatest common real part. Other branch points must be located strictly to the left of the pair $(b, \bar{b})$. Furthermore, assume that $F(\bar{s}) = \overline{F(s)}$ and that the constant λ in the asymptotic expansion*

$$F(s) \sim \sum_{k=0}^{\infty} a_k (s-b)^{(k+\lambda-\mu)/\mu} \qquad (s \to b)$$

is real ($\mu > 0$). Then for $(t \to \infty,\ t \in \mathbb{R})$ (6.24) is replaced by

$$f(t) \sim \sum_{\nu=1}^{m} f_\nu(t) + 2\, e^{\chi t} \sum_{k=0}^{\infty} \frac{A_k \cos \omega t - B_k \sin \omega t}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right) t^{(k+\lambda)/\mu}} \tag{6.25}$$

where $b = \chi + i\omega$, $a_k = A_k + iB_k$.

Example. In 6.2.3.1 we derived the asymptotic expansion for large values of t of the Bessel function $J_n(t)$ when t is restricted to the upper complex half-plane. By use of Watson's lemma for loop integrals, we shall now derive the asymptotic expansion for large real t. We start from the correspondence

$$t^n J_n(t) \circ\!\!-\!\!\bullet \frac{\Gamma(n+\frac{1}{2})\, 2^n}{\sqrt{\pi}} \frac{1}{(s^2+1)^{n+1/2}} =: F(s)$$

and assume that $n + 1/2$ is not an integer. At $b = i$, the upper branch point, we have the expansion

$$\frac{1}{(s^2+1)^{n+1/2}} = \frac{1}{(s+i)^{n+1/2}\,(s-i)^{n+1/2}} = \frac{1}{(s-i+2i)^{n+1/2}\,(s-i)^{n+1/2}}$$

$$= \frac{1}{(2i)^{n+1/2}\,(1+\frac{s-i}{2i})^{n+1/2}\,(s-i)^{n+1/2}}$$

$$= \frac{e^{-i(\pi/2)(n+1/2)}}{2^{n+1/2}} \sum_{k=0}^{\infty} \binom{-1/2-n}{k} \frac{e^{-i(\pi/2)k}}{2^k} (s-i)^{k-n-1/2}$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \binom{-1/2-n}{k} \frac{e^{-i(\pi/2)(n+k+1/2)}}{2^{n+k}} (s-i)^{k-n-1/2},$$

hence

$$F(s) = \sum_{k=0}^{\infty} \binom{-1/2-n}{k} \Gamma\left(n+\frac{1}{2}\right) \frac{e^{-i(\pi/2)(n+k+1/2)}}{2^k \sqrt{2\pi}} (s-i)^{k-n-1/2}.$$

The use of the formula

$$\binom{-1/2-n}{k} = (-1)^k \frac{\Gamma(n+1/2-k)\ 2^k}{\Gamma(n+1/2)} A_k(n)$$

($n+1/2$ is not an integer) with

$$A_k(n) := \binom{n-1/2}{k} \frac{\Gamma(n+1/2+k)}{\Gamma(n+1/2)\ 2^k}$$

$$= \frac{(4n^2-1^2)(4n^2-3^2)\ldots(4n^2-(2k-1)^2)}{k!\,8^k}$$

implies

$$F(s) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k A_k(n) \Gamma(n+1/2-k)\ e^{-i(\pi/2)(n+k+1/2)} (s-i)^{k-n-1/2}.$$

This is a convergent power series at $b = i$ and therefore also an asymptotic expansion.

Since $\lambda = -n+1/2$, $\mu = 1$,

$$a_k = \frac{A_k(n)\,\Gamma(n+1/2-k)\ e^{-i(\pi/2)(n+2k+1/2)}}{\sqrt{2\pi}}$$

$$= \frac{A_k(n)\,\Gamma(n+1/2-k)}{\sqrt{2\pi}} \left\{\cos\left[\frac{\pi}{2}(n+k+1/2)\right] - i \sin\left[\frac{\pi}{2}(n+k+1/2)\right]\right\}.$$

With $\omega = 1$ and $\chi = 0$, we have

$$\frac{\alpha_k \cos\omega t - i\beta_k \sin\omega t}{\Gamma\left(\frac{\mu-\lambda-k}{\mu}\right)} = \frac{A_k(n)\,\Gamma(n+1/2-k)}{\Gamma(n+1/2-k)\sqrt{2\pi}}$$

$$\times\left\{\cos\left[\frac{\pi}{2}(n+k+1/2)\right]\cos t + \sin\left[\frac{\pi}{2}(n+k+1/2)\right]\sin t\right\}$$

$$= \frac{A_k(n)}{\sqrt{2\pi}} \cos\left[t - \frac{\pi}{2}(n+k+1/2)\right].$$

Finally, we obtain the asymptotic expansion for t (real) $\to \infty$

$$J_n(t) \sim \frac{2}{\sqrt{2\pi t}} \sum_{k=0}^{\infty} A_k(n) \frac{\cos[t - \frac{\pi}{2}(n+k+1/2)]}{t^k}$$

or

$$J_n(t) \sim \sqrt{\frac{2}{\pi t}}\,\cos\Big[t - \frac{\pi}{2}n - \frac{\pi}{4}\Big] \times \sum_{k=0}^{\infty} (-1)^k \frac{A_{2k}(n)}{t^{2k}} - \sqrt{\frac{2}{\pi t}}\,\sin\Big[t - \frac{\pi}{2}n - \frac{\pi}{4}\Big] \times \sum_{k=0}^{\infty} (-1)^k \frac{A_{2k+1}(n)}{t^{2k+1}}. \tag{6.26}$$

■ 6.2.4 Laplace's Method

Laplace's method basically results from Watson's lemma by a substitution process. We illustrate this by a special simple case. Let

$$F(s, \nu) := \int_0^{\infty} e^{-s t^{\nu}} f(t)\, dt \quad (\nu > 0)$$

be a generalisation of Laplace's integral. Assume that $f(t)$ has the asymptotic expansion

$$f(t) \sim \sum_{k=0}^{\infty} a_k\, t^{(k+\lambda-\mu)/\mu} \quad (t \to 0+,\ t \in \mathbb{R}),$$

where μ is a positive constant and λ a complex constant with positive real part. By a change of variables $\tau = t^{\nu}$, we obtain

$$F(s, \nu) := \int_0^{\infty} e^{-s\tau}\, \frac{f(\tau^{1/\nu})}{\nu\, \tau^{(\nu-1)/\nu}}\, d\tau$$

with

$$\frac{f(\tau^{1/\nu})}{\nu\, \tau^{(\nu-1)/\nu}} \sim \sum_{k=0}^{\infty} \frac{a_k}{\nu}\, \tau^{(k+\lambda-\mu\nu)/(\mu\nu)} \quad (\tau \to 0+,\ \tau \in \mathbb{R}).$$

By Watson's lemma, we obtain

$$F(s, \nu) \sim \sum_{k=0}^{\infty} \frac{a_k}{\nu}\, \Gamma\Big(\frac{k+\lambda}{\mu\nu}\Big)\, \frac{1}{s^{(k+\lambda)/(\mu\nu)}} \quad \Big(s \to \infty,\ |\arg s| \le \frac{\pi}{2} - \delta\Big). \tag{6.27}$$

Consider now a generalisation of Laplace's integral

$$I(x) := \int_a^b e^{-x\,p(t)}\, q(t)\, dt, \tag{6.28}$$

where x is a large positive parameter. The functions $p(t)$ and $q(t)$ are independent of the parameter x. Either a or b or both may be infinite. We assume that the function $p(t)$ has a global minimum at $t = t_0$. A powerful method for approximating $I(x)$ for large values of x originates from Laplace (1820). The idea is that the peak of $\exp[-x\,p(t)]$, for large x, occurs at the minimum of $p(t)$. This means that the major contribution to the integral comes from a small neighborhood of $t = t_0$. Accordingly, we can replace the functions $p(t)$ and $q(t)$ by their convergent or asymptotic series expansions at $t = t_0$ and then substitute the variable in the

integral appropriately in order to apply Watson's lemma. Finally, we obtain an asymptotic expansion for the integral $I(x)$ for large values of x. We shall distinguish three cases:

- the global minimum of $p(t)$ is at the left endpoint of the integration interval: $t_0 = a$,
- the global minimum of $p(t)$ is at the right endpoint of the integration interval: $t_0 = b$,
- the global minimum of $p(t)$ is at an interior point $t = t_0$ of the integration interval.

An integral of the type (6.28) may occur, for example, when Efros's rule is applied. This in general, may turn the original function into an integral expression that cannot be evaluated in a closed manner. Laplace's method can then be applied to obtain an asymptotic expansion for the original function.

6.2.4.1 Minimum Value of $p(t)$ at the Left Endpoint of $[a, b]$

This is the basic case. Assume that $p(t)$ and $q(t)$ have the following asymptotic expansions for $t \to a+$:

$$p(t) \sim p(a) + \sum_{k=0}^{\infty} p_k (t-a)^{k+\mu}, \qquad p_0 > 0,\ \mu > 0,$$

$$q(t) \sim \sum_{k=0}^{\infty} q_k (t-a)^{k+\lambda-1}, \qquad q_0 \neq 0,\ \lambda > 0.$$

Also assume that the first asymptotic relation can be differentiated, i.e.,

$$p'(t) \sim \sum_{k=0}^{\infty} p_k\,(k+\mu)\,(t-a)^{k+\mu-1} \qquad (t \to a+).$$

In order to simplify the computations, assume that $v := p(t) - p(a)$ is an increasing function on (a, b) and therefore has an inverse function $t - a =: \tau(v)$. By making the substitution $v = p(t) - p(a)$, we have

$$I(x) := \int_a^b e^{-x\,p(t)}\, q(t)\, dt = e^{-x\,p(a)} \int_a^b e^{-x\,[p(t)-p(a)]}\, q(t)\, dt$$

$$= e^{-x\,p(a)} \int_0^k e^{-x\,v}\, q(a+\tau(v))\, \frac{dv}{p'(a+\tau(v))}$$

with $k = p(b) - p(a) > 0$. Now

$$\frac{q(a+\tau(v))}{p'(a+\tau(v))} \sim \frac{\sum_{k=0}^{\infty} q_k\, \tau^{k+\lambda-1}}{\sum_{k=0}^{\infty} p_k\,(k+\mu)\,\tau^{k+\mu-1}} = \tau^{\lambda-\mu}\, \frac{\sum_{k=0}^{\infty} q_k\, \tau^k}{\sum_{k=0}^{\infty} p_k\,(k+\mu)\,\tau^k}.$$

Let

$$\frac{\sum_{k=0}^{\infty} q_k\, \tau^k}{\sum_{k=0}^{\infty} p_k\,(k+\mu)\,\tau^k} = \sum_{k=0}^{\infty} r_k\, \tau^k$$

where the coefficients r_k are recursively determined by the division algorithm for power series

$$r_0 = \frac{q_0}{\mu\, p_0},$$
$$r_k = \frac{1}{\mu\, p_0}\left\{q_k - \sum_{j=0}^{k-1} r_j\, p_{k-j}\,(k-j+\mu)\right\} \quad (k = 1, 2, 3, \ldots). \tag{6.29}$$

We then have

$$\frac{q(a+\tau(v))}{p'(a+\tau(v))} \sim \sum_{k=0}^{\infty} r_k\, \tau^{k+\lambda-\mu} \quad (\tau \to 0+).$$

In order to expand the right-hand side in the variable v, we write

$$v = p(t) - p(a) = \sum_{k=0}^{\infty} p_k (t-a)^{k+\mu} = \sum_{k=0}^{\infty} p_k\, \tau^{k+\mu}$$
$$= p_0\, \tau^{\mu}\left[1 + \sum_{k=1}^{\infty} \frac{p_k}{p_0}\, \tau^k\right],$$
$$z := v^{1/\mu} = {p_0}^{1/\mu}\, \tau \left[1 + \sum_{k=1}^{\infty} \frac{p_k}{p_0}\, \tau^k\right]^{1/\mu} = {p_0}^{1/\mu}\, \tau\, B_{1/\mu}\left(\sum_{k=1}^{\infty} \frac{p_k}{p_0}\, \tau^k\right),$$

with the binomial series $B_{1/\mu}(x) := (1+x)^{1/\mu}$.

A power series $Q(\tau)$ with $Q(0) = 0$ can be substituted into a binomial series and the coefficients of the resulting series can be computed by the J.C.P. Miller formula, see [Henrici (1974)]. We obtain

$$z = {p_0}^{1/\mu}\, \tau\, B_{1/\mu}(Q(\tau)) = {p_0}^{1/\mu}\, \tau\, B_{1/\mu}\left(\sum_{k=1}^{\infty} \frac{p_k}{p_0}\, \tau^k\right)$$
$$= {p_0}^{1/\mu}\, \tau \sum_{k=0}^{\infty} c_k\, \tau^k = \sum_{k=1}^{\infty} {p_0}^{1/\mu}\, c_{k-1}\, \tau^k.$$

The c_k are determined recursively by the J.C.P. Miller formula

$$c_0 = 1; \quad c_k = \frac{1}{k} \sum_{j=0}^{k-1} \left[\frac{k-j}{\mu} - j\right] c_j\, \frac{p_{k-j}}{p_0}. \tag{6.30}$$

The power series

$$z = \sum_{k=1}^{\infty} a_k\, \tau^k, \quad a_k = {p_0}^{1/\mu}\, c_{k-1}, \; a_1 = {p_0}^{1/\mu} \tag{6.31}$$

can be reversed. The inverse series is

$$\tau = \sum_{n=1}^{\infty} b_n\, z^n, \quad b_1 = \frac{1}{{p_0}^{1/\mu}}.$$

and we obtain, by the Schur-Jabotinsky lemma [Henrici (1974)],

$$b_n = \frac{1}{n}\, a_{-1}^{(-n)} . \tag{6.32}$$

The coefficients $a_{-k}^{(-n)}$ come from the expansion

$$z^n = \sum_{k=n}^{\infty} a_k^{(n)}\, \tau^k \qquad (n \in \mathbb{Z})$$

where the $a_k^{(n)}$ are determined recursively by the generalized J.P.C. Miller formula [Henrici (1974)]. For n $\in \mathbb{Z}$ we have

$$a_n^{(n)} = {a_1}^n ,$$

$$a_k^{(n)} = \frac{1}{a_1\,(k-n)} \sum_{j=1}^{k-n} [(n+1)\,j - (k-n)]\, a_{1+j}\, a_{k-j}^{(n)} \qquad (k = n+1,\ n+2,\ \ldots).$$

In order to compute the b_n, we need $a_{-1}^{(-n)}$. This is obtained recursively from the above formula, if we replace n by $-n$ and k by $-k$,

$$a_{-n}^{(-n)} = {a_1}^{-n} = 1 / {a_1}^n$$

$$a_{-k}^{(-n)} = \frac{1}{a_1\,(n-k)} \times \sum_{j=1}^{n-k} [(-n+1)\,j - (n-k)]\; a_{1+j}\; a_{-k-j}^{(-n)} \tag{6.33}$$

$$(k = n-1,\ n-2,\ \ldots,\ 1).$$

Once $a_{-1}^{(-n)}$ is computed, we have $b_n = (1/n)\, a_{-1}^{(-n)}$. From

$$\tau = \sum_{n=1}^{\infty} b_n\, z^n$$

$$= b_1\, z \left[1 + \sum_{n=2}^{\infty} \frac{b_n}{b_1}\, z^{n-1}\right] = b_1\, z \left[1 + \sum_{n=1}^{\infty} \frac{b_{n+1}}{b_1}\, z^n\right]$$

we obtain

$$\tau^{k+\lambda-\mu} = {b_1}^{k+\lambda-\mu}\, z^{k+\lambda-\mu} \left[1 + \sum_{n=1}^{\infty} \frac{b_{n+1}}{b_1}\, z^n\right]^{k+\lambda-\mu}$$

$$= {b_1}^{k+\lambda-\mu}\, z^{k+\lambda-\mu}\, B_{k+\lambda-\mu}\left(\sum_{n=1}^{\infty} \frac{b_{n+1}}{b_1}\, z^n\right)$$

$$= : {b_1}^{k+\lambda-\mu}\, z^{k+\lambda-\mu} \sum_{n=0}^{\infty} f_n^{(k)}\, z^n = \sum_{n=0}^{\infty} f_n^{(k)}\, {b_1}^{k+\lambda-\mu}\, z^{n+k+\lambda-\mu}$$

$$= \sum_{m=k}^{\infty} f_{m-k}^{(k)}\, {b_1}^{k+\lambda-\mu}\, z^{m+\lambda-\mu} ,$$

where the coefficients $f_n^{(k)}$ again are recursively determined by the J.P.C. Miller formula

$$f_0^{(k)} = 1; \quad f_n^{(k)} = \frac{1}{n} \sum_{j=0}^{n-1} [(n-j)(k+\lambda-\mu) - j] f_j^{(k)} \frac{b_{n+1-j}}{b_1} \qquad (n = 1, 2, \ldots). \tag{6.34}$$

Here we have set $n + k = m$, i.e. $n = m - k$. Now we have

$$\frac{q(a+\tau(v))}{p'(a+\tau(v))} \sim \sum_{k=0}^{\infty} r_k \tau^{k+\lambda-\mu} = \sum_{k=0}^{\infty} r_k \sum_{m=k}^{\infty} f_{m-k}^{(k)} b_1{}^{k+\lambda-\mu} z^{m+\lambda-\mu}.$$

By interchanging the order of summation in the double series, we finally obtain

$$\sum_{k=0}^{\infty} r_k \tau^{k+\lambda-\mu} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} r_k f_{m-k}^{(k)} b_1{}^{k+\lambda-\mu} z^{m+\lambda-\mu} = \sum_{m=0}^{\infty} A_m v^{(m+\lambda-\mu)/\mu}$$

with

$$A_m = \sum_{k=0}^{m} r_k f_{m-k}^{(k)} b_1{}^{k+\lambda-\mu}.$$

Let us summarize the procedure. The r_k's are determined from (6.29). The $f_n^{(k)}$'s follow fron (6.34) which uses (6.32) and hence (6.33). (6.33) uses (6.31) and (6.30). From these formulas t three coefficients can be computed by hand:

$$A_0 = \frac{q_0}{\mu\, p_0{}^{\lambda/\mu}}, \quad A_1 = \frac{1}{p_0{}^{(1+\lambda)/\mu}} \left\{ \frac{q_1}{\mu} - \frac{q_0\, p_1(\lambda+1)}{p_0\, \mu^2} \right\}$$

$$A_2 = \frac{1}{p_0{}^{(2+\lambda)/\mu}} \tag{6.35}$$

$$\left\{ \frac{q_2}{\mu} - \frac{q_1\, p_1(\lambda+2)}{p_0\, \mu^2} + [(\lambda+\mu+2)\, p_1^2 - 2\mu\, p_0\, p_1] \frac{q_0\,(\lambda+2)}{2\mu^3\, p_0^2} \right\}.$$

The computation of higher coefficients by hand becomes rapidly cumbersome. But the procedure chosen can be programmed, which is done in the Package. We now have the situation

$$I(x) := \int_a^b e^{-x\,p(t)}\, q(t)\, dt$$

$$= e^{-x\,p(a)} \int_0^{\infty} e^{-x\,v} \frac{q(a+\tau(v))}{p'(a+\tau(v))}\, dv - e^{-x\,p(a)} \int_k^{\infty} e^{-x\,v} \frac{q(a+\tau(v))}{p'(a+\tau(v))}\, dv$$

with

$$\frac{q(a+\tau(v))}{p'(a+\tau(v))} \sim \sum_{m=0}^{\infty} A_m v^{(m+\lambda-\mu)/\mu} \quad (v \to 0+).$$

For an asymptotic expansion for large x, it can be shown that the contribution of the second integral on the right-hand side can be neglected. We apply Watson's lemma to the first integral and obtain the

Proposition. (*Laplace's Method I*) *If $I(x)$ converges absolutely for sufficiently large real x, and if $p(t)$ attains a global minimum at $t = a$, where the asymptotic expansions*

$$p(t) \sim p(a) + \sum_{k=0}^{\infty} p_k (t-a)^{k+\mu}, \quad p_0 > 0,\ \mu > 0,$$

$$q(t) \sim \sum_{k=0}^{\infty} q_k (t-a)^{k+\lambda-1}, \quad q_0 \neq 0,\ \lambda > 0$$

hold, then for $x \to \infty$,

$$I(x) \sim e^{-x\,p(a)} \sum_{m=0}^{\infty} \Gamma\Big(\frac{m+\lambda}{\mu}\Big) \frac{A_m}{x^{(m+\lambda)/\mu}} \quad . \tag{6.36}$$

The coefficients A_m are determined by the equations (6.29) through (6.34).

6.2.4.2 Minimum Value of $p(t)$ at the Right Endpoint of $[a, b]$

This case can be transformed into the former one. We assume that $p(t)$ attains a global minimum at the right finite limit $t = b$ with the asymptotic expansions

$$p(t) \sim p(b) + \sum_{k=0}^{\infty} p_k (b-t)^{k+\mu}, \quad p_0 > 0,\ \mu > 0, \quad (t \to b-),$$

$$q(t) \sim \sum_{k=0}^{\infty} q_k (b-t)^{k+\lambda-1}, \quad q_0 \neq 0,\ \lambda > 0, \quad (t \to b-).$$

In the integral

$$I(x) = \int_a^b e^{-x\,p(t)}\, q(t)\, dt$$

we make a change of variables $t = -\tau$ and obtain

$$I(x) = \int_{-b}^{-a} e^{-x\,p(-\tau)}\, q(-\tau)\, d\tau$$

where the function

$$\hat{p}(\tau) := p(-\tau) \sim p(b) + \sum_{k=0}^{\infty} p_k (b+\tau)^{k+\mu}$$

$$= p(b) + \sum_{k=0}^{\infty} p_k (\tau - (-b))^{k+\mu}$$

now has its global minimum at the left endpoint $-b$. Similarly,

$$\hat{q}(t) = q(-\tau) \sim \sum_{k=0}^{\infty} q_k (b+\tau)^{k+\lambda-1} = \sum_{k=0}^{\infty} q_k (\tau - (-b))^{k+\lambda-1}.$$

Using the result of the last section, we obtain the

Proposition. (*Laplace's Method II*) *If $I(x)$ converges absolutely for sufficiently large real x, and if $p(t)$ attains a global minimum at the finite right endpoint $t = b$ of the interval with the asymptotic expansions for $t \to b-$,*

$$p(t) \sim p(b) + \sum_{k=0}^{\infty} p_k (b-t)^{k+\mu}, \quad p_0 > 0,\ \mu > 0,$$

$$q(t) \sim \sum_{k=0}^{\infty} q_k (b-t)^{k+\lambda-1}, \quad q_0 \neq 0,\ \lambda > 0,$$

then for $x \to \infty$,

$$I(x) \sim e^{-x\,p(b)} \sum_{m=0}^{\infty} \Gamma\Big(\frac{m+\lambda}{\mu}\Big) \frac{A_m}{x^{(m+\lambda)/\mu}}. \tag{6.37}$$

The coefficients A_m are determined by the equations (6.29) through (6.34).

6.2.4.3 Minimum Value of $p(t)$ at an Interior Point of $[a, b]$

We now assume that $p(t)$ attains a global minimum at the interior point $t = t_0$ of the interval (a, b). First consider the general case where we may have different asymptotic expansions for $t \to t_0 -$ and for $t \to t_0 +$:

$$p(t) \sim p(t_0) + \sum_{k=0}^{\infty} p_k^- (t_0 - t)^{k+\mu}, \quad p_0^- > 0,\ \mu^- > 0 \qquad (t \to t_0 -),$$

$$q(t) \sim \sum_{k=0}^{\infty} q_k^- (t_0 - t)^{k+\lambda-1}, \quad q_0^- \neq 0,\ \lambda^- > 0 \qquad (t \to t_0 -),$$

$$p(t) \sim p(t_0) + \sum_{k=0}^{\infty} p_k^+ (t - t_0)^{k+\mu}, \quad p_0^+ > 0,\ \mu^+ > 0 \qquad (t \to t_0 +),$$

$$q(t) \sim \sum_{k=0}^{\infty} q_k^+ (t - t_0)^{k+\lambda-1}, \quad q_0^+ \neq 0,\ \lambda^+ > 0 \qquad (t \to t_0 +).$$

This is the case if, for example, the function $p(t)$ is continuous at $t = t_0$ but $p'(t_0 -) \neq p'(t_0 +)$. We then obtain the compound asymptotic expansion for $x \to \infty$

$$I(x) \sim e^{-x\,p(t_0)} \left\{ \sum_{m=0}^{\infty} \Gamma\Big(\frac{m+\lambda^-}{\mu^-}\Big) \frac{A_m^-}{x^{(m+\lambda^-)/\mu^-}} + \sum_{m=0}^{\infty} \Gamma\Big(\frac{m+\lambda^+}{\mu^+}\Big) \frac{A_m^+}{x^{(m+\lambda^+)/\mu^+}} \right\}. \tag{6.38}$$

More frequent and more interesting are power series that converge in a neighborhood of $t = t_0$:

$$\begin{aligned} p(t) &= p(t_0) + \sum_{k=0}^{\infty} p_k (t - t_0)^{k+2\ell}, \quad p_0 > 0,\ \ell \in \mathbb{N}, \\ q(t) &= \sum_{k=0}^{\infty} q_k (t - a)^{k+n-1}, \quad q_0 \neq 0,\ n \in \mathbb{N}. \end{aligned} \tag{6.39}$$

This implies

$$\mu^- = \mu^+ = 2\ell,\ \lambda^- = \lambda^+ = n,\ p_k^+ = p_k = (-1)^k p_k^-,\ q_k^+ = q_k = (-1)^{k+n-1} q_k^-$$

which gives

$$I(x) \sim e^{-x p(b)} \sum_{k=0}^{\infty} \Gamma\Big(\frac{k+n}{2\ell}\Big) \frac{A_k^- + A_k^+}{x^{(k+n)/(2\ell)}} \qquad (x \to \infty).$$

From (6.35) we obtain

$$A_0^- + A_0^+ = \frac{(-1)^{n-1} q_0}{2\ell [(-1)^0 p_0]^{n/(2\ell)}} + \frac{q_0}{2\ell p_0^{n/(2\ell)}}$$

$$= \frac{q_0}{2\ell p_0^{n/(2\ell)}} [(-1)^{n-1} + 1],$$

$$A_1^- + A_1^+ = \frac{1}{p_0^{(1+n)/(2\ell)}} \left\{ \frac{(-1)^n q_1}{\mu} - \frac{(-1)^{n-1} q_0 (-1) p_1 (\lambda + 1)}{p_0 \mu^2} \right\}$$

$$+ \frac{1}{p_0^{(1+\lambda)/\mu}} \left\{ \frac{q_1}{\mu} - \frac{q_0 p_1 (\lambda + 1)}{p_0 \mu^2} \right\}$$

$$= \frac{1}{p_0^{(1+\lambda)/\mu}} \left\{ \frac{q_1}{\mu} - \frac{q_0 p_1 (\lambda + 1)}{p_0 \mu^2} \right\} [(-1)^n + 1]$$

and so on. In general,

$$A_k^- + A_k^+ = A_k [(-1)^{k+n-1} + 1] \qquad (k = 0, 1, 2, \ldots).$$

Proposition. (Laplace's Method III) *If $I(x)$ converges absolutely for sufficiently large real x and if $p(t)$ attains a global minimum at the interior point $t = t_0$ of the interval where the convergent power series (6.39) hold, then for $x \to \infty$,*

$$I(x) \sim e^{-x p(t_0)} \sum_{k=0}^{\infty} \Gamma\Big(\frac{k+n}{2\ell}\Big) [(-1)^{k+n-1} + 1] \frac{A_k}{x^{(k+n)/(2\ell)}}.$$

The following example taken from Olver's Book [Olver (1974)], explains another feature. Often an integral that depends on a parameter is not from the beginning given in a form where Laplace's method can be applied. Take for example

$$I(x) = \int_0^{\infty} e^{x t - (t-1) \ln t}\, dt.$$

The obvious choice of $p(t) = -t$ and $q(t) = t^{-(t-1)}$ is not useful because $p(t)$ has no minimum on $[0, \infty)$. Therefore we must look for the peak of the whole integrand. This occurs where the derivative of the exponent vanishes:

$$x - \ln t - (t-1)/t = x - 1 - \ln t + 1/t = 0.$$

This is a transcendental equation for t which, for large x, has the approximate solution

$$t \approx \xi(x) = e^{x-1},$$

which is the approximate global minimum of the integrand. In order to make the peak value independent from x, we substitute $\tau = t/\xi(x)$. This gives

$$I(x) = \xi(x)^2 \int_0^\infty e^{-\xi(x)\,p(\tau)}\, q(\tau)\, d\tau$$

with

$$p(\tau) = \tau(\ln \tau - 1), \quad q(\tau) = \tau.$$

The function $p(\tau)$ now has a global minimum at the interior point $\tau_0 = 1$ with the expansions

$$p(\tau) = -1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)} (\tau - 1)^{k+2}, \quad q(\tau) = 1 + (\tau - 1).$$

Thus $\ell = 1$, $n = 1$, $q_0 = q_1 = 1$, $q_k = 0$ $(k \geq 2)$, $p_k = (-1)^k / ((k+1)(k+2))$. This implies

$$A_k^- + A_k^+ = 2A_k \ (k \text{ even}), \quad A_k^- + A_k^+ = 0 \ (k \text{ odd}),$$

$$2A_0 = \frac{1}{\sqrt{2}}, \quad 2A_2 = \frac{1}{48\sqrt{2}}.$$

Therefore

$$\int_0^\infty e^{-\xi(x)\,p(\tau)}\, q(\tau)\, d\tau \sim \frac{\sqrt{2\pi}}{\sqrt{\xi}} \left\{1 + \frac{1}{192}\,\frac{1}{\xi} + \ldots\right\} \qquad (\xi \to \infty)$$

or, after restoring the original variable x, we obtain for $x \to \infty$,

$$I(x) \sim \sqrt{2\pi}\, e^{3(x-1)/2} \exp(e^{x-1}) \left\{1 + \frac{e^{-(x-1)}}{192} + \cdots\right\}.$$

The preliminary steps in this example should be clearly understood since they recur often in other examples. Let us summarize:

- Set up an equation for the abscissa t of the global minimum of the whole integrand.
- Solve the equation, which in general is transcendental, asymptotically for large x, giving $t = \xi(x)$, say.
- Introduce a new integration variable $\tau = t/\xi(x)$ that makes the approximate location of the new minimum independent of the parameter x.

■ 6.2.5 Use of the Package

See Section 3.1.1 and Section 3.1.2 for the installation and loading of the Package.

6.2.5.1 Asymptotic Series

In order to obtain an asymptotic expansion of $f(t)$ at $t = a$, up to the order n, use `AsymptoticSeries[f[t],{t,a,n}]`. For example:

```
f = 1 / Sqrt[a t] / (1 + Sqrt[a t]) ^ 2
```

$$\frac{1}{\sqrt{a\,t}\,\left(1+\sqrt{a\,t}\right)^2}$$

```
AsymptoticSeries[f, {t, 0, 4}]
```

$$-2+\frac{1}{\sqrt{a}\,\sqrt{t}}+3\sqrt{a}\,\sqrt{t}-4\,a\,t+5\,a^{3/2}\,t^{3/2}-6\,a^2\,t^2+$$
$$7\,a^{5/2}\,t^{5/2}-8\,a^3\,t^3+9\,a^{7/2}\,t^{7/2}-10\,a^4\,t^4+11\,a^{9/2}\,t^{9/2}+\text{LandauO}[t^5]$$

where $a\,t > 0$ is assumed. If $f(t)$ contains $|t-a|$, use

`AsymptoticSeries[f[t],{t,a,n},Direction->1]` and

`AsymptoticSeries[f[t],{t,a,n},Direction->-1]`

according to whether the expansion at $t = a$ is from the left or from the right. For example:

```
AsymptoticSeries[1 / (1 - Abs [t]) ^ (3 / 2), {t, 0, 4}, Direction → 1]
```

$$1-\frac{3\,t}{2}+\frac{15\,t^2}{8}-\frac{35\,t^3}{16}+\frac{315\,t^4}{128}+\text{LandauO}[t^5]$$

```
AsymptoticSeries[1 / (1 - Abs [t]) ^ (3 / 2), {t, 0, 4}, Direction → -1]
```

$$1+\frac{3\,t}{2}+\frac{15\,t^2}{8}+\frac{35\,t^3}{16}+\frac{315\,t^4}{128}+\text{LandauO}[t^5]$$

6.2.5.2 Watson's Lemma

Let, for example,

```
f = Cos[Sqrt[t]] / (Sqrt[t] (1 + Sqrt[t]) ^ 2)
```

$$\frac{\text{Cos}\left[\sqrt{t}\right]}{\left(1+\sqrt{t}\right)^2\sqrt{t}}$$

What is the asymptotic series of its image function for $s \to \infty$? Watson's lemma gives an answer by use of the asymptotic expansion of $f(t)$ at $t \to 0+$. The asymptotic expansion of the image function is obtained by substituting `LaplaceIntegral[f[t],{t,0}][s]` into `AsymptoticSeries`.

```
LI = LaplaceIntegral[f, {t, 0}][s];
```

```
AsymptoticSeries[LI, {s, Infinity, 5}]
```

$$\frac{753507929\sqrt{\pi}}{589824\,s^{13/2}} - \frac{13290659}{15120\,s^6} + \frac{24620669\sqrt{\pi}}{122880\,s^{11/2}} - \frac{41963}{280\,s^5} + \frac{229993\sqrt{\pi}}{6144\,s^{9/2}} - \frac{1859}{60\,s^4} + \frac{3329\sqrt{\pi}}{384\,s^{7/2}} - \frac{49}{6\,s^3} + \frac{85\sqrt{\pi}}{32\,s^{5/2}} - \frac{3}{s^2} + \frac{5\sqrt{\pi}}{4\,s^{3/2}} - \frac{2}{s} + \frac{\sqrt{\pi}}{\sqrt{s}} + \text{Landau0}\left[\frac{1}{s^7}\right]$$

To get rid of the `Landau O`-term use the NamedRule `RemoveLandauO`:

```
% /. RemoveLandauO
```

$$\frac{753507929\sqrt{\pi}}{589824\,s^{13/2}} - \frac{13290659}{15120\,s^6} + \frac{24620669\sqrt{\pi}}{122880\,s^{11/2}} - \frac{41963}{280\,s^5} + \frac{229993\sqrt{\pi}}{6144\,s^{9/2}} - \frac{1859}{60\,s^4} + \frac{3329\sqrt{\pi}}{384\,s^{7/2}} - \frac{49}{6\,s^3} + \frac{85\sqrt{\pi}}{32\,s^{5/2}} - \frac{3}{s^2} + \frac{5\sqrt{\pi}}{4\,s^{3/2}} - \frac{2}{s} + \frac{\sqrt{\pi}}{\sqrt{s}}$$

Example. Let us revisit the example at the end of Section 6.1.3.3, where the solution of the initial value problem

$$y'' + 2\kappa y' + (1+\kappa^2)\,y(t) = \frac{1}{\sqrt{t}}\,e^{bt} \qquad (b \in \mathbb{R})$$
$$y(0+) = y'(0) = 0$$

was found to be

$$y(t) = \sqrt{\pi}\,\frac{e^{-\kappa t}}{\sqrt{A^2+1}}\left\{-\sqrt{\frac{\sqrt{A^2+1}+A}{2}}\cos t + \sqrt{\frac{\sqrt{A^2+1}-A}{2}}\sin t\right\}$$
$$+ \frac{e^{bt}}{\sqrt{\pi}}\int_0^\infty \frac{e^{-xt}}{(x^2 - 2Ax + 1 + A^2 - b^2)\sqrt{x}}\,dx.$$

The last term is a Laplace integral for which an asymptotic expansion for large values of t can be found.

```
f = ((x^2 - 2 A x + 1 + A^2 - b^2) Sqrt[x] Sqrt[π])^-1;

LI = LaplaceIntegral[f, {x, 0}][t];

AsymptoticSeries[LI, {t, Infinity, 2}]
```

$$\frac{3\left(\frac{4A^2}{(1+A^2-b^2)^2} - \frac{1}{1+A^2-b^2}\right)}{4\,(1+A^2-b^2)\,t^{5/2}} + \frac{A}{(1+A^2-b^2)^2\,t^{3/2}} + \frac{1}{(1+A^2-b^2)\sqrt{t}} + \text{Landau0}\left[\frac{1}{t^3}\right]$$

with $A = a + \kappa$.

6.2.5.3 Watson's Lemma for Loop Integrals

A tool for finding the asymptotic expansion of an original function for large values of t is to use the asymptotic expansion of the image function at a branch point. For example,

```
F = (s + 2) / ((s - 4) (s - 1)) / Sqrt[s] / Sqrt[s + 1]
```

$$\frac{2+s}{(-4+s)\,(-1+s)\,\sqrt{s}\,\sqrt{1+s}}$$

We first compute the contribution of the poles at $s = 1$ and $s = 4$. The residues of $e^{st} F(s)$ at these points are evaluated.

```
f1 = ResidueLaplace[F, {1, 4}, s, t]
```

$$-\frac{e^{t}}{\sqrt{2}} + \frac{e^{4t}}{\sqrt{5}}$$

The asymptotic behavior for large values of t is determined by the right-most branch point, in this case the branch point at $s = 0$. The vertical integration path can be bent to the left (the Closing Contour Condition (6.3) is satisfied), and we use Watson's lemma for loop integrals (Hankel-type integrals). The asymptotic expansion of $F(s)$ at $a = 0$ is then used. `HankelIntegral[F[s],{s,a}][t]` is substituted into `AsymptoticSeries`.

```
HI = HankelIntegral[F, {s, 0}][t];

f2 = AsymptoticSeries[HI, {t, Infinity, 5}]
```

$$-\frac{1327725}{65536\sqrt{\pi}\,t^{11/2}} + \frac{39165}{8192\sqrt{\pi}\,t^{9/2}} - \frac{1305}{1024\sqrt{\pi}\,t^{7/2}} + \frac{69}{128\sqrt{\pi}\,t^{5/2}} - \frac{5}{16\sqrt{\pi}\,t^{3/2}} + \frac{1}{2\sqrt{\pi}\sqrt{t}} + \text{LandauO}\left[\frac{1}{t^6}\right]$$

The complete expression governing the original for large values of t then is

```
f1 + f2
```

$$-\frac{e^{t}}{\sqrt{2}} + \frac{e^{4t}}{\sqrt{5}} - \frac{1327725}{65536\sqrt{\pi}\,t^{11/2}} + \frac{39165}{8192\sqrt{\pi}\,t^{9/2}} - \frac{1305}{1024\sqrt{\pi}\,t^{7/2}} + \frac{69}{128\sqrt{\pi}\,t^{5/2}} - \frac{5}{16\sqrt{\pi}\,t^{3/2}} + \frac{1}{2\sqrt{\pi}\sqrt{t}} + \text{LandauO}\left[\frac{1}{t^6}\right]$$

In the next example we compare the approximation of the computed asymptotic expansion with the exact inverse Laplace transform.

```
F = 1 / (s - 1 / 10) / Sqrt[s + 2 / 10]
```

$$\frac{1}{\left(-\frac{1}{10} + s\right)\sqrt{\frac{1}{5} + s}}$$

The contribution from the pole at $s = 1/10$ is

```
f1 = ResidueLaplace[F, {1 / 10}, s, t]
```

$$\sqrt{\frac{10}{3}}\, e^{t/10}$$

The asymptotic expansion for large values of t is governed by the branch point at $s = -1/5$:

```
HI = HankelIntegral[F, {s, -1 / 5}][t];

f2 = AsymptoticSeries[HI, {t, Infinity, 2}]
```

$$e^{-t/5}\left(-\frac{250}{9\sqrt{\pi}\,t^{5/2}} + \frac{50}{9\sqrt{\pi}\,t^{3/2}} - \frac{10}{3\sqrt{\pi}\sqrt{t}} + \text{LandauO}\left[\frac{1}{t^3}\right]\right)$$

The expression of the original for large values of t, therefore, becomes

```
f1 + f2;

fasmp = % /. RemoveLandau0;
```

On the other hand, the exact inverse Laplace transform with its plot is

```
fexact = InverseLaplaceTransform[F, s, t]
```

$$\sqrt{\frac{10}{3}}\, e^{t/10}\, \mathrm{Erf}\left[\sqrt{\frac{3}{10}}\,\sqrt{t}\,\right]$$

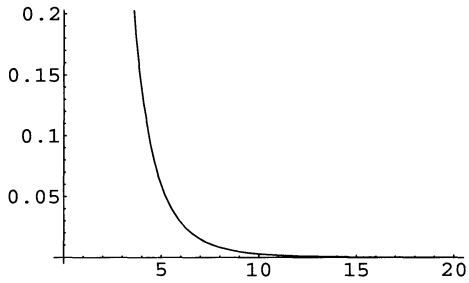
```
Plot[fexact, {t, 1, 20}];
```

The plot of the asymptotic expression of the original is

```
Plot[fasmp, {t, 1, 20}, PlotRange → {0, 13.5}];
```

The plot of the difference of the exact solution and the asymptotic expression clearly shows that t need not be very large for a reasonable approximation by the asymptotic expansion. $t > 5$ suffices.

```
Plot[fexact - fasmp, {t, 3, 20}];
```



Example. Reconsider the example at the end of Section 6.2.5.2 where we found an asymptotic expression for the solution of

$$y'' + 2\kappa y' + (1+\kappa^2)\, y(t) = \frac{1}{\sqrt{t}}\, e^{bt} \qquad (b \in \mathbb{R}),$$
$$y(0+) = y'(0) = 0.$$

We now use the representation of the integral term by the Hankel-type integral (see Section 6.1.3.3):

$$y(t) = \sqrt{\pi}\, \frac{e^{-\kappa t}}{\sqrt{A^2+1}} \left\{ -\sqrt{\frac{\sqrt{A^2+1}+A}{2}}\, \cos t + \sqrt{\frac{\sqrt{A^2+1}-A}{2}}\, \sin t \right\}$$
$$+ \sqrt{\pi}\, \frac{e^{bt}}{2\pi i} \int_{-\infty}^{(0+)} \frac{e^{st}}{(s^2 + 2As + 1 + A^2 - b^2)\sqrt{s}}\, ds\,.$$

An asymptotic expansion for large values of t is obtained by

```
g = Sqrt[Pi] ((s^2 + 2 A s + 1 + A^2 - b^2) Sqrt[s]) ^-1;

HI = HankelIntegral[g, {s, 0}][t];
```

```
AsymptoticSeries[HI, {t, Infinity, 2}]
```

$$\frac{3\left(\frac{4A^2}{(1+A^2-b^2)^2}-\frac{1}{1+A^2-b^2}\right)}{4\,(1+A^2-b^2)\,t^{5/2}}+\frac{A}{(1+A^2-b^2)^2\,t^{3/2}}+\frac{1}{(1+A^2-b^2)\,\sqrt{t}}+\text{LandauO}\left[\frac{1}{t^3}\right]$$

in agreement with the result of Section 6.2.5.2.

Example. In the last example of Section 6.1.3.3 we found the temperature distribution inside a semi-infinite solid if a periodic temperature is applied to the left boundary. We obtained

$$y(x,t)=A\,e^{-x\sqrt{\omega}/\sqrt{2}\,\kappa}\cos\left[t\,\omega-\frac{x\sqrt{\omega}}{\sqrt{2}\,\kappa}\right]+A\int_{-\infty}^{(0+)}e^{st}\,\frac{s}{s^2+\omega^2}\,e^{-x\sqrt{s}/\kappa}\,ds\,.$$

Let us now compute the asymptotic expansion of the integral of the Hankel-type.

```
F = A s / (s^2 + ω^2) Exp[-x Sqrt[s] / κ];

HI = HankelIntegral[F, {s, 0}][t];

AsymptoticSeries[HI, {t, Infinity, 3}]
```

$$\frac{105\,A\left(\frac{x}{\kappa\,\omega^4}-\frac{x^5}{120\,\kappa^5\,\omega^2}\right)}{16\,\sqrt{\pi}\,t^{9/2}}+\frac{5\,A\,x^3}{16\,\sqrt{\pi}\,t^{7/2}\,\kappa^3\,\omega^2}-\frac{3\,A\,x}{4\,\sqrt{\pi}\,t^{5/2}\,\kappa\,\omega^2}+\text{LandauO}\left[\frac{1}{t^5}\right]$$

Thus, we obtain the temperature distribution in the solid for large values of t:

$$y(x,t)=A\left\{e^{-x\sqrt{\omega}/\sqrt{2}\,\kappa}\cos\left[t\,\omega-\frac{x\sqrt{\omega}}{\sqrt{2}\,\kappa}\right]-\frac{3\,x}{4\,\sqrt{\pi}\,t^{5/2}\,\kappa\,\omega^2}+\cdots\right\}.$$

6.2.5.4 Laplace's Method

In Section 6.2.4.1 we explained an algorithm for finding the coefficients that are used in Laplace's method. The procedure allows a computation by hand only for the first few coefficients. The first three are given in (6.35). The advantage of the method presented is that i can be programmed in *Mathematica.* This is done in the Package. The command `LaplaceMethodCoefficients[{p0,p1,...,pn},{q0,q1,...,qn},μ,λ]` computes the first n coefficients {A0, A1, ..., An}. For example, in order to get a list of the general formulas for the first four coefficients:

```
LaplaceMethodCoefficients[{p0, p1, p2}, {q0, q1, q2}, μ, λ]
```

$$\left\{\frac{p_0^{-\frac{\lambda}{\mu}}\,q_0}{\mu},\ \frac{p_0^{-\frac{1+\lambda+\mu}{\mu}}\,(-(1+\lambda)\,p_1\,q_0+\mu\,p_0\,q_1)}{\mu^2},\right.$$
$$\left.\frac{1}{2\,\mu^3}\left(p_0^{-\frac{2+\lambda+2\,\mu}{\mu}}\,((2+\lambda)\,(2+\lambda+\mu)\,p_1^2\,q_0-2\,(2+\lambda)\,\mu\,p_0\,p_1\,q_1+2\,\mu\,p_0\,(-(2+\lambda)\,p_2\,q_0+\mu\,p_0\,q_2))\right)\right\}$$

In the three following commands `LaplaceTypeIntegral1...3` the substitution algorithm for computing the coefficients is used extensively. Assume that in the integral

$$I(x) := \int_a^b e^{-xP(t)}\, Q(t)\, dt$$

the function $P(t)$ has a global minimum at the left endpoint a of the interval and that the asymptotic expressions

$$P(t) \sim c + (t-a)^{\mu}\, p(t), \quad p(a) \neq 0, \qquad (t \to a+),$$

$$Q(t) \sim (t-a)^{k+\lambda}\, q(t), \qquad q(a) \neq 0, \qquad (t \to a+)$$

hold. In order to compute the asymptotic expansion up to the order n and for large values of x use

`LplT=LaplaceTypeIntegral1[c+(t-a)^μ p(t),(t-a)^λ q(t),{t,a}]`
which is inserted into `AsymptoticSeries[LplT[x],{x,Infinity,n}]`.

Example.

$$I(x) = \int_0^{\pi^2/4} e^{x\cos\sqrt{t}}\, dt \qquad (x \to \infty).$$

Here $P(t) = -\cos\sqrt{t}$ and the global minimum is at $t = 0$. Let us transform the function $P(t)$ into the demanded form.

```
Series[- Cos[Sqrt[t]], {t, 0, 5}] // Normal
```

$$-1 + \frac{t}{2} - \frac{t^2}{24} + \frac{t^3}{720} - \frac{t^4}{40320} + \frac{t^5}{3628800}$$

Make sure that t is factored out, as we must respect $p(0) \neq 0$.

```
P = ((% + 1) /. FactorOutOfSum[t]) - 1
```

$$-1 + t\left(\frac{1}{2} - \frac{t}{24} + \frac{t^2}{720} - \frac{t^3}{40320} + \frac{t^4}{3628800}\right)$$

```
LplT = LaplaceTypeIntegral1[ P, 1, {t, 0}];
AsymptoticSeries[LplT[x], {x, Infinity, 5}]
```

$$e^x\left(\frac{128}{105\,x^5} + \frac{24}{35\,x^4} + \frac{8}{15\,x^3} + \frac{2}{3\,x^2} + \frac{2}{x} + \text{Landau0}\left[\frac{1}{x^6}\right]\right)$$

Now assume that in the integral

$$I(x) := \int_a^b e^{-xP(t)}\, Q(t)\, dt$$

the function $P(t)$ has a global minimum at the right endpoint b of the interval and that the asymptotic expressions

$$P(t) \sim c + (b-t)^{\mu}\, p(t), \quad p(b) \neq 0, \qquad (t \to b-),$$

$$Q(t) \sim (b-t)^{k+\lambda}\, q(t), \qquad q(b) \neq 0, \qquad (t \to b-)$$

hold. In order to compute the asymptotic expansion up to the order n, and for large values of x, use

`LplT=LaplaceTypeIntegral2[c+(b-t)^μ p(t),(b-t)^λ q(t),{t,b}]`
which is inserted into `AsymptoticSeries[LplT[x],{x,Infinity,n}]`.

Example.

$$I(x) = \int_0^{\pi^2/4} e^{-x\cos\sqrt{t}}\, dt \qquad (x \to \infty).$$

We have $P(t) = \cos\sqrt{t}$ and the global minimum now is at the right limit $t = \pi^2/4$. First transform the function $P(t)$ into the demanded form (in order to save space we don't display all the intermediate steps)

```
Series[Cos[Sqrt[t]], {t, Pi^2/4, 5}] // Normal;
```

```
% /. {(-Pi^2/4 + t)^n_. -> (-1)^n (Pi^2/4 - t)^n};
```

```
P = % /. FactorOutOfSum[Pi^2/4 - t]
```

$$\left(\frac{1}{\pi} + \frac{\frac{\pi^2}{4} - t}{\pi^3} - \left(-\frac{2}{\pi^5} + \frac{1}{6\,\pi^3}\right)\left(\frac{\pi^2}{4} - t\right)^2 + \left(\frac{5}{\pi^7} - \frac{1}{2\,\pi^5}\right)\left(\frac{\pi^2}{4} - t\right)^3 - \left(-\frac{14}{\pi^9} + \frac{3}{2\,\pi^7} - \frac{1}{120\,\pi^5}\right)\cdot\left(\frac{\pi^2}{4} - t\right)^4\right)\left(\frac{\pi^2}{4} - t\right)$$

```
LplT = LaplaceTypeIntegral2[ P, 1, {t, Pi^2/4}];
AsymptoticSeries[LplT[x], {x, Infinity, 5}]
```

$$\frac{9\,\pi}{x^5} - \frac{8}{x^4} + \frac{\pi}{x^3} - \frac{2}{x^2} + \frac{\pi}{x} + \text{LandauO}\left[\frac{1}{x^6}\right]$$

Assume now that in the integral

$$I(x) := \int_a^b e^{-x\,P(t)}\, Q(t)\, dt$$

the function $P(t)$ has a global minimum at the interior point t_0 of the interval and that the expansions

$$P(t) = p(t_0) + (t - t_0)^{2\ell}\, p(t), \quad p(t_0) \neq 0, \quad \ell \in \mathbb{N},$$
$$Q(t) = (t - t_0)^n\, q(t), \quad q(t_0) \neq 0, \quad n \in \mathbb{N} \cup \{0\}$$

hold, where now $p(t)$ and $q(t)$ are assumed to be analytic functions at t_0. In order to compute the asymptotic expansion for large values of x up to the order n, use `LplT =`

`LaplaceTypeIntegral3[p(t0) + (t - t0)^μ p(t), (t - t0)^λ q(t), {t, t0}][x]` which is inserted into `AsymptoticSeries[LplT, {x, Infinity, n}]`.

Example. As an example we shall derive *Stirling's Formula*. We start with Euler's integral for the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \qquad (x > 0).$$

Integration by parts yields

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-t}\, t^x\, dt = \frac{1}{x}\int_0^\infty e^{-t+x\operatorname{Ln} t}\, dt.$$

First we determine the global minimum of the whole integrand. It is located at $t = x$. Secondly, we introduce the new integration variable $\tau = t/x$. After a change of variable we get

$$\Gamma(x) = x^x \int_0^\infty e^{-x(\tau-\ln\tau)}\, d\tau =: x^x I(x).$$

We now have $p(\tau) = \tau - \ln\tau$, $q(\tau) = 1$. $p(\tau)$ has a global minimum at $\tau_0 = 1$.

```
Series[τ - Log[τ], {τ, 1, 10}] // Normal;

P = ((% - 1) /. FactorOutOfSum[(-1 + τ)^2]) + 1
```

$$1 + \left(\frac{1}{2} + \frac{1-\tau}{3} + \frac{1}{4}(-1+\tau)^2 - \frac{1}{5}(-1+\tau)^3 + \frac{1}{6}(-1+\tau)^4 - \frac{1}{7}(-1+\tau)^5 + \frac{1}{8}(-1+\tau)^6 - \frac{1}{9}(-1+\tau)^7 + \frac{1}{10}(-1+\tau)^8\right)(-1+\tau)^2$$

```
LplT = LaplaceTypeIntegral3[ P, 1, {τ, 1}];
AsymptoticSeries[LplT[x], {x, Infinity, 10}]
```

$$e^{-x}\left(-\frac{571\sqrt{\frac{\pi}{2}}}{1244160\,x^{9/2}} - \frac{139\sqrt{\frac{\pi}{2}}}{25920\,x^{7/2}} + \frac{\sqrt{\frac{\pi}{2}}}{144\,x^{5/2}} + \frac{\sqrt{\frac{\pi}{2}}}{6\,x^{3/2}} + \frac{\sqrt{2\pi}}{\sqrt{x}} + \text{LandauO}\left[\frac{1}{x^5}\right]\right)$$

Finally we obtain Stirling's formula for $x \to \infty$ up to five terms:

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3} + \frac{571}{2488320x^4} + \ldots\right\}$$

6.3 Differential Equations

■ 6.3.1 Normal Systems

Let us consider the system of second order matrix differential equations

$$\mathbf{A}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{B}\frac{d\mathbf{y}}{dt} + \mathbf{C}\,\mathbf{y}(t) = \mathbf{f}(t), \tag{6.40}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are constant $n \times n$-matrices and $\mathbf{y}(t)$ and $\mathbf{f}(t)$ are vector-valued functions of dimension n. Let the initial conditions be

$$\mathbf{y}(0+) = \mathbf{y}_0, \quad \mathbf{y}'(0+) = \mathbf{y}_0', \tag{6.41}$$

with given n-vectors on the right-hand side. By taking the Laplace transform of (6.40) with the initial conditions (6.41), we obtain the image system

$$(s^2\mathbf{A} + s\mathbf{B} + \mathbf{C})\,\mathbf{Y}(s) = \mathbf{F}(s) + (s\mathbf{A} + \mathbf{B})\,\mathbf{y}_0 + \mathbf{A}\,\mathbf{y}_0'.$$

System (6.40) with $\det \mathbf{A} \neq 0$ is called a *normal system.* In this case the inverse matrix $\mathbf{A}^{-1}$ exists. Thus the last equation can be written as

$$s\mathbf{A}\left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right) s\mathbf{Y}(s)$$
$$= \mathbf{F}(s) + s\mathbf{A}\left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{y}_0 + \mathbf{A}\,\mathbf{y}_0'$$

and, if $\Re s$ is sufficiently large and after a minor simplification,

$$s\mathbf{Y}(s) =$$
$$\left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right)^{-1}\left\{\frac{1}{s}\mathbf{A}^{-1}\mathbf{F}(s) + \left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{y}_0 + \frac{1}{s}\mathbf{y}_0'\right\}.$$

Consequently by the initial value theorem, we conclude from the last equation

$$\lim_{s\to\infty} s\mathbf{Y}(s) = \mathbf{y}(0+) = \mathbf{y}_0. \tag{6.42}$$

Similarly, we write

$$s[s\mathbf{Y}(s) - \mathbf{y}_0] = s\left\{\left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right)^{-1}\right.$$
$$\times\left\{\frac{1}{s}\mathbf{A}^{-1}\mathbf{F}(s) + \left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{y}_0 + \frac{1}{s}\mathbf{y}_0'\right\}$$
$$\left. - \left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right)^{-1}\left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right)\mathbf{y}_0\right\}$$
$$= \left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right)^{-1}$$
$$\times\left[\mathbf{A}^{-1}\mathbf{F}(s) + (s\mathbf{E} + \mathbf{A}^{-1}\mathbf{B})\mathbf{y}_0 + \mathbf{y}_0' - \left(s\mathbf{E} + \mathbf{A}^{-1}\mathbf{B} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{C}\right)\mathbf{y}_0\right]$$
$$= \left(\mathbf{E} + \frac{1}{s}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{s^2}\mathbf{A}^{-1}\mathbf{C}\right)^{-1}\left[\mathbf{A}^{-1}\mathbf{F}(s) + \mathbf{y}_0' - \frac{1}{s}\mathbf{A}^{-1}\mathbf{C}\mathbf{y}_0\right].$$

Using the fact that $\mathbf{F}(s) \to 0$ for $\Re s \to \infty$, which is satisfied by the Laplace transform of an ordinary function, we conclude

$$\lim_{s\to\infty} s[s\mathbf{Y}(s) - \mathbf{y}_0] = \mathbf{y}'(0+) = \mathbf{y}_0'. \tag{6.43}$$

Equations (6.42) and (6.43) show that the computed solution of the initial value problem indeed meet the initial conditions. Furthermore, to prove this property, we need that $\det \mathbf{A} \neq 0$, i.e. the system must be normal.

For an nth order matrix system the situation is analogous: The initial conditions are met by the computed solution, in the sense of limits from the right, provided that the determinant of the matrix-coefficient of the highest derivative does not vanish, i.e. if the system is normal.

Starting again from

$$(s^2\mathbf{A} + s\mathbf{B} + \mathbf{C})\mathbf{Y}(s) = \mathbf{F}(s) + (s\mathbf{A} + \mathbf{B})\mathbf{y}_0 + \mathbf{A}\,\mathbf{y}_0'.$$

we write

$$\mathbf{Y}(s) = (s^2\,\mathbf{A} + s\,\mathbf{B} + \mathbf{C})^{-1}\,\mathbf{F}(s) + (s^2\,\mathbf{A} + s\,\mathbf{B} + \mathbf{C})^{-1}\,(s\,\mathbf{A} + \mathbf{B})\,\mathbf{y}_0 + (s^2\,\mathbf{A} + s\,\mathbf{B} + \mathbf{C})^{-1}\,\mathbf{A}\,\mathbf{y}_0'.$$

Since

$$\det(s^2\,\mathbf{A} + s\,\mathbf{B} + \mathbf{C}) = s^{2n}\det\mathbf{A} + s^{2n-1}\cdots + \cdots + \det\mathbf{C},$$

we see that for a normal system $\det(s^2\,\mathbf{A} + s\,\mathbf{B} + \mathbf{C})$ is a polynomial of maximum degree $2n$. Because all minors have degrees $2n-2$ or smaller, we conclude that in the case of a normal system the entries of the inverse matrix

$$(s^2\,\mathbf{A} + s\,\mathbf{B} + \mathbf{C})^{-1}$$

are proper rational functions. This implies that even if the right-hand side $\mathbf{f}(t)$ of (6.40) has a discontinuity at $t = 0$, the response

$$\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{Y}(s)]\,(t)$$

will be continuous at $t = 0$. Normal systems have a smoothing effect on the input.

We conclude that a normal system of a matrix differential equation of order n behaves like a scalar differential equation of order n. The setback is that normal systems are rather an exception in real world problems.

■ 6.3.2 Anomalous Systems

Modeling of a real world problem often leads to an anomalous system. A matrix differential system is said to be *anomalous,* if the coefficient matrix of the highest derivative has a vanishing determinant. Let

$$\mathbf{A}\,\frac{d^2\,\mathbf{y}}{dt^2} + \mathbf{B}\,\frac{dy}{dt} + \mathbf{C}\,\mathbf{y}(t) = \mathbf{f}(t) \qquad (\det\mathbf{A} = 0),$$
$$\mathbf{y}(0+) = \mathbf{y}_0,\quad \mathbf{y}'(0+) = \mathbf{y}_0'$$

be a given initial value problem for an anomalous system. The inverse matrix $\mathbf{A}^{-1}$ does not exist. Hence we cannot prove that the given initial values are satisfied by the solution as in the case of a normal system. The rank of the $n \times n$-matrix $\mathbf{A}$ is strictly smaller than n and the adjoint linear system

$$\mathbf{A}^t\,\boldsymbol{x} = \mathbf{0}$$

has $\nu := n - \operatorname{rank}(\mathbf{A})$ linear independent non-trivial solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_\nu$ (the rank of the adjoint matrix $\mathbf{A}^t$ equals the rank of $\mathbf{A}$). Therefore, $\mathbf{x}_k^t\,\mathbf{A} = \mathbf{0}$ for $k = 1, 2, \ldots, \nu$. Let us form the ν matrix products

$$\mathbf{x}_k^t\left\{\mathbf{A}\,\frac{d^2\,\mathbf{y}}{dt^2} + \mathbf{B}\,\frac{dy}{dt} + \mathbf{C}\,\mathbf{y}(t)\right\} = \mathbf{x}_k^t\,\mathbf{f}(t) \qquad (k = 1, 2, \ldots \nu),$$

which, by the associative law of matrix multiplication, can be written as

$$(\mathbf{x}_k^t\ \mathbf{A})\frac{d^2\mathbf{y}}{dt^2} + (\mathbf{x}_k^t\ \mathbf{B})\frac{d\mathbf{y}}{dt} + (\mathbf{x}_k^t\ \mathbf{C})\,\mathbf{y}(t)$$
$$= \mathbf{x}_k^t\,\mathbf{f}(t) \qquad (k = 1, 2, \ldots, \nu).$$

This yields ν relations

$$(\mathbf{x}_k^t\ \mathbf{B})\frac{d\mathbf{y}}{dt} + (\mathbf{x}_k^t\ \mathbf{C})\,\mathbf{y}(t) = \mathbf{x}_k^t\,\mathbf{f}(t) \qquad (k = 1, 2, \ldots, \nu). \tag{6.44}$$

By setting $t = 0$, this implies the following $\nu := n - \operatorname{rank}(\mathbf{A})$ *compatibility conditions*

$$(\mathbf{x}_k^t\ \mathbf{B})\,\mathbf{y}'(0+) + (\mathbf{x}_k^t\ \mathbf{C})\,\mathbf{y}(0+) = \mathbf{x}_k^t\,\mathbf{f}(0+) \qquad (k = 1, 2, \ldots, \nu). \tag{6.45}$$

The specified initial conditions of the solution are now related with the initial value of the given right-hand side $\mathbf{f}(0+)$. We have established:

- In an anomalous second order matrix differential system, the $2n$ initial conditions cannot be chosen independently. They have to satisfy $\nu := n - \operatorname{rank}(\mathbf{A})$ *compatibility conditions (6.45)*.
- Relation (6.44) shows that if the right-hand side $f(t)$ has a jump discontinuity at the origin, the solution also may have a jump there. This is in contrast to a normal system which exerts a smoothing effect on the input.
- Another particularity is the following: Again, start with

$$\mathbf{Y}(s) = (s^2\mathbf{A} + s\mathbf{B} + \mathbf{C})^{-1}\,\mathbf{F}(s) + (s^2\mathbf{A} + s\mathbf{B} + \mathbf{C})^{-1}(s\mathbf{A} + \mathbf{B})\,\mathbf{y}_0 + (s^2\mathbf{A} + s\mathbf{B} + \mathbf{C})^{-1}\mathbf{A}\,\mathbf{y}_0'$$

and

$$\det(s^2\mathbf{A} + s\mathbf{B} + \mathbf{C}) = s^{2n}\det\mathbf{A} + s^{2n-1}\ldots + \ldots + \det\mathbf{C} = s^{2n-1}\cdots + \cdots + \det\mathbf{C}.$$

The determinant is a polynomial of degree $\leq 2n - 1$. Since the minors of $(s^2\mathbf{A} + s\mathbf{B} + \mathbf{C})$ may have the degree $2n - 2$, the entries of the image matrix $\mathbf{Y}(s)$ may be non-proper rational functions. This implies that the solution may consist of Dirac impulses and/or its derivatives.

In the case of a first order anomalous system

$$\mathbf{A}\frac{d\mathbf{y}}{dt} + \mathbf{B}\,\mathbf{y}(t) = \mathbf{f}(t) \qquad (\det\mathbf{A} = 0),$$
$$\mathbf{y}(0+) = \mathbf{y}_0,$$

we obtain

$$\mathbf{Y}(s) = (s\mathbf{A} + \mathbf{B})^{-1}\,\mathbf{F}(s) + (s\mathbf{A} + \mathbf{B})^{-1}\,\mathbf{A}\,\mathbf{y}_0$$

with

$$\det(s\mathbf{A} + \mathbf{B}) = s^n\det\mathbf{A} + s^{n-1}\cdots + \cdots + \det\mathbf{B} = s^{n-1}\cdots + \cdots + \det\mathbf{B}.$$

The determinant is a polynomial of degree $\leq n-1$, while the minors of $s\mathbf{A}+\mathbf{B}$ may be of the same degree. This implies that the forced response

$$\mathbf{y}(t) = \mathcal{L}^{-1}[(s\mathbf{A}+\mathbf{B})^{-1}\mathbf{F}(s)](t)$$

may produce a term containing the input $\mathbf{f}(t)$. This again shows that a jump of the input function propagates through the system and reappears in the output function. Anomalous systems do not have the smoothing effect of the normal ones.

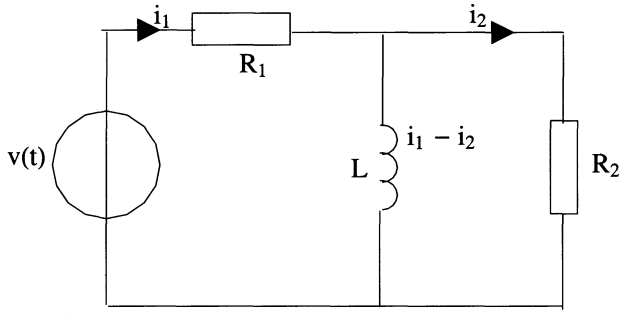


Fig.6.6: Example of an anomalous system

Example. We consider the simple electrical network shown in Fig.6.6. Kirchhoff's laws give the equations of motion

$$R_1\, i_1(t) + L\frac{d}{dt}[i_1(t)-i_2(t)] = v(t),$$
$$-L\frac{d}{dt}[i_1(t)-i_2(t)] + R_2\, i_2(t) = 0,$$

with the two initial conditions $i_1(0+)=i_{10}$, $i_2(0+)=i_{20}$. By setting $y_1(t)=i_1(t)$ $y_2(t)=i_2(t)$, we arrive at the first order matrix system

$$\begin{pmatrix} L & -L \\ -L & L \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} y_1(0+) \\ y_2(0+) \end{pmatrix} = \begin{pmatrix} i_{10} \\ i_{20} \end{pmatrix}.$$

The system is anomalous.

$$\det(s\mathbf{A}+\mathbf{B}) = \det\begin{pmatrix} sL+R_1 & -sL \\ -sL & sL+R_2 \end{pmatrix} = sL(R_1+R_2) + R_1R_2,$$

This determinant is not of full degree 2. The inverse matrix is

$$(s\mathbf{A}+\mathbf{B})^{-1} = \frac{1}{sL(R_1+R_2)+R_1R_2}\begin{pmatrix} sL+R_2 & sL \\ sL & sL+R_1 \end{pmatrix}$$

whose entries are non-proper rational functions. We decompose this matrix as follows:

$$(s\mathbf{A}+\mathbf{B})^{-1} = \frac{1}{R_1+R_2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{L(R_1+R_2)^2}\begin{pmatrix} R_2^2 & -R_1R_2 \\ -R_1R_2 & R_1^2 \end{pmatrix}\frac{1}{s+\alpha},$$

$$\alpha := \frac{R_1 R_2}{L(R_1 + R_2)}.$$

We then have

$$(s\mathbf{A} + \mathbf{B})^{-1}\mathbf{F}(s) = \frac{1}{R_1 + R_2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}V(s)\\ 0\end{pmatrix}$$
$$+ \frac{1}{L(R_1 + R_2)^2}\begin{pmatrix}R_2^2 & -R_1 R_2\\ -R_1 R_2 & R_1^2\end{pmatrix}\begin{pmatrix}V(s)\\ 0\end{pmatrix}\frac{1}{s+\alpha}$$
$$= \frac{1}{R_1 + R_2}\begin{pmatrix}V(s)\\ V(s)\end{pmatrix} + \frac{R_2}{L(R_1 + R_2)^2}\begin{pmatrix}R_2\\ -R_1\end{pmatrix}\frac{V(s)}{s+\alpha},$$

$$(s\mathbf{A} + \mathbf{B})^{-1}\mathbf{A}\mathbf{y}_0 = \frac{L}{R_1 + R_2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}y_{10}\\ y_{20}\end{pmatrix}$$
$$+ \frac{1}{(R_1 + R_2)^2}\begin{pmatrix}R_2^2 & -R_1 R_2\\ -R_1 R_2 & R_1^2\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}y_{10}\\ y_{20}\end{pmatrix}\frac{1}{s+\alpha}$$
$$= \frac{1}{R_1 + R_2}\begin{pmatrix}R_2 & -R_2\\ -R_1 & R_1\end{pmatrix}\begin{pmatrix}y_{10}\\ y_{20}\end{pmatrix}\frac{1}{s+\alpha}.$$

Finally,

$$\mathbf{Y}(s) = (s\mathbf{A} + \mathbf{B})^{-1}\mathbf{F}(s) + (s\mathbf{A} + \mathbf{B})^{-1}\mathbf{A}\mathbf{y}_0$$
$$= \frac{1}{R_1 + R_2}$$
$$\times \left\{\begin{pmatrix}1\\ 1\end{pmatrix}V(s) + \frac{R_2}{L(R_1 + R_2)}\begin{pmatrix}R_2\\ -R_1\end{pmatrix}\frac{V(s)}{s+\alpha} + \begin{pmatrix}R_2\\ -R_1\end{pmatrix}\frac{y_{10} - y_{20})}{s+\alpha}\right\}$$

or

$$\begin{pmatrix}i_1(t)\\ i_2(t)\end{pmatrix} =$$
$$\frac{v(t)}{R_1 + R_2}\left\{\begin{pmatrix}1\\ 1\end{pmatrix} + \left[\frac{R_2}{L(R_1 + R_2)}\int_0^t e^{-\alpha\tau}\, v(t-\tau)\, d\tau + (i_{10} - i_{20})\, e^{-\alpha t}\right]\begin{pmatrix}R_2\\ -R_1\end{pmatrix}\right\}.$$

If $t \to 0+$, we obtain the compatibility relation

$$\begin{pmatrix}i_{10}\\ i_{20}\end{pmatrix} = \frac{v(0+)}{R_1 + R_2}\left\{\begin{pmatrix}1\\ 1\end{pmatrix} + (y_{10} - y_{20})\begin{pmatrix}R_2\\ -R_1\end{pmatrix}\right\}.$$

The two components yield the same relation: $R_1\, i_{10} + R_2\, i_{20} = v(0+)$. There are physical reasons in this case (Kirchhoff's law) that the initial conditions cannot be arbitrarily chosen. Also observe that if $v(t)$ has a jump at $t = 0$, the jump will appear in $i_1(t)$ and $i_2(t)$, too. This collides with the fact that these two functions should be continuous since they are even assumed to be differentiable.

This situation is annoying. Modeling real world problems often results in anomalous matrix differential systems (as in the above example). Nothing prevents us from assigning arbitrary initial conditions. But by doing so, we create a mathematical problem which may not have a solution in the classical sense. We certainly will not satisfy the compatibility conditions randomly when we assign the initial values. However, the physical system will move without

any difficulties. Furthermore, there is a regularity problem for jump inputs. What is the way out of this conflict?

In [Doetsch (1976)] or [Doetsch and Nader (1974)], for example, one can find a thorough discussion of this problem. The authors suggest that the classical modeling process with ordinary function must be abandoned and that the differential system must be regarded as one in the distributional sense. Firstly, this solves the regularity problem, since distributions are as many times differentiable as one wishes. Secondly, the notion of initial values has to be reexamined in the sense that the initial values need to be regarded as initial values from the left. The initial values can be arbitrarily chosen irrespective of any compatibility conditions. In the case of a normal system, we see that the right-sided limit for $t \to 0+$ of the solution coincides with the given initial conditions, while for an anomalous system the right-sided limits need not coincide with the assumed initial values (interpreted as left-sided limits), i.e. the solution will have a discontinuity at the initial instant. Furthermore, interpreting the given initial conditions as limits from the left at $t = 0$ is quite a natural assumption, since they can be regarded as a reminiscence of the state of the system from its past.

The fact that for a normal system (or a scalar differential equation) it doesn't matter if the specified initial conditions are regarded as limits from the right (as we did) or as limits from the left, allows that the "+" in $y(0+)$, $y'(0+)$, ... can be dropped and the simpler notation $y(0)$, $y'(0)$, ... adopted. But for an anomalous system we have to write $y(0-)$, $y'(0-)$, ..., otherwise we must respect compatibility conditions! To specify initial conditions in the form $y(0)$, $y'(0)$, ... does not make sense for an anomalous system.

■ 6.3.3 Use of the Package

6.3.3.1 AnomalousSystemQ

In the following example we test whether a system of coupled differential equations is anomalous or normal.

$$\begin{aligned} y_1' + y_3' + y_1(t) &= f_1(t), \\ -y_1' + 2y_2' + y_2(t) &= f_2(t), \\ y_1' + 2y_2' + 2y_3(t) &= f_3(t). \end{aligned}$$

With the operational notation, we write

```
dop = {{p + 1, 0, p}, {-p, 2 p + 1, 0}, {p, 2 p, 2 p}};

MatrixForm[dop]
```

$$\begin{pmatrix} 1+p & 0 & p \\ -p & 1+2p & 0 \\ p & 2p & 2p \end{pmatrix}$$

```
sys = (dop[{y1, y2, y3}] == {f1[t], f2[t], f3[t]});
```

This tests whether the system is anomalous or not.

```
AnomalousSystemQ[sys, p, {y1[t], y2[t], y3[t]}, t]

True
```

6.3.3.2 Compatibility Conditions

The following command gives the compatibility conditions.

```
CompatibilityConditions[sys, p, {y1[t], y2[t], y3[t]}, t]
```

```
{-2 y1[0] - y2[0] == -2 f1[0] - f2[0] + f3[0]}
```

This relation has to be satisfied if the initial values are interpreted as limits from the right. Let us choose a set of initial conditions and a vector for the right-hand side such that the compatibility condition is satisfied. For example:

$$f_1(t) = t,\ f_2(t) = e^{-t},\ f_3(t) = 2 - t^2\ ;\ y_1(0+) = -3,\ y_2(0+) = 5.$$

The solution is computed by

```
LinearDOpSolve[dop[{y1, y2, y3}] = {t, Exp[-t], 2 - t},
 {y1[0] -> -3, y2[0] -> 5}, p, {y1[t], y2[t], y3[t]}, t]
```

$$\left\{y3[t] \to \frac{23}{8} - \frac{23}{8}\,e^{-2t/5} + \frac{t}{4} - \frac{t^2}{4},\right.$$
$$\left. y1[t] \to -\frac{7}{4} + \frac{2\,e^{-t}}{3} - \frac{23}{12}\,e^{-2t/5} + \frac{3t}{2},\ y2[t] \to \frac{3}{2} - \frac{e^{-t}}{3} + \frac{23}{6}\,e^{-2t/5}\right\}$$

The following confirms that the initial conditions are satisfied.

```
% /. t -> 0
```

```
{y3[0] -> 0, y1[0] -> -3, y2[0] -> 5}
```

Next we choose a set of initial conditions that does not satisfy the compatibility condition and that, therefore, must be interpreted as limits from the left. Take

$$y_1(0-) = 1,\ y_2(0-) = 1,\ y_3(0-) = 1.$$

```
LinearDOpSolve[dop[{y1, y2, y3}] = {t, Exp[-t], 2 - t},
 {y1[0] -> 1, y2[0] -> 1, y3[0] -> 1}, p, {y1[t], y2[t], y3[t]}, t]
```

$$\left\{y3[t] \to \frac{15}{8} + \frac{29}{40}\,e^{-2t/5} + \frac{t}{4} - \frac{t^2}{4},\right.$$
$$\left. y1[t] \to -\frac{7}{4} + \frac{2\,e^{-t}}{3} + \frac{29}{60}\,e^{-2t/5} + \frac{3t}{2},\ y2[t] \to \frac{3}{2} - \frac{e^{-t}}{3} - \frac{29}{30}\,e^{-2t/5}\right\}$$

Let us compute the right-sided initial values as $t \to 0+$ of the solution

```
% /. t -> 0
```

$$\left\{y3[0] \to \frac{13}{5},\ y1[0] \to -\frac{3}{5},\ y2[0] \to \frac{1}{5}\right\}$$

They are all different from the given initial values, i.e. the computed solution of the anomalous system has a discontinuity at the origin:

$y_1(0+) - y_1(0-) = -8/5,\ y_2(0+) - y_2(0-) = -4/5,\ y_3(0+) - y_3(0-) = 8/5$.

In order to overcome this difficulty, the derivatives in the differential equations must be interpreted in the distributional sense. The computed solutions then are distributions with support $\mathbb{R}_+$.

■ 6.3.4 Backward Initial Value Problems

Most initial value problems are forward initial value problems, i.e. for a given differential equation, say of second order, and specified initial values, say $y(0+)$, $y'(0+)$, find the solution $y(t)$ for $t > 0$.

When constructing Green's function (see Chapter 11), for example, another type of initial value problem (which could also be called a terminal value problem) occurs. For given initial conditions (terminal conditions), say $y(T-)$, $y'(T-)$, where T is a specified instant, the solution $y(t)$ is found backwards, i.e. for time values $t < T$. Such a problem is called a *backward initial value problem.* To be more specific, a backward initial value problem for a linear differential equation with constant coefficients of order n is given by

$$\frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0\, y(t) = f(t), \qquad (t < T), \tag{6.46}$$

$$\begin{aligned} &y(T-) = y_0,\ y'(T-) = y_0',\ y''(T-) = y_0'', \\ &\ldots \\ &y^{(n-1)}(T-) = y_0^{(n-1)}\,. \end{aligned} \tag{6.47}$$

The right-hand side $f(t)$ is assumed to be defined on the interval $(-\infty, T)$. The method of Laplace transformation can readily be used to solve backward initial value problems. Let $x(\tau) := y(T-\tau)$, $y(t) = x(T-t)$, i.e. $\tau = T - t$. We then have

$$\frac{d^k y}{dt^k} = (-1)^k \frac{d^k x}{d\tau^k} \qquad (k = 0, 1, \ldots, n)$$

and the backward initial value problem (6.46), (6.47) transforms into the ordinary forward initi value problem

$$(-1)^n \frac{d^n x}{d\tau^n} + (-1)^{n-1} a_{n-1} \frac{d^{n-1} x}{d\tau^{n-1}} + \cdots - a_1 \frac{dx}{d\tau} + a_0\, x(\tau) = g(\tau) \quad (\tau > 0),$$

$$\begin{aligned} &x(0+) = y_0, \\ &x'(0+) = -y_0',\ x''(0+) = y_0'', \\ &\ldots \\ &x^{(n-1)}(0+) = (-1)^{n-1}\, y_0^{(n-1)}, \end{aligned}$$

where $g(\tau) := f(T-\tau)$, $\tau > 0$. This problem is solved as before. The solution of the original backward initial value problem is obtained by $y(t) = x(T-t)$, $t < T$.

6.3.4.1 Use of the Package

The solution of a backward initial value problem for a scalar differential equation is obtained by

```
LinearBackwardDSolve[eqn, {y[T]->A, y'[T] -> B,...},y[t],t,T].
```

Example. Let the backward initial value problem be

$$y'' - 2y' + 5y(t) = \cos t \qquad (t < T),$$

$$y(T) = A, \quad y'(T) = B.$$

```
LinearBackwardDSolve[y''[t] - 2 y'[t] + 5 y[t] == Cos[ t],
 {y[T] → A, y'[T] → B}, y[t], t, T]
```

$$\left\{y[t] \to 2\left(\text{Sin}[t-T]\left(-\frac{\text{Cos}[T]}{20}-\frac{\text{Sin}[T]}{10}\right)+\text{Cos}[t-T]\left(\frac{\text{Cos}[T]}{10}-\frac{\text{Sin}[T]}{20}\right)\right)+\right.$$
$$2\,e^{t-T}\left(\text{Cos}[2\,(-t+T)]\left(\frac{1}{10}\,(5\,A-\text{Cos}[T])+\frac{\text{Sin}[T]}{20}\right)-\right.$$
$$\left.\left.\left(\frac{1}{40}\,(-10\,A+10\,B+3\,\text{Cos}[T])+\frac{\text{Sin}[T]}{40}\right)\text{Sin}[2\,(-t+T)]\right)\right\}$$

Let us check the solution:

```
y = y[t] /. %;

y /. t → T // Simplify
```

```
A
```

```
D[y, t] /. t → T // Simplify
```

```
B
```

```
D[y, {t, 2}] - 2 D[y, t] + 5 y // Simplify
```

```
Cos[t]
```

Chapter 7
z-Transformation: Further Topics

7.1 The Advanced z-Transformation

The z-transform of a sequence $f(kT)$ was introduced in Chapter 2. It is a function of the complex variable z and is defined by

$$F_T(z) = F(z) := \sum_{k=0}^{\infty} f(kT)\, z^{-k}\,.$$

The domain of $F(z)$ is the exterior of some circle centered at the origin. In the applications to control problems, the values $f(kT)$ are the sampled-data values of a time function $f(t)$, sampled with the sampling period $T > 0$. Instead of sampling a time function at the discrete times $0,\ T,\ 2T$, etc., we may sample at the instants $\tau,\ \tau + T,\ \tau + 2T,\ \ldots$, where we consider τ as a fixed parameter restricted to $0 \le \tau < T$. The z-transform of the sequence $f(kT + \tau)$ then is simply

$$F_{T,\tau}(z) := \sum_{k=0}^{\infty} f(kT + \tau)\, z^{-k}$$

and nothing is really new. Thus, the z-transformation is a mapping which maps a sequence to a function. Things may be different when we do not consider τ as a fixed parameter anymore, but as a new independent variable, ranging in the interval $[0,\ T)$. From this new point of view the original object is no longer the sequence $f(kT + \tau)$ with $k = 0,\ 1,\ 2,\ \ldots$ but again the continuous time function $f(t)$, where t varies over the entire real axis. As for the Laplace transformation, we assume that $f(t) = 0$ on the negative part of the real axis. A new name now is justified.

Definition. The mapping which associates to a given original function $f(t)$ the function

$$F_T(z,\ \tau) = F(z,\ \tau) := \sum_{k=0}^{\infty} f(kT + \tau)\, z^{-k} \tag{7.1}$$

with $0 \le \tau < T$ is called the *advanced z-transformation.* The function $F(z,\ \tau)$ of the two variables $(z,\ \tau)$ is called the *advanced z-transform* of $f(t)$.

When we refer to the former z-transformation, we call it the ordinary z-transformation from now on. Thus, if the second variable τ is considered as a fixed parameter, the advanced z-transform becomes the ordinary z-transform, and we have

$$F(z,\ 0+) := \sum_{k=0}^{\infty} f(kT)\, z^{-k} \;=\; F(z)\,,$$

for an original $f(t)$, assumed to be everywhere continuous from the right. We agree, that if $f(t)$ has a discontinuity at a sample point $t = kT + \tau$, we set $f(kT + \tau) := f((kT + \tau) +)$. We shall use the following notation, too:

$$f(t) \square\!-\!\triangleright F(z, \tau), \quad F(z, \tau) = Z_A[f(t)](z, \tau), \quad f(t) = Z_A^{-1}[F(z, \tau)](t).$$

If the original $f(t)$ is of bounded exponential growth, i.e. there are some constants $K > 0$ and β such that $|f(t)| < K e^{\beta t}$, the advanced z-transform of $f(t)$ exists and is a holomorphic function in the domain $|z| > \exp(\beta T)$. This is seen as follows:

$$|F(z, \tau)| \leq \sum_{k=0}^{\infty} |f(kT + \tau)| \, |z|^{-k} \leq K e^{\beta\tau} \sum_{k=0}^{\infty} e^{\beta k T} \, |z|^{-k}$$

$$\leq K e^{|\beta| T} \sum_{k=0}^{\infty} \left(\frac{e^{\beta T}}{|z|} \right)^k .$$

The geometric series in the last expression converges if $|z| > \exp(\beta T)$ and the estimate

$$|F(z, \tau)| \leq K e^{|\beta| T} \frac{1}{1 - e^{\beta T}/|z|}$$

holds for all $0 \leq \tau < T$. It is also seen that on the assumption of bounded exponential growth the series converges absolutely and uniformly with respect to $\tau \in [0, T)$ and $|z| \geqslant \rho > |z| > \exp(\beta T)$. After some preliminary manipulations, advanced z-transforms can often be computed as ordinary z-transforms.

Example. $f(t) = \cos \omega t$. The ordinary z-transform of the sequence $y(k) := f(kT + \tau)$ is

$$\begin{aligned} F(z, \tau) &= Z[\cos(\omega(kT + \tau))](z) \\ &= Z[\cos(\omega kT)\cos(\omega\tau) - \sin(\omega kT)\sin(\omega\tau)](z) \\ &= \cos(\omega\tau)\, Z[\cos(\omega kT)](z) - \sin(\omega\tau)\, Z[\sin(\omega kT)](z) \\ &= \cos(\omega\tau) \frac{z(z - \cos(\omega T))}{z^2 - 2z\cos(\omega T) + 1} - \sin(\omega\tau) \frac{z \sin(\omega T)}{z^2 - 2z\cos(\omega T) + 1} \\ &= \frac{z(z\cos(\omega\tau) - \cos(\omega(T - \tau)))}{z^2 - 2z\cos(\omega T) + 1} . \end{aligned}$$

■ 7.1.1 Basic Properties

An advanced z-transform can be considered as an ordinary z-transform when the second variable τ only enters as a fixed parameter. Therefore all properties of the ordinary z-transformation also apply to the advanced z-transformation. Let us summarize these properties.

Proposition. *If* $f(t) \square\!-\!\triangleright F(z, \tau)$ *or* $f_i(t) \square\!-\!\triangleright F_i(z, \tau)$, *then*

Linearity:

$$\sum_{i=1}^{m} c_i \, f_i(t) \square\!-\!\triangleright \sum_{i=1}^{m} c_i \, F_i(z, \tau) . \tag{7.2}$$

Original Shift Rules I and II: For any positive integer m, we have

$$u(t - mT)\, f(t - mT) \;\square\!-\!\triangleright\; z^{-m} F(z, \tau)\,, \tag{7.3}$$

$$f(t + mT) \;\square\!-\!\triangleright\; z^{m} F(z, \tau) - \sum_{k=0}^{m-1} f(kT + \tau)\, z^{m-k}\,. \tag{7.4}$$

Damping Rule:

$$f(t)\, e^{-at} \;\square\!-\!\triangleright\; e^{-a\tau} F(e^{aT} z, \tau). \tag{7.5}$$

t-Multiplication Rule: For any positive integer m, we have

$$t^m f(t) \;\square\!-\!\triangleright\; \left(-Tz \frac{\partial}{\partial z} + \tau\right)^m F(z, \tau)\,. \tag{7.6}$$

Example. The correspondence

$$\cos \omega t \;\square\!-\!\triangleright\; \frac{z\,[z \cos(\omega\tau) - \cos(\omega(T - \tau))]}{z^2 - 2z \cos(\omega T) + 1}$$

implies

$$e^{-at} \cos \omega t \;\square\!-\!\triangleright\; e^{-a\tau} \frac{z\, e^{aT}(z\, e^{aT} \cos(\omega\tau) - \cos(\omega(T - \tau)))}{z^2\, e^{2aT} - 2z\, e^{aT} \cos(\omega T) + 1}$$

$$= e^{-a\tau} \frac{z\,[z \cos(\omega\tau) - e^{-aT} \cos(\omega(T - \tau))]}{z^2 - 2z\, e^{-aT} \cos(\omega T) + e^{-2aT}}\,.$$

The Final Value Theorem of Section 2.1.2.9 now takes the following form:

Final Value Theorem. *Assume that* $\lim_{k\to\infty} f(kT + \tau)$ *exists, then*

$$g(\tau) := \lim_{k\to\infty} f(kT + \tau) = \lim_{z\to 1+} (z - 1) F(z, \tau)\,. \tag{7.7}$$

If the parameter τ is considered as a second variable, there is another set of rules. These rules have no counterpart for the ordinary z-transformation.

Proposition. *Assume* $f(t) \;\square\!-\!\triangleright\; F(z, \tau)$ *, then*

Original Shift Rule III: If δ *is any real number with* $0 < \delta \le T$*, then*

$$\begin{cases} u(t - \delta)\, f(t - \delta) \;\square\!-\!\triangleright\; F(z, \tau - \delta), & 0 < \delta \le \tau \\ u(t - \delta)\, f(t - \delta) \;\square\!-\!\triangleright\; z^{-1} F(z, \tau - \delta + T), & \tau < \delta \le T, \end{cases} \tag{7.8}$$

$$\begin{cases} f(t + \delta) \;\square\!-\!\triangleright\; F(z, \tau + \delta), & 0 < \delta \le T - \tau \\ f(t + \delta) \;\square\!-\!\triangleright\; z\,[F(z, \tau + \delta - T) - f(\tau + \delta - T)], & T - \tau < \delta \le T\,. \end{cases} \tag{7.9}$$

Proof.

$$u(t - \delta)\, f(t - \delta) \;\square\!-\!\triangleright\; \sum_{k=0}^{\infty} u(kT + \tau - \delta)\, f(kT + \tau - \delta)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(kT+\tau-\delta) \;=\; z^{-k}\, F(z, \tau-\delta)\,,$$

provided $0 \le \tau - \delta < T$. This proves the first line in the first statement. For the second line,

$$u(t-\delta)\, f(t-\delta) \;\square\!-\!\triangleright\; \sum_{k=0}^{\infty} u(kT+\tau-\delta)\, f(kT+\tau-\delta)\, z^{-k}$$

$$= \sum_{k=1}^{\infty} f(kT+\tau-\delta)\, z^{-k}$$

$$= z^{-1} \sum_{k=1}^{\infty} f((k-1)T+\tau-\delta+T)\, z^{-(k-1)}$$

$$= z^{-1} \sum_{k=0}^{\infty} f(kT+\tau-\delta+T)\, z^{-k} \;=\; z^{-1}\, F(z, \tau-\delta+T),$$

provided $0 \le \tau - \delta + T < T$. The second statement is proved similarly. □

The Shift Rule III implies that if $|\,\delta\,|$ is sufficiently small we have, by use of the linearity,

$$\frac{f(t+\delta)-f(t)}{\delta} \;\square\!-\!\triangleright\; \frac{F(z, \tau+\delta)-F(z, \tau)}{\delta}$$

which implies the

Proposition. *(Differentiation Rule) If* $f(t) \;\square\!-\!\triangleright\; F(z, \tau)$ *and* $f(t)$ *is continuously differentiable for all* $t > 0$, *then*

$$f'(t) \;\square\!-\!\triangleright\; \frac{\partial F}{\partial \tau}. \tag{7.10}$$

Example. Since

$$\cos \omega t \;\square\!-\!\triangleright\; \frac{z(z\cos(\omega\tau) \;-\; \cos(\omega(T-\tau)))}{z^2 - 2z\cos(\omega T) + 1}$$

we have

$$-\omega \sin \omega t \;\square\!-\!\triangleright\; -\omega\, \frac{z(z\sin(\omega\tau) \;+\; \sin(\omega(T-\tau)))}{z^2 - 2z\cos(\omega T) + 1}$$

or

$$\sin \omega t \;\square\!-\!\triangleright\; \frac{z(z\sin(\omega\tau) \;+\; \sin(\omega(T-\tau)))}{z^2 - 2z\cos(\omega T) + 1}\,.$$

In the context of Laplace transformation, the convolution of two original functions is defined for $t > 0$ as

$$(f * g)(t) := \int_0^t f(t-x)\, g(x)\, \boldsymbol{d}x\,.$$

In Chapter 2 the convolution of two original sequences (also called the discrete convolution) is introduced as

$$(f * g)(k) := \sum_{\nu=0}^{k} f(k-\nu)\, g(\nu) \qquad (k = 0, 1, 2, \ldots).$$

For both types similar convolution theorems hold: the convolution of two originals corresponds to the ordinary product of their images. For the advanced z-transformation, we now introduce a third type of convolution. We shall call it the discrete composition in order to distinguish it from the former notions of convolution.

Definition. Let $f(t)$ and $g(t)$ be two original functions. The *discrete composition* or *composition product of* $f(t)$ *and* $g(t)$ is the original defined and denoted by

$$(f \otimes g)(t) := \sum_{\nu=0}^{[t/T]} f(\nu T)\, g(t - \nu T) \qquad (t \geq 0). \tag{7.11}$$

$[x]$ denotes the integer part of x. For $t < 0$, we define $(f \otimes g)(t) := 0$.

The discrete composition is distributive but not commutative. It is introduced because again a convolution or composition theorem holds. In order to show this, we form

$$(f \otimes g)(kT + \tau) := \sum_{\nu=0}^{[(kT+\tau)/T]} f(\nu T)\, g(kT + \tau - \nu T)$$

$$= \sum_{\nu=0}^{k} f(\nu T)\, g((k-\nu)T + \tau) \qquad (k = 0, 1, 2, \ldots).$$

The last expression is the discrete convolution of the two sequences $f(k)$ and $g(kT + \tau)$. Therefore, by the convolution theorem for the ordinary z-transform, we have

$$\sum_{\nu=0}^{k} f(\nu T)\, g((k-\nu)T + \tau) \circ\!-\!\diamond Z[f(kT)](z)\, Z[g(kT + \tau)](z) = F(z)\, G(z, \tau).$$

Hence, we have proved the

Composition Theorem. *If* $f(kT) \circ\!-\!\diamond F(z)$ *and* $g(t) \square\!-\!\triangleright G(z, \tau)$, *then*

$$(f \otimes g)(t) \square\!-\!\triangleright F(z)\, G(z, \tau). \tag{7.12}$$

Let us now find the integration rule, i.e. the advanced z-transform of

$$g(t) := \int_0^t f(x)\, \boldsymbol{d}x.$$

We may write

$$G(z,\tau) = \sum_{k=0}^{\infty} \int_0^{kT+\tau} f(x)\,dx\; z^{-k}$$

$$= \int_0^{\tau} f(x)\,dx + \sum_{k=1}^{\infty} \int_0^{kT+\tau} f(x)\,dx\; z^{-k}$$

$$= \int_0^{\tau} f(x)\,dx + \sum_{k=1}^{\infty} \left\{ \int_0^{kT} f(x)\,dx + \int_{kT}^{kT+\tau} f(x)\,dx \right\} z^{-k}$$

$$= \int_0^{\tau} f(x)\,dx + \sum_{k=1}^{\infty} \left\{ \sum_{\nu=0}^{k-1} \int_{\nu T}^{(\nu+1)T} f(x)\,dx + \int_{kT}^{kT+\tau} f(x)\,dx \right\} z^{-k}$$

$$= \int_0^{\tau} f(x)\,dx + \sum_{k=1}^{\infty} \left\{ \sum_{\nu=0}^{k-1} \int_0^{T} f(x+\nu T)\,dx + \int_0^{\tau} f(x+kT)\,dx \right\} z^{-k}.$$

By interchanging the order of integration and infinite summation, we have

$$G(z,\tau) = \int_0^{\tau} f(x)\,dx + \int_0^{\tau} \sum_{k=1}^{\infty} f(x+kT)\,z^{-k}\,dx + \int_0^{T} \left(\sum_{k=1}^{\infty} \sum_{\nu=0}^{k-1} f(x+\nu T)\,z^{-k} \right) dx$$

$$= \int_0^{\tau} \sum_{k=0}^{\infty} f(x+kT)\,z^{-k}\,dx + \int_0^{T} \left(\sum_{k=0}^{\infty} y(k,x)\,z^{-k} \right) dx$$

with the sequence

$$y(k,x) := \begin{cases} 0, & k = 0 \\ \sum_{\nu=0}^{k-1} f(x+\nu T), & k = 1, 2, \ldots . \end{cases}$$

For the ordinary z-transform correspondence $f(x+kT) \circ\!-\!\diamond\, F(z,x)$ the summation rule implies

$$\sum_{\nu=0}^{k} f(x+\nu T) \circ\!-\!\diamond\, \frac{z}{z-1} F(z,x),$$

whereas the original shift rule gives

$$u(k-1) \sum_{\nu=0}^{k-1} f(x+\nu T) = \sum_{\nu=1}^{k-1} f(x+\nu T) \circ\!-\!\diamond\, z^{-1} \frac{z}{z-1} F(z,x).$$

Therefore,

$$y(k,x) \circ\!-\!\diamond\, \frac{1}{z-1} F(z,x) = \sum_{k=0}^{\infty} y(k,x)\,z^{-k}$$

and finally,

$$G(z,\tau) = \int_0^{\tau} F(z,x)\,dx + \frac{1}{z-1} \int_0^{T} F(z,x)\,dx.$$

We have proved the *Integration Rule*:

Proposition. (*Integration Rule*) *If* $f(t) \square\!-\!\triangleright F(z, \tau)$, *then*

$$\int_0^t f(x)\,dx \square\!-\!\triangleright \int_0^\tau F(z, x)\,dx + \frac{1}{z-1}\int_0^T F(z, x)\,dx\,. \tag{7.13}$$

7.1.1.1 Application to Sampled-Data Systems

We consider the following sampled-data system (see also Section 5.6). An analog signal $f(t)$ is sampled by a pulse-sampler whose output is fed into a plant described by its transfer function $G(s)$, see Fig.7.1. If there is a hold-device between the pulse-sampler and the plant, the transfer function $G(s)$ is simply replaced by $Q(s)\,G(s)$, where $Q(s)$ is the transfer function of the hold-device. For a zero-hold device, we have

$$Q(s) = \frac{1 - e^{Ts}}{s}.$$

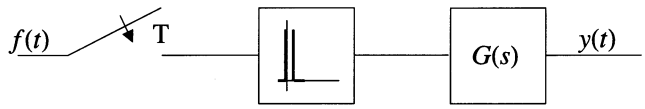


Fig.7.1: Sampled-data system

The analog input signal $f(t)$ is sampled with the sampling period T producing the discrete signal $f(kT)$. The second block represents a D/A converter transforming the discrete signal $f(kT)$ into the sampled-signal

$$f^*(t) = T\sum_{k=0}^{\infty} f(kT)\,\delta(t - kT)\,.$$

This represents a special type of an analog signal, also called an impulse-modulated signal. It is a generalized function. The Laplace transform of the sampled-signal is

$$F^*(s) = T\sum_{k=0}^{\infty} f(kT)\,e^{-kTs} = T\,F^\dagger(e^{Ts}),$$

where $F^\dagger(z)$ is the z-transform of the discrete signal $f(kT)$. If no hold-device is used, the Laplace transform of the output $y(t)$ is given by

$$Y(s) = F^*(s)\,G(s) = T\;F^\dagger(e^{Ts})\,G(s),$$

and by

$$Y(s) = F^*(s)\,Q(s)\,G(s) = F^\dagger(e^{Ts})\,Q(s)\,G(s)$$

if a hold-device is inserted (see Section 5.6.1). We can also write

$$Y(s) = F^*(s)\,G(s) = T\sum_{k=0}^{\infty} f(kT)\,e^{-kTs}\,G(s),$$

and, by taking the inverse Laplace transform, and using the Original Shift Rule, we obtain

$$y(t) = T \sum_{k=0}^{\infty} f(kT)\, u(t-kT)\, g(t-kT),$$

with $g(t) = \mathcal{L}^{-1}[G(s)](t)$. The last expression is equal to

$$y(t) = T \sum_{k=0}^{[t/T]} f(kT)\, g(t-kT) = T\,(f \otimes g)(t).$$

The composition theorem now implies

$$Y(z,\tau) = T\, F^{\dagger}(z)\, G(z,\tau), \tag{7.14}$$

where $Y(z,\tau)$ is the advanced z-transform of the output $y(t)$, and $F^{\dagger}(z)$ is the ordinary z-transform of the input sequence $f(kT)$. $G(z,\tau)$ is the advanced z-transform of $g(t) = \mathcal{L}^{-1}[G(s)](t)$. If a hold-device is used, we write $g(t) = \mathcal{L}^{-1}[Q(s)\,G(s)](t)$ and the factor T is omitted. In contrast to the use of the ordinary z-transformation for the description of a sampled-data control system, permitting only the control of $y(t)$ at the discrete instants kT, $(k = 0, 1, 2, \ldots)$, the advanced z-transform allows that $y(t)$ can be obtained for all instants t via

$$y(t) = T\, Z_A^{-1}[F^{\dagger}(z)\, G(z,\tau)](t). \tag{7.15}$$

In the above process, we also see the importance of the operation

$$G(z,\tau) = Z_A[\mathcal{L}^{-1}[G(s)](t)](z,\tau),$$

i.e. of the correspondence G(s) ■–▷ G(z,τ), which maps a Laplace image function to its counterpart, an advanced z-image. The composed operation $Z_A \circ \mathcal{L}^{-1}$ is called the advanced z-transform of a Laplace image. Let us investigate specifically the case where the hold-device is a zero-hold. By the above we must then write

$$Y(z,\tau) = F^{\dagger}(z)\, G(z,\tau)$$

with

$$G(z,\tau) = Z_A\left[\mathcal{L}^{-1}\left[\frac{1-e^{-Ts}}{s}\, G(s)\right](t)\right](z,\tau).$$

Let

$$H(s) = \frac{G(s)}{s} \;\blacksquare\!-\!\square\; h(t),$$

then

$$\begin{aligned} G(z,\tau) &= Z_A[h(t) - u(t-T)\,h(t-T)](z,\tau) \\ &= Z_A[h(t)](z,\tau) - Z_A[u(t-T)\,h(t-T)](z,\tau). \end{aligned}$$

By the Original Shift Rule I, the second expression on the right-hand side implies

$$Z_A[u(t-T)\,h(t-T)](z,\tau) = z^{-1}\, Z_A[h(t)](z,\tau).$$

Therefore,

$$G(z,\tau) = (1-z^{-1})\,Z_A[h(t)](z,\tau) = \frac{z-1}{z}\,Z_A[h(t)](z,\tau)$$

holds, and finally

$$Y(z,\tau) = \frac{z-1}{z}\,F^{\dagger}(z)\,Z_A\Big[\mathcal{L}^{-1}\Big[\frac{G(s)}{s}\Big](t)\Big](z,\tau). \tag{7.16}$$

This formula gives the advanced z-transform of the output when a zero-hold between the pulse-sampler and the plant is inserted. Without zero-hold, we simply have

$$Y(z,\tau) = T\,F^{\dagger}(z)\,Z_A[\mathcal{L}^{-1}[G(s)](t)](z,\tau). \tag{7.17}$$

■ 7.1.2 Inversion of an Advanced z-Transform

An advanced z-transform $F(z,\tau)$ is a holomorphic function in the exterior of some circle of radius ρ centered at the origin. For example, if the original $f(t)$ is of bounded exponential growth, i.e. if

$$|f(t)| < K e^{\beta t},$$

the radius of convergence can be taken as $\rho = \exp(\beta T)$. The Laurent series expansion of $F(z,\tau)$ is

$$F(z,\tau) = \sum_{k=0}^{\infty} f(k,\tau)\,z^{-k},$$

where the formula for the coefficients is

$$f(kT+\tau) = :f(k,\tau) = \frac{1}{2\pi i}\oint_C F(z,\tau)\,z^{k-1}\,dz. \tag{7.18}$$

The integration contour C is a simple closed path, traversed in the positive sense, and entirely located in the exterior of the circle of convergence. If $F(z,\tau)$ is a meromorphic function with respect to z, for example a rational function, we have only poles as singularities inside C, and (7.18) can be evaluated by the calculus of residues:

$$f(kT+\tau) = :f(k,\tau) = \sum \operatorname*{res}_{z_i\,(\text{pole})} \{F(z,\tau)\,z^{k-1}\}.$$

But in most cases, it is quicker to compute the ordinary inverse z-transform with respect to z, for example:

$$F(z,\tau) = e^{-a\tau}\left\{T\,e^{-aT}\,\frac{z}{(z-e^{-aT})^2} + \tau\,\frac{z}{z-e^{-aT}}\right\},$$

$$Z^{-1}\left[e^{-a\tau}\left\{T\,e^{-aT}\,\frac{z}{(z-e^{-aT})^2} + \tau\,\frac{z}{z-e^{-aT}}\right\}\right](k)$$

$$= e^{-a\tau}\,T\,e^{-aT}\,Z^{-1}\left[\frac{z}{(z-e^{-aT})^2}\right](k) + e^{-a\tau}\,\tau\,Z^{-1}\left[\frac{z}{z-e^{-aT}}\right](k)$$

$$= e^{-a\tau}\, T e^{-aT}\, k e^{-akT+aT} + e^{-a\tau}\, \tau e^{-akT}$$
$$= e^{-a(kT+\tau)}[kT + \tau] = f(k, \tau).$$

Thus

$$t e^{-at} \;\square\!-\!\triangleright\; e^{-a\tau}\left\{ T e^{-aT} \frac{z}{(z-e^{-aT})^2} + \tau \frac{z}{z-e^{-aT}} \right\}.$$

When applying the inverse advanced z-transform, the functional expression $f(t)$ is not directly obtained, but rather $f(k, \tau)$. The graph $f(t)$ can be computed from $f(k, \tau)$, once the sampling period T is numerically given.

■ 7.1.3 Advanced z-Transforms of a Laplace Image

At the end of Section 7.1.1.1 we emphasized the importance of the mapping $Z_A \circ \mathcal{L}^{-1}$, i.e. to find the advanced z-transform of a time function given by its Laplace transform. We shall use for this operation the symbol $F(s) \blacksquare\!-\!\triangleright\; F(z, \tau)$ and also the shorter notation $Z_{\mathcal{L}} := Z_A \circ \mathcal{L}^{-1}$. The operational rules for the Laplace and advanced z-transformation now imply the following rules for the operator $Z_{\mathcal{L}}$.

7.1.3.1 Operational Rules for $Z_{\mathcal{L}}$

Linearity:

$$Z_{\mathcal{L}}[c_1 F_1(s) + c_2 F_2(s)] = c_1 Z_{\mathcal{L}}[F_1(s)] + c_2 Z_{\mathcal{L}}[F_2(s)]. \tag{7.19}$$

Integer Shift Rule: For any positive integer m,

$$Z_{\mathcal{L}}[e^{-mTs} F_T(s)] = z^{-m} Z_{\mathcal{L}}[F_T(s)]\,. \tag{7.20}$$

Noninteger Shift Rule: For any δ with $0 \le \delta \le T$,

$$Z_{\mathcal{L}}[e^{-\delta s} F_T(s)]\,(z, \tau) = \begin{cases} Z_{\mathcal{L}}[F_T(s)]\,(z, \tau - \delta), & 0 < \delta \le \tau \\ z^{-1} Z_{\mathcal{L}}[F_T(s)]\,(z, \tau - \delta + T), & \tau < \delta \le T. \end{cases} \tag{7.21}$$

Damping Rule: For any complex constant a,

$$Z_{\mathcal{L}}[F_T(s-a)]\,(z, \tau) = e^{a\tau} Z_{\mathcal{L}}[F_T(s)]\,(e^{-aT} z, \tau). \tag{7.22}$$

Division Rule: The correspondence $F(s) \blacksquare\!-\!\triangleright\; F(z, \tau)$ implies

$$\frac{F(s)}{s} \;\square\!-\!\triangleright\; \int_0^{\tau} F(z, x)\, \mathbf{d}x + \frac{1}{z-1} \int_0^{T} F(z, x)\, \mathbf{d}x\,. \tag{7.23}$$

Differentiation Rule: The correspondence $F(s) \blacksquare\!-\!\triangleright\; F(z, \tau)$ implies

$$\frac{d^m}{ds^m} F(s) \;\blacksquare\!-\!\triangleright\; \left(Tz \frac{\partial}{\partial z} - \tau\right)^m F(z, \tau). \tag{7.24}$$

s-Multiplication Rule: If $\lim_{s\to\infty} s\, F(s) = 0$, then $F(s) \blacksquare\!-\!\triangleright\; F(z, \tau)$ implies

$$s\,F(s) \blacksquare\!-\!\triangleright \frac{\partial F}{\partial \tau}\,, \tag{7.25}$$

provided that $f(0+)$ exists.

Zero-Hold Device Rule:

$$Z_{\mathcal{L}}\left[\frac{1-e^{-Ts}}{s}\,F(s)\right] \blacksquare\!-\!\triangleright \frac{z-1}{z}\,Z_{\mathcal{L}}\left[\frac{F(s)}{s}\right]. \tag{7.26}$$

The rules (7.26) and (7.20) are special cases of the following rule that will be proved at the end of the next Section.

Exp(s t)-Rule: If $h(z)$ is a rational function, then

$$Z_{\mathcal{L}}[\,h(e^{sT})\,G(s)] = h(z)\,Z_{\mathcal{L}}[\,G(s)]. \tag{7.27}$$

The *Exp(s t)-Rule* can be used to compute inverse Laplace transforms of image functions of the type

$$F(s) = h(e^{sT})\,G(s) \blacksquare\!-\!\square f(t) = Z_A^{-1}[\,h(z)\,Z_{\mathcal{L}}[\,G(s)]]\,(t)\,,$$

see Section 7.3.4.1.

7.1.3.2 Generalized Discrete Laplace Transforms

In Section 2.1.1.1 the (generalized) discrete Laplace transform as the discrete Laplace transform of the sequence $f(kT+\tau)$ is introduced, i.e. with $0 \le \tau < T$,

$$F^*(s,\tau) = \sum_{k=0}^{\infty} f(kT+\tau)\,e^{-kTs} = Z_A[f(t)]\,(e^{Ts},\tau)\,.$$

The discrete Laplace transform is a periodic function with respect to s with the period $2\pi i/T$:

$$F^*\left(s+\frac{2\pi}{T}\,i,\tau\right) = F^*(s,\tau)\,.$$

Therefore, it is sufficient to study its behaviour on the horizontal strip $-\pi i/T \le \mathrm{I}s \le \pi i/T$.

Now assume that the original $f(t)$ has an absolute convergent Laplace integral and is locally of bounded variation. Let x_1 be the abscissa of absolute convergence of the Laplace integral and let $c > x_1$. Furthermore assume that $f(t)$ is continuous at all instants $t = kT+\tau$ $k = 1, 2, 3, \ldots$ but may have a discontinuity of the first kind at $t = \tau$ with the jump $f(\tau+) - f(\tau-)$. The complex inversion formula then gives

$$\frac{f(\tau+)+f(\tau-)}{2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{p\tau}\,F(p)\,dp\,,$$

$$f(kT+\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{p(kT+\tau)}\,F(p)\,dp$$

for $k = 1, 2, 3, \ldots$. Let us compute the discrete Laplace transform.

$$F^*(s,\tau) = \sum_{k=0}^{\infty} f(kT+\tau)\, e^{-kTs} = f(\tau+) + \sum_{k=1}^{\infty} f(kT+\tau)\, e^{-kTs}$$

$$= \frac{f(\tau+) - f(\tau-)}{2} + \frac{f(\tau+) + f(\tau-)}{2} + \sum_{k=1}^{\infty} f(kT+\tau)\, e^{-kTs}.$$

The last two terms on the right-hand side combine to

$$\frac{f(\tau+) + f(\tau-)}{2} + \sum_{k=1}^{\infty} f(kT+\tau)\, e^{-kTs}$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{c-i\infty}^{c+i\infty} e^{p(kT+\tau)}\, F(p)\, dp\ e^{-kTs}.$$

Interchanging summation and integration, we obtain

$$\frac{f(\tau+) + f(\tau-)}{2} + \sum_{k=1}^{\infty} f(kT+\tau)\, e^{-kTs}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau}\, F(p) \sum_{k=0}^{\infty} e^{-kT(s-p)}\, dp$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p\tau}\, F(p)}{1 - e^{-(s-p)T}}\, dp,$$

provided $\Re[(s-p)\,T] > 0 \Longleftrightarrow \Re s > c$. Hence,

$$F^*(s,\tau) = \frac{f(\tau+) - f(\tau-)}{2} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p\tau}\, F(p)}{1 - e^{-(s-p)T}}\, dp. \tag{7.28}$$

The zeros of the denominator of the integrand are

$$p_k = s + \frac{2\pi i}{T} k,\ k \in \mathbb{Z}.$$

Assuming $\Re s > c$, the zeros are located in the right half-plane where $F(s)$ is holomorphic. Furthermore, we can also write

$$F^*(s,\tau) = \frac{f(\tau+) - f(\tau-)}{2} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{-p(T-\tau)}\, F(p)}{e^{-pT} - e^{-sT}}\, dp$$

with $T - \tau > 0$. Let us now consider

$$\frac{1}{2\pi i} \int_{c-iR_n}^{c+iR_n} \frac{e^{-p(T-\tau)}\, F(p)}{e^{-pT} - e^{-sT}}\, dp$$

where $R_n = \pi(2n+1)/T$. We close the integration contour by two horizontal segments $\Im p = s \pm iR_n$, $c \le \Re p \le \Re s$ and the right half-circle

$$C_n = s + R_n e^{i\varphi} \quad (\pi/2 \ge \varphi \ge -\pi/2).$$

The circle C_n will pass between two consecutive poles where $1 - e^{-(s-p)T}$ takes the value 2 for $\varphi = \pm\pi/2$. If in addition, we assume that

$$\left| \frac{F(p)}{e^{-pT} - e^{-sT}} \right|$$

tends uniformly to zero, with respect to the argument of p, as $R_n \to \infty$, then, by Jordan's lemma (see Section 6.1.1), we have

$$\frac{1}{2\pi i} \int_{C_n} \frac{e^{-p(T-\tau)} F(p)}{e^{-pT} - e^{-sT}} \, dp \to 0 .$$

It can readily be shown that the contribution on the two horizontal segments also tends to zero. We then have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p\tau} F(p)}{1 - e^{-(s-p)T}} \, dp = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{-p(T-\tau)} F(p)}{e^{-pT} - e^{-sT}} \, dp$$

$$= -\sum_{k=-\infty}^{\infty} \operatorname*{res}_{p_k} \left\{ \frac{e^{p\tau} F(p)}{1 - e^{-(s-p)T}} \right\} = -\sum_{k=-\infty}^{\infty} \frac{e^{p_k \tau} F(p_k)}{-T e^{-(s-p_k)T}}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{e^{\left(s + \frac{2\pi i}{T} k\right)\tau} F\left(s + \frac{2\pi i}{T} k\right)}{1} .$$

We leave it as an exercise to the reader to verify that on the right half-circles C_n the factor

$$\left| \frac{1}{e^{-pT} - e^{-sT}} \right|$$

remains bounded as $n \to \infty$. Because $F(p)$, as a Laplace image function, tends uniformly to zero on the right half-circles C_n as $R_n \to \infty$, the above assumption is satisfied, and we have established the formula

$$F^*(s, \tau) = \frac{f(\tau+) - f(\tau-)}{2} + \operatorname{pv} \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(s + \frac{2\pi i}{T} k\right) e^{\left(s + \frac{2\pi i}{T} k\right)\tau} \tag{7.29}$$

which is valid for $0 \le \tau < T$. If $f(t)$ is continuous at $t = \tau > 0$ we simply obtain for $\tau > 0$,

$$F^*(s, \tau) = \operatorname{pv} \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(s + \frac{2\pi i}{T} k\right) e^{\left(s + \frac{2\pi i}{T} k\right)\tau} \tag{7.30}$$

and for $\tau = 0$,

$$F^*(s, 0) = \frac{f(0+)}{2} + \operatorname{pv} \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(s + \frac{2\pi i}{T} k\right) . \tag{7.31}$$

These formulas were already established in Section 2.1.1.1 on the basis of Poisson's summation formula.

Now assume that $F(s)$ is of the form $F(s) = \Pi(s)\, G(s)$ where

$$\Pi\Big(s + \frac{2\pi}{T}\, i\Big) = \Pi(s)$$

is a periodic function with the period $2\pi i/T$, and $G(s)$ is the Laplace transform of an original $g(t)$. We wish to prove the formula

$$F^*(s, \tau) = \Pi(s)\; G^*(s, \tau).$$

Proof. For $\tau > 0$ we can write

$$F^*(s, \tau) = \operatorname{pv} \frac{1}{T} \sum_{k=-\infty}^{\infty} \Pi\Big(s + \frac{2\pi i}{T} k\Big)\; G\Big(s + \frac{2\pi i}{T} k\Big)\; e^{\left(s + \frac{2\pi i}{T} k\right)\tau}$$

$$= \operatorname{pv} \frac{1}{T} \sum_{k=-\infty}^{\infty} \Pi(s)\; G\Big(s + \frac{2\pi i}{T} k\Big)\; e^{\left(s + \frac{2\pi i}{T} k\right)\tau}$$

$$= \Pi(s) \operatorname{pv} \frac{1}{T} \sum_{k=-\infty}^{\infty} G\Big(s + \frac{2\pi i}{T} k\Big)\; e^{\left(s + \frac{2\pi i}{T} k\right)\tau}$$

$$= \Pi(s)\, G^*(s, \tau)\,. \;\square$$

A special case is the *Exp(s T)-Rule* of the last section: *If $h(z)$ is a rational function, then*

$$F(s) = h(e^{sT})\, G(s) \Longrightarrow F^\dagger(z, \tau) = h(z)\, G^\dagger(z, \tau). \tag{7.32}$$

Proof. For the function $\Pi(s)$ we may take $h(e^{sT})$, for

$$\Pi\Big(s + \frac{2\pi}{T}\, i\Big) = h\Big[\exp\Big(\Big(s + \frac{2\pi}{T}\, i\Big)T\Big)\Big]$$
$$= h[\exp((s\,T + 2\pi\, i))] = h[\exp(s\,T)\, \exp(2\pi\, i)\,] = h[\exp(s\,T)\,] = \Pi(s).$$

Therefore,

$$F(s) = h(e^{sT})\, G(s) \Longrightarrow F^*(s, \tau) = h(e^{sT})\, G^*(s, \tau)\,,$$

but $F^*(s, \tau) = F^\dagger(e^{sT}, \tau)$ and $G^*(s, \tau) = G^\dagger(e^{sT}, \tau)$ if $f(t) \square\!-\!\triangleright F^\dagger(z, \tau)$ and $g(t) \square\!-\!\triangleright G^\dagger(z, \tau)$. We can therefore write

$$F(s) = h(e^{sT})\, G(s) \Longrightarrow F^\dagger(e^{sT}, \tau) = h(e^{sT})\, G^\dagger(e^{sT}, \tau)\,. \;\square$$

Example.

$$F(s) = \frac{G(s)}{1 - e^{-sT}} = \frac{e^{sT}\, G(s)}{e^{sT} - 1}$$
$$= h(e^{sT})\, G(s) \Longrightarrow F^\dagger(z, \tau) = \frac{z}{z-1}\, G^\dagger(z, \tau).$$

Example. Let us illustrate the *Exp(s T)-Rule* with a case where the above assumptions are not necessarily satisfied. We consider the following original (a step-function)

$$f(t) := g(k)\,, \quad k\,T \le t < (k+1)\,T \qquad (t \ge 0),$$

where $g(k)$ is a given sequence. The function has an infinite number of discontinuities. For the Laplace transform we have

$$f(t) \circ\!-\!\bullet\ F(s) = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} e^{-st}\, g(k)\, dt$$

$$= \frac{1 - e^{-sT}}{s} \sum_{k=0}^{\infty} g(k)\, e^{-ksT} = \frac{1 - e^{-sT}}{s} Z[g(k)]\,(e^{sT}).$$

The advanced z-transform of $f(t)$ is

$$f(t) \square\!-\!\triangleright\ F^{\dagger}(z, \tau) = \sum_{k=0}^{\infty} f(kT + \tau)\, z^{-k} = \sum_{k=0}^{\infty} g(k)\, z^{-k} = Z[g(k)]\,(z).$$

Let us take, for example,

$$g(k) = k^2 \circ\!-\!\diamond\ \frac{z(z+1)}{(z-1)^3} = Z[k^2]\,(z),$$

$$F(s) = \frac{1 - e^{-sT}}{s} \frac{e^{sT}(e^{sT}+1)}{(e^{sT}-1)^3}$$

$$= \frac{e^{sT}-1}{e^{sT}} \frac{e^{sT}(e^{sT}+1)}{(e^{sT}-1)^3} \frac{1}{s} = \frac{(e^{sT}+1)}{(e^{sT}-1)^2} \frac{1}{s} = h(e^{sT}) \frac{1}{s}.$$

Applying the *Exp(s t)-Rule*,

$$F^{\dagger}(z, \tau) = \frac{z+1}{(z-1)^2} Z_{\mathcal{L}}\left[\frac{1}{s}\right](z, \tau) = \frac{z+1}{(z-1)^2} \frac{z}{z-1} = \frac{z(z+1)}{(z-1)^3}$$

in agreement with the above directly computed $F(z, \tau)$.

7.1.3.3 $Z_{\mathcal{L}}$ for Meromorphic Laplace Image Functions

By (7.28) we have

$$F(e^{Ts}, \tau) = F^*(s, \tau) = \frac{f(\tau+) - f(\tau-)}{2} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p\tau} F(p)}{1 - e^{-Ts}\, e^{pT}}\, dp$$

(we again write $F(z, \tau)$ for the advanced z-transform of $f(t)$). Hence

$$F(z, \tau) = \frac{f(\tau+) - f(\tau-)}{2} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p\tau} F(p)}{1 - z^{-1}\, e^{pT}}\, dp \tag{7.33}$$

is a general formula for the advance z-transform in terms of the Laplace image function. If $F(s)$ is a meromorphic function, we can evaluate the above integral by the calculus of residues. By closing the integration contour by a half-circle to the left and by Jordan's lemma, we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p\tau} F(p)}{1 - z^{-1}\, e^{pT}}\, dp = \sum_{i} \operatorname*{res}_{a_i} \left\{ \frac{e^{p\tau} F(p)}{1 - z^{-1}\, e^{pT}} \right\},$$

where the summation runs over all poles a_i of $F(s)$. The reason is that the zeros of $1 - z^{-1}\, e^{pT}$ are all located to the right of c if $|z| > \exp(cT)$, so that the only singularities, located to the left of the vertical $\Re s = c$, are the poles of $F(s)$. We wish to apply Jordan's lemma. Therefore, we assume $\tau > 0$, and, if the radius of the half-circle tends to infinity, that the expression

$$\left| \frac{1}{1 - z^{-1}\, e^{pT}} \right|$$

remains bounded on the left half-circle, and that $F(s)$ tends to zero, uniformly with respect to the argument of s. It is left as an exercise to the reader to show that the above factor remains indeed bounded.

Proposition. *Let $f(t)$ be continuous for $t > 0$, and let its Laplace image $F(s)$ be a meromorphic function that tends to zero uniformly with respect to the argument of s, as $|s| \to \infty$ and $\pi/2 \le \arg s \le 3\pi/2$. Then the advanced z-transform of the Laplace image $F(s)$ is for $\tau > 0$,*

$$F(z, \tau) = Z_{\mathcal{L}}[F(s)](z, \tau) = \sum_i \operatorname*{res}_{a_i} \left\{ \frac{e^{s\tau}\, F(s)}{1 - z^{-1}\, e^{sT}} \right\} \tag{7.34}$$

where the summation runs over all poles of $F(s)$.

Example. For $F(s) = 1/(s-a)^m$ we have immediately

$$F(z, \tau) = \frac{z}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left\{ \frac{e^{s\tau}}{z - e^{sT}} \right\}_{s=a} .$$

$F(s) = 1/s$ gives $F(z, \tau) = z/(z-1)$. $F(s) = 1/(s-a)^2$ gives

$$F(z, \tau) = z \frac{d}{ds} \left\{ \frac{e^{s\tau}}{z - e^{sT}} \right\}_{s=a} = e^{a\tau} \left\{ \tau \frac{z}{z - e^{aT}} + T\, e^{aT} \frac{z}{(z - e^{aT})^2} \right\}.$$

7.2 Applications

■ 7.2.1 Difference Equations with a Continuous Variable

The advanced z-transformation may be used to solve difference equations in a continuous variable. Let us show the principle with the difference equation

$$y(t + 2T) + 3\, y(t + T) + 2\, y(t) = t$$

with the initial conditions

$$y(\tau) = T - \tau, \; y(\tau + T) = 0, \qquad 0 \le \tau < T.$$

The assumption $y(t) \; \square\!\!-\!\!\triangleright \; Y(z, \tau)$ implies

$$\begin{aligned} &y(t + T) \; \square\!\!-\!\!\triangleright \; z\{Y(z, \tau) - y(\tau)\} = z\, Y(z, \tau) - z(T - \tau), \\ &y(t + 2T) \; \square\!\!-\!\!\triangleright \; z^2 \{Y(z, \tau) - y(\tau) - z^{-1}\, y(\tau + T)\} \\ &= z^2\, Y(z, \tau) - z^2 (T - \tau) . \end{aligned}$$

By using the correspondence

$$t \square\!\!-\!\!\triangleright\; T\frac{z}{(z-1)^2} + \tau\frac{z}{z-1}$$

the image equation becomes

$$(z^2 + 3z + 2)\,Y(z,\tau) = z^2(T-\tau) + 3z(T-\tau) + T\frac{z}{(z-1)^2} + \tau\frac{z}{z-1}.$$

Solving for $Y(z,\tau)$ gives

$$Y(z,\tau) = (T-\tau)\frac{z(z+3)}{(z+1)(z+2)} + \frac{z[T+\tau z-\tau]}{(z+1)(z+2)(z-1)^2}.$$

The partial fraction decomposition yields

$$Y(z,\tau) = (T-\tau)\left\{\frac{2z}{1+z} - \frac{z}{2+z}\right\}$$
$$+ \frac{Tz}{6(-1+z)^2} + \frac{z(T-2\tau)}{4(1+z)} + \frac{z(-T+3\tau)}{9(2+z)} + \frac{z(-5T+6\tau)}{36(-1+z)},$$

and, by taking the inverse z-transform, we finally obtain

$$y(kT+\tau) = (T-\tau)\{2(-1)^k - (-2)^k\} + \frac{T}{6}k$$
$$+ \frac{(T-2\tau)}{4}(-1)^k + \frac{(-T+3\tau)}{9}(-2)^k + \frac{(-5T+6\tau)}{36}.$$

■ 7.2.2 Sampled-Data Control Systems

The discussion of Section 7.1.1.1 is continued to consider the following feedback sampled-data control system (Fig.7.2). Here the digital controller is inserted into the forward-path (cascade controller configuration).

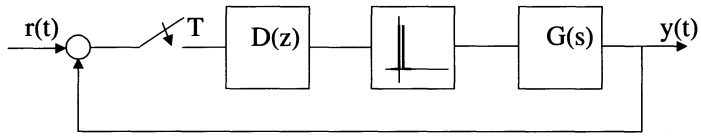


Fig.7.2: Sampled-data control system with digital controller

The input to the digital controller is the discrete signal $r(kT) - y(kT)$. Its z-transform is $R^{\dagger}(z) - Y^{\dagger}(z)$. The z-transform of the output of the digital controller is $D(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)]$, which is the input to the D/A converter whose output is the input to the plant. The latter is described by its transfer function $G(s)$. If a hold device between the D/A converter and the plant is inserted, its transfer function simply is absorbed into $G(s)$. Due to Section 7.1.1.1, the advanced z-transform of the response $y(t)$, with no hold-device, is

$$Y(z,\tau) = T\,D(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)]\;Z_A[\mathcal{L}^{-1}[G(s)]\,(t)]\,(z,\tau).$$

If a zero-hold device is used we have

$$Y(z,\tau) = \frac{z-1}{z} D(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)]\; Z_A\Big[\mathcal{L}^{-1}\Big[\frac{G(s)}{s}\Big](t)\Big](z,\tau).$$

With

$$H_0(z,\tau) := T\, Z_A[\mathcal{L}^{-1}[G(s)]\,(t)]\,(z,\tau),$$

$$H_1(z,\tau) := \frac{z-1}{z} Z_A\Big[\mathcal{L}^{-1}\Big[\frac{G(s)}{s}\Big](t)\Big](z,\tau)$$

respectively, we have

$$Y(z,\tau) = D(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)]\,H_i(z,\tau) \qquad (i = 0,\ 1).$$

With $\tau = 0$, we obtain for $i = 0,\ 1$,

$$Y^{\dagger}(z) = Y(z,0) = D(z)\,[R^{\dagger}(z) - Y^{\dagger}(z)]\,H_i(z,0)$$

and from this, with $H_i(z) = H_i(z,0)$,

$$Y^{\dagger}(z) = \frac{D(z)\,H_i(z)}{1 + D(z)\,H_i(z)}\,R^{\dagger}(z).$$

Thus

$$R^{\dagger}(z) - Y^{\dagger}(z)$$
$$= R^{\dagger}(z)\Big\{1 - \frac{D(z)\,H_i(z)}{1 + D(z)\,H_i(z)}\Big\} = \frac{1}{1 + D(z)\,H_i(z)}\,R^{\dagger}(z),$$

that finally yields

$$Y(z,\tau) = \frac{D(z)\,H_i(z,\tau)}{1 + D(z)\,H_i(z)}\,R^{\dagger}(z) \qquad (i = 1,\ 2). \tag{7.35}$$

This implies that the system is asymptotically stable, if for any $\tau \in [0,\ T)$ all poles of the closed-loop transfer function

$$\frac{D(z)\,H_i(z,\tau)}{1 + D(z)\,H_i(z)}$$

are entirely located inside the unit-circle of the z-plane.

Remark. If the analysis is only done with the ordinary z-transform, and the closed-loop transfer function

$$\frac{D(z)\,H_i(z)}{1 + D(z)\,H_i(z)}$$

has all poles inside the unit-circle, it may still happen that while the system behaves well at the sampled instants kT, it produces oscillations with significant amplitudes between two consecutive sampled instants (so-called hidden oscillations). This generally happens if the plant itself, i.e. the open-loop system, is in itself unstable .

Example. Let us consider the sampled-data system with the unstable plant

$$G(s) = \frac{1}{s\,(s+a)}\,.$$

If no hold-device is used, we have

$$H_0(z,\tau) := T\,Z_A\Big[\mathcal{L}^{-1}\Big[\frac{1}{s\,(s+a)}\Big](t)\Big](z,\tau)$$

$$= \frac{T}{a}\Big\{Z_A\Big[\mathcal{L}^{-1}\Big[\frac{1}{s}\Big](t)\Big](z,\tau) - Z_A\Big[\mathcal{L}^{-1}\Big[\frac{1}{s+a}\Big](t)\Big](z,\tau)\Big\}$$

$$= \frac{T}{a}\Big\{\frac{z}{z-1} - e^{-a\tau}\,\frac{z}{z-e^{-aT}}\Big\}.$$

The closed-loop transfer function becomes, if we assume $D(z) = k = \text{constant}$,

$$\frac{D(z)\,H_0(z,\tau)}{1 + D(z)\,H_0(z)} = K\,\frac{z\,(z-e^{-aT})}{z^2 + [(K-1) - e^{-aT}(K+1)]\,z + e^{-aT}}$$

$$- e^{-a\tau}\,\frac{z(z-1)}{z^2 + [(K-1) - e^{-aT}(K+1)]\,z + e^{-aT}}$$

with $K = k\,T/a$. The closed-loop system is asymptotically stable, if all roots of

$$z^2 + [(K-1) - e^{-aT}(K+1)]\,z + e^{-aT} = 0$$

are located inside the unit-circle. This is the case (see Section 2.2.1.1) if and only if

$$e^{-aT} + (K-1) - e^{-aT}(K+1) + 1 > 0 \;\;\&\;\; e^{-aT} < 1$$
$$\text{and } e^{-aT} - [(K-1) - e^{-aT}(K+1)] + 1 > 0.$$

Hence a has to be a positive quantity. The first inequality requires $K > 0$ and the third one $K < 2\,(1 + \exp(-a\,T))\,/\,(1 - \exp(-a\,T))$. Thus if $a > 0$, a simple proportional controller yields an asymptotic stable closed-loop system, if the gain k satisfies

$$0 < k < \frac{2\,a}{T}\,\frac{1 + e^{-aT}}{1 - e^{-aT}}.$$

■ 7.2.3 Closed-Form Solutions to a Periodic Input

Let a linear system be given by its transfer function $G(s)$. The system is assumed to be stable (all poles of $G(s)$ are located in the left half-plane). If a periodic input $f(t) = f(t+T)$ is applied, the stationary part of the response will again have the period T. Normally, you get this output in the form of an infinite series which corresponds to the Fourier series of the input. The advanced z-transform technique allows us to obtain a closed-form expression for the stationary part of the response. For this the $Z_{\mathcal{L}}$ transformation, the *Exp[s T]-Rule*, and the final-value theorem for the advanced z-transform are used. We illustrate the method with the following example. For another approach see [Weber H. (1987)]. Let us consider the simple electrical network of Fig.7.3. As input voltage, a half rectified sine wave with period $T = 2\,a$ is applied.

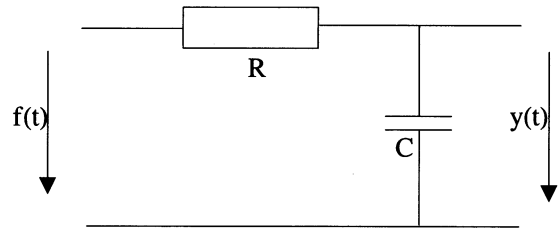
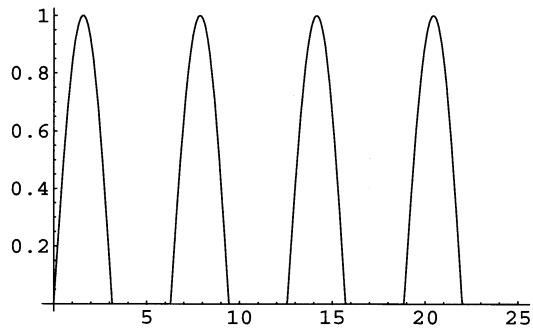


Fig.7.3: Electrical network with input voltage $f(t)$ and output voltage $y(t)$.

```
g1 = Plot[(Sin[t] + Abs[Sin[t]]) / 2, {t, 0, 25}];
```



The Laplace transform of the input is with $\omega = \pi / a = 2\pi / T$

$$f(t) \circ\!-\!\bullet\, F(s) = \omega \frac{1}{(s^2 + \omega^2)(1 - e^{-s\pi/\omega})}.$$

The transfer function of the system is

$$G(s) = \frac{b}{s+b}, \qquad \left(b = \frac{1}{RC}\right).$$

Thus, the forced response is given by

$$Y(s) = \omega b \frac{1}{(s+b)(s^2 + \omega^2)(1 - e^{-as})} = \omega b \frac{(1 + e^{-sT/2})}{(s+b)(s^2 + \omega^2)(1 - e^{-sT})}.$$

This forced response is not yet the stationary part, due to the presence of the pole at $s = -b$. We write

$$\frac{Y(s)}{\omega b} = \frac{1}{(s+b)(s^2 + \omega^2)(1 - e^{-Ts})} + e^{-sT/2} \frac{1}{(s+b)(s^2 + \omega^2)(1 - e^{-Ts})} = Y_1(s) + e^{-sT/2} Y_1(s)$$

with

$$Y_1(s) = \frac{e^{sT}}{e^{sT} - 1} \frac{1}{(s+b)(s^2 + \omega^2)}.$$

By applying the *Exp[s T]-Rule*, the advanced z-transform of $y_1(t)$ is

$$Y_1(z,\tau) = Z_{\mathcal{L}}[Y_1(s)](z,\tau) = \frac{z}{z-1}\, Z_{\mathcal{L}}\Big[\frac{1}{(s+b)(s^2+\omega^2)}\Big](z,\tau)$$

$$= \frac{1}{(b^2+\omega^2)}\Big(\frac{z}{z-1}\Big)^2\Big\{(z-1)\,\frac{e^{-b\tau}}{z-e^{-bT}} - \cos\omega\tau + \frac{b}{\omega}\sin\omega\tau\Big\}.$$

By the *Original Shift Rule III*, we obtain for $\tau < a < T = 2a$,

$$Z_{\mathcal{L}}[e^{-sa}\,Y_1(s)] = z^{-1}\,Z_{\mathcal{L}}[Y_1(s)](z,\tau+a),$$

$$Z_{\mathcal{L}}\Big[\frac{Y(s)}{\omega b}\Big](z,\tau) = \frac{1}{(b^2+\omega^2)}\,\frac{z}{(z-1)^2}$$

$$\times\Big\{z(z-1)\,\frac{e^{-b\tau}}{z-e^{-bT}} - z\cos\omega\tau$$

$$+\frac{b}{\omega}\,z\sin\omega\tau + (z-1)\,\frac{e^{-b(\tau+a)}}{z-e^{-bT}} + \cos\omega\tau - \frac{b}{\omega}\sin\omega\tau +\Big\},$$

hence,

$$Z_{\mathcal{L}}\Big[\frac{Y(s)}{\omega b}\Big](z,\tau) = \frac{1}{(b^2+\omega^2)}\,\frac{z}{z-1}$$

$$\times\Big\{e^{-b\tau}\,\frac{z+e^{-ba}}{z-e^{-bT}} - \cos\omega\tau + \frac{b}{\omega}\sin\omega\tau\Big\}.$$

For $0 < a \le \tau < T = 2a$ the *Original Shift Rule III* gives

$$Z_{\mathcal{L}}[e^{-sa}\,Y_1(s)] = Z_{\mathcal{L}}[Y_1(s)](z,\tau-a),$$

$$Z_{\mathcal{L}}\Big[\frac{Y(s)}{\omega b}\Big](z,\tau) = \frac{1}{(b^2+\omega^2)}\,\frac{z^2}{(z-1)^2}$$

$$\times\Big\{(z-1)\,\frac{e^{-b\tau}}{z-e^{-bT}} - \cos\omega\tau$$

$$+\frac{b}{\omega}\sin\omega\tau + (z-1)\,\frac{e^{-b(\tau-a)}}{z-e^{-bT}} + \cos\omega\tau - \frac{b}{\omega}\sin\omega\tau\Big\}$$

hence

$$Z_{\mathcal{L}}\Big[\frac{Y(s)}{\omega b}\Big](z,\tau) = \frac{1}{(b^2+\omega^2)}\,\frac{z^2}{(z-1)}\Big\{e^{-b\tau}\,\frac{1+e^{ab}}{z-e^{-bT}}\Big\}.$$

The final-value theorem now implies for $0 \le \tau < a$,

$$y_{\mathrm{st}}(\tau) := \lim_{k\to\infty} y(kT+\tau) = \lim_{z\to 1+}(z-1)\,Z_{\mathcal{L}}[Y(s)](z,\tau) =$$

$$\frac{\omega b}{(b^2+\omega^2)}\,\Big\{e^{-b\tau}\,\frac{1+e^{-ba}}{1-e^{-bT}} - \cos\omega\tau + \frac{b}{\omega}\sin\omega\tau\Big\},$$

and, for $a \le \tau < T$,

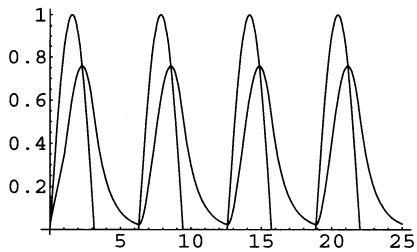
$$y_{st}(\tau) := \lim_{k\to\infty} y(kT+\tau) = \lim_{z\to 1+} (z-1)\, Z_{\mathcal{L}}[Y(s)]\,(z,\tau)$$
$$= \frac{\omega b}{(b^2+\omega^2)} \left\{ e^{-b\tau}\, \frac{1+e^{ab}}{1-e^{-bT}} \right\}.$$

The closed-form solution of the stationary part of the response therefore is given on the interval of periodicity $[0, T = 2a]$ by

$$y_{st}(\tau) = \frac{\omega b}{(b^2+\omega^2)} \times$$
$$\begin{cases} e^{-b\tau}(1+e^{-bT/2})/(1-e^{-bT}) - \cos\omega\tau + \frac{b}{\omega}\sin\omega\tau, & 0 \le \tau < T/2 \\ e^{-b\tau}(1+e^{bT/2})/(1-e^{-bT}), & T/2 \le \tau < T. \end{cases}$$

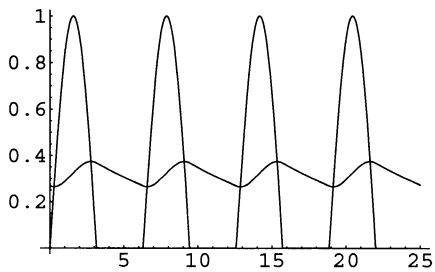
Note that $y_{st}(0) = y_{st}(T)$, as expected for an expression that represents a periodic continuous function defined by periodic extension to the right. The two plots below show the input signal together with the stationary part of the output, first for $b = 1$ and then for $b = 0.1$:

```
y11[t_, b_] :=
 b / (b^2 + 1) * If[t < Pi, Exp[-b t] (1 + Exp[-b Pi]) / (1 - Exp[-b 2 Pi]) -
    Cos[t] + b Sin[t], Exp[-b t] (1 + Exp[b Pi]) / (1 - Exp[-b 2 Pi])]

f[t_, b1_] := Module[{k},
  k = Floor[t / 2 / Pi];
  y11[t - k 2 Pi, b1]
 ]

g2 = Plot[f[t, 0.1], {t, 0, 25}];

Show[g1, g2];
```



Input and output signals for $b = 1$

```
Show[g1, g2];
```



Input and output signals for $b = 0.1$

7.3 Use of the Package

■ 7.3.1 Advanced z-Transforms

The advanced z-transform of an original function $f(t)$ with the sampling period T is computed with the command `AdvancedZTransform[f[t],t,T,τ,z]`. For example:

```
F = AdvancedZTransform[t^3, t, T, τ, z]
```

$$\frac{T^3\, z\, (1 + 4\, z + z^2)}{(-1 + z)^4} + \frac{3\, T^2\, z\, (1 + z)\, \tau}{(-1 + z)^3} + \frac{3\, T\, z\, \tau^2}{(-1 + z)^2} + \frac{z\, \tau^3}{-1 + z}$$

Since an advanced z-transform can be regarded as an ordinary z-transform of a sequence depending on a supplementary parameter τ, the sequence can be restored by

```
InverseZTransform[F, z, k]
```

```
τ^3 + 3 T τ^2 GenFac[2, 1, 1][k] +
 3 T^2 τ (GenFac[2, 1, 1][k] + 2 GenFac[3, 1, 1][k]) +
 T^3 (GenFac[2, 1, 1][k] + 6 GenFac[3, 1, 1][k] + 6 GenFac[4, 1, 1][k])
```

```
f[k_, T_, τ_] = % /. ExpandGenFac
```

$$(k + 3\, (-1 + k)\, k + (-2 + k)\, (-1 + k)\, k)\, T^3 + 3\, (k + (-1 + k)\, k)\, T^2\, \tau + 3\, k\, T\, \tau^2 + \tau^3$$

By applying `InverseZTransform` to an advanced z-image, the sequence $f_T(k, \tau)$ is obtained which represents the original function, i.e. the inverse advanced z-transform, in a special manner as $f_T(k, \tau)$, where $f_T(k, \tau) := f(kT + \tau)$, $0 \le \tau < T$, $k = 0, 1, 2, \ldots$. If the functional expression $f(t)$ is wanted, i.e. the inverse advanced z-transform, you can try to replace τ by $t - kT$ and apply the `Expand` command. For simple expressions it may work.

```
% /. τ → (t - k T) // Expand
```

$$t^3$$

If a numerical value is assigned to the sampling period T the original function $f(t)$ can numerically be obtained. For this use the following command `InverseAdvancedZ-Transform[F[z,τ],z,f,k,T,τ]`, where a symbol f is assigned to the sequence to be computed, for example,

```
F = AdvancedZTransform[t Cos[t], t, T, τ, z]
```

$$\frac{\tau\, (-z\, \mathrm{Cos}[T - \tau] + z^2\, \mathrm{Cos}[\tau])}{1 + z^2 - 2\, z\, \mathrm{Cos}[T]} -$$
$$T\, z \left(\frac{-\mathrm{Cos}[T - \tau] + 2\, z\, \mathrm{Cos}[\tau]}{1 + z^2 - 2\, z\, \mathrm{Cos}[T]} - \frac{(2\, z - 2\, \mathrm{Cos}[T])\, (-z\, \mathrm{Cos}[T - \tau] + z^2\, \mathrm{Cos}[\tau])}{(1 + z^2 - 2\, z\, \mathrm{Cos}[T])^2} \right)$$

```
Clear[f]; InverseAdvancedZTransform[F, z, f, k, T, τ] // Short
```

$$\left\{ f[k, T, \tau] \to \tau \left(\frac{2^{-1-k}\ \langle\langle 1 \rangle\rangle\ (\langle\langle 1 \rangle\rangle)}{\sqrt{-1 + \mathrm{Cos}[\langle\langle 1 \rangle\rangle]^2}} + \frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle} \right) - \langle\langle 1 \rangle\rangle \right\}$$

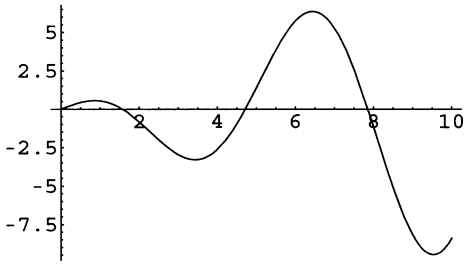
The output of `InverseAdvancedZTransform` is then inserted as the first argument in the `EvaluateToFunction` command. The second argument is the assigned numerical value for the sampling period T, for example $\pi/2$.

```
EvaluateToFunction[%, Pi / 2] // Short
```

$$-\frac{1}{8}\pi\ (\text{<<1>>}) + \left(\text{<<1>>} - \frac{1}{2}\,i\,i^{\text{Floor[<<1>>]}}\ (\text{<<1>>})\right)\ (\text{<<1>>} + \#1)\ \&$$

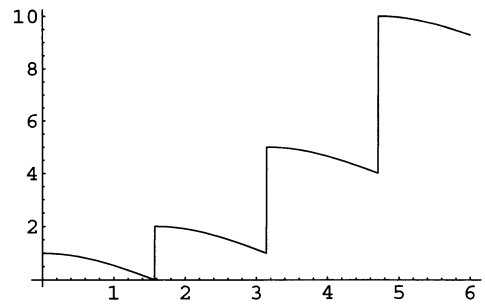
In this way, the inverse advanced z-transform is obtained in the form of a pure function that can be plotted.

```
Plot[%[t], {t, 0, 10}];
```



Next, let us define the following step function and compute its advanced z-transform. The sampling period equals the step interval.

```
f = StepFunction[#^2 &, π / 2, Cos[#] &];

Plot[f[t], {t, 0, 6}];
```




```
AdvancedZTransform[f[t], t, π / 2, τ, z]
```

$$\frac{\pi^2\ z\ (1+z)}{4\ (-1+z)^3} + \frac{z\ \text{Cos}[\tau]}{-1+z}$$

The given original function can be restored by executing the following steps.

```
F = InverseAdvancedZTransform[%, z, g, k, π / 2, τ];

EvaluateToFunction[F, π / 2];

Plot[%[t], {t, 0, 6}];
```

■ 7.3.2 Operational Rules

The linearity rule, the t-multiplication rule, the damping rule, and the t-differentiation rule of Section 7.1.1 are automatically applied during the evaluation of an advanced z-transform. For example:

```
f = c1 t Cos[2 t] + c2 Exp[a t] t^2;

AdvancedZTransform[f, t, T, τ, z]
```

$$c2\, e^{a\tau}\left(\frac{e^{-aT}T^2 z\,(1+e^{-aT}z)}{(-1+e^{-aT}z)^3}+\frac{2e^{-aT}T z\tau}{(-1+e^{-aT}z)^2}+\frac{e^{-aT}z\tau^2}{-1+e^{-aT}z}\right)+$$
$$c1\left(\frac{\tau\,(-z\,\mathrm{Cos}[2\,(T-\tau)]+z^2\,\mathrm{Cos}[2\tau])}{1+z^2-2z\,\mathrm{Cos}[2T]}-Tz\left(\frac{-\mathrm{Cos}[2\,(T-\tau)]+2z\,\mathrm{Cos}[2\tau]}{1+z^2-2z\,\mathrm{Cos}[2T]}-\right.\right.$$
$$\left.\left.\frac{(2z-2\,\mathrm{Cos}[2T])\,(-z\,\mathrm{Cos}[2\,(T-\tau)]+z^2\,\mathrm{Cos}[2\tau])}{(1+z^2-2z\,\mathrm{Cos}[2T])^2}\right)\right)$$

```
AdvancedZTransform[(t Exp[-t])', t, T, τ, z]
```

$$\frac{e^{T-\tau}z}{-1+e^T z}-e^{-\tau}\left(\frac{e^T T z}{(-1+e^T z)^2}+\frac{e^T z\tau}{-1+e^T z}\right)$$

In order to apply the shift rules, you have to distinguish whether or not the shift is an integral multiple of the sampling period T.

Integral T-shift to the right:

```
DefInteger[m]; DefPositive[m];

AdvancedZTransform[Sin[t] // RightShift[m, t], t, T, τ, z]
```

$$\frac{z^{-m}\,(z\,\mathrm{Sin}[T-\tau]+z^2\,\mathrm{Sin}[\tau])}{1+z^2-2z\,\mathrm{Cos}[T]}$$

Non-integral T-shift to the right: The sampling period T and the shift τ must now be numerical and non-integral.

```
AdvancedZTransform[Sin[t] // RightShift[4.25, t], t, 1, 0.2, z]
```

$$\frac{0.0499792\,z+0.813416\,z^2}{z^5\,(1+z^2-2z\,\mathrm{Cos}[1])}$$

Integral T-shifts to the left are treated similarly by using `LeftShift[T,t]` instead of `RightShift[T,t]`.

In order to apply the Integration Rule:

```
AdvancedZTransform[t^3 // IntegrationTo[t], t, T, τ, z]
```

$$\frac{\frac{T^4 z}{(-1+z)^2}+\frac{T^4 z}{4\,(-1+z)}+\frac{3T^4 z\,(1+z)}{2\,(-1+z)^3}+\frac{T^4 z\,(1+4z+z^2)}{(-1+z)^4}}{-1+z}+$$
$$\frac{T^3 z\,(1+4z+z^2)\,\tau}{(-1+z)^4}+\frac{3T^2 z\,(1+z)\,\tau^2}{2\,(-1+z)^3}+\frac{Tz\tau^3}{(-1+z)^2}+\frac{z\tau^4}{4\,(-1+z)}$$

Let us check that it is the same as

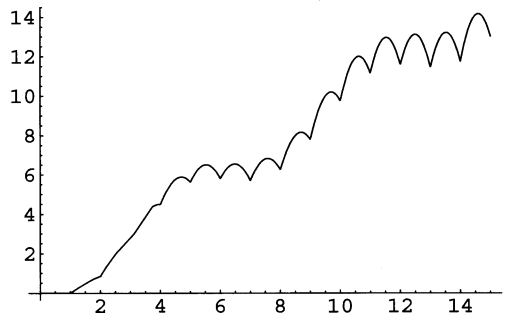
```
AdvancedZTransform[t^5 / 5 , t, T, τ, z];

% - %% // Simplify

0
```

The discrete composition $(f \otimes g)(t)$ of two originals is represented in the form of a pure function by `DiscreteComposition[f[x],g[x],x,T]`. By applying the NamedRule `EvaluateDiscreteComposition` the discrete composition product then is evaluated by the use of its definition.

```
DiscreteComposition[x, Sin[x], x, T][t] /.
  EvaluateDiscreteComposition // Short
```

$$-\frac{1}{2}\,T\,(\text{<<11>>} + \text{<<1>>})$$

```
Plot[% /. T → 1, {t, 0, 15}];
```



If the discrete composition should be evaluated by using the Discrete Composition Theorem, use the NamedRule `ComputeDiscreteComposition`.

```
DiscreteComposition[x, Sin[x], x, T][t, τ] /.
  ComputeDiscreteComposition // Short
```

$$\left\{\text{y5}[t, T, \tau] \to T\left(-\frac{\text{Sin}[\tau]}{2\,(-1+\text{Cos}[T])} - \frac{\text{<<1>>}}{\text{<<1>>}} + \frac{\text{<<1>>}}{\text{<<1>>}} - \text{<<1>>}\right)\right\}$$

Since the `InverseAdvancedZTransform` command was used the result is returned as a replacement list. In order to plot the result:

```
EvaluateToFunction[%, 1] // Short
```

$$\text{<<5>>} + \frac{2^{\text{<<1>>}}\ \text{<<2>>}}{(\text{<<1>>})\ \text{<<1>>}\ (\text{<<1>>})}\ \&$$

```
Plot[%[t], {t, 0, 15}];
```

The Summation Rule is a special case of the Discrete Composition Rule. If $g(t) \square\!-\!\triangleright\ G_T(z, \tau)$ the following discrete summation

$$\sum_{k=0}^{[t/T]} g(t - kT) \square\!-\!\triangleright\ \frac{z}{z-1}\,G_T(z, \tau)$$

can be evaluated by choosing the unit-step function as the first factor. For example:

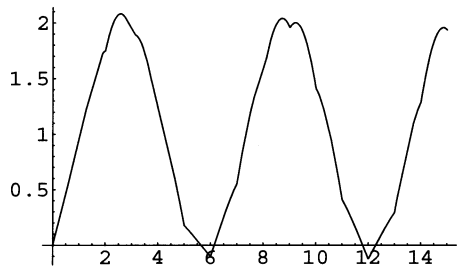
```
DiscreteComposition[UnitStep[x], Sin[x], x, T][t, τ] /.
  ComputeDiscreteComposition // Short
```

$$\left\{y6[t, T, \tau] \to -\frac{\text{Sin}[T-\tau]+\text{Sin}[\tau]}{2\,(-1+\text{Cos}[T])} + \text{<<1>>} + \frac{\text{<<1>>}}{\text{<<1>>}}\right\}$$

```
EvaluateToFunction[%, 1] // Short
```

$$-\frac{\text{<<1>>}}{2\,(-1+\text{Cos}[1])} + \frac{\text{<<1>>}}{\text{<<1>>}} + \text{<<1>>}\ \&$$

```
Plot[%[t], {t, 0, 15}];
```



■ 7.3.3 Initial and Final Behavior

For an advanced z-transform, one can rapidly compute the list of the first $n+1$ values of the corresponding original, i.e.,

$$\{f(\tau),\ f(T+\tau),\ f(2T+\tau),\ \ldots,\ f(nT+\tau)\}.$$

This is done by using the `InitialBehavior[F[z,τ],z,T,τ,n]` command. For example,

```
F = AdvancedZTransform[t Sin[t], t, T, τ, z];

InitialBehavior[F, z, T, τ, 4]

{τ Sin[τ], (T + τ) Sin[T + τ], (2 T + τ) Sin[2 T + τ],
 (3 T + τ) Sin[3 T + τ], (4 T + τ) Sin[4 T + τ]}
```

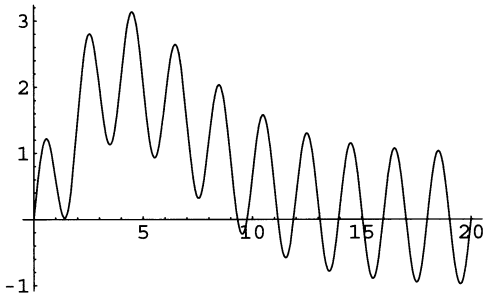
On the assumption that the limit

$$g(\tau) := \lim_{k\to\infty} f(kT+\tau) \qquad (0 \le \tau < T)$$

exists, it can be computed by the Final Value Theorem. This is, for example, the case if $f(t)$ is composed of a transitory part and of a periodic steady-state part. The sampling period is chosen equal to the period of the steady-state part. This happens for the output of a linear stable system, when the input is a periodic signal. The Final Value Theorem can then be used to compute the periodic steady-state part. It is even possible to obtain a closed-form solution for it. To obtain the final behavior, use the `FinalBehavior[F[z,τ],z,τ]` command. A simple example is the signal

```
f = t^2 Exp[-t / 2] + Sin[π t];
```

```
Plot[f, {t, 0, 20}];
```




```
F = AdvancedZTransform[f, t, T, τ, z]
```

$$e^{-\tau/2}\left(\frac{e^{T/2}\,T^2\,z\,(1+e^{T/2}\,z)}{(-1+e^{T/2}\,z)^3}+\frac{2\,e^{T/2}\,T\,z\,\tau}{(-1+e^{T/2}\,z)^2}+\frac{e^{T/2}\,z\,\tau^2}{-1+e^{T/2}\,z}\right)+\frac{z\,\mathrm{Sin}[\pi\,(T-\tau)]+z^2\,\mathrm{Sin}[\pi\,\tau]}{1+z^2-2\,z\,\mathrm{Cos}[\pi\,T]}$$

```
FinalBehavior[F /. T → 2, z, τ]
```

```
Sin[π τ]
```

For such problems the sampling period must be chosen equal to the period of the steady-state part !

■ 7.3.4 Advanced z-Transforms of a Laplace Image

In sampled-data control theory the computation of the advanced z-transform of a Laplace image function, i.e. the evaluation of the operator $Z_{\mathcal{L}} := Z_A \circ \mathcal{L}^{-1}$, is often an important step. The *Exp(s t)-Rule* of Section 7.1.3.1 can be helpful for this. The rule says that if $h(z)$ is a rational function, then

$$Z_{\mathcal{L}}[\,h(e^{sT})\,G(s)] = h(z)\,Z_{\mathcal{L}}[\,G(s)]\,(z,\tau)\,.$$

$G(s)$ often is a rational function. If `LaplaceToZImage[F[s], s,T,τ,z]` is used, the Package first computes the inverse Laplace transform $f(t)$ of $F(s)$ and then takes the advanced z-transform of $f(t)$. For example,

```
LaplaceToZImage[s / ((s + 1) (s^2 + 2 s + 2)), s, T, τ, z]
```

$$-\frac{e^{T-\tau}\,z}{-1+e^{T}\,z}+2\,e^{-\tau}\left(\frac{-e^{T}\,z\,\mathrm{Cos}[T-\tau]+e^{2T}\,z^2\,\mathrm{Cos}[\tau]}{2\,(1+e^{2T}\,z^2-2\,e^{T}\,z\,\mathrm{Cos}[T])}+\frac{e^{T}\,z\,\mathrm{Sin}[T-\tau]+e^{2T}\,z^2\,\mathrm{Sin}[\tau]}{2\,(1+e^{2T}\,z^2-2\,e^{T}\,z\,\mathrm{Cos}[T])}\right)$$

More economical is the use of `ZLaplace[F[s],s,T,τ,z]` which is an implementation of the method of Section 7.1.3.

```
F = ZLaplace[s / ((s^2 + 25) (s^2 + 2 s + 26)), s, T, τ, z]
```

$$\left(2\,e^{-\tau}\,z\left(-e^{-T}\left(-\frac{1}{202}\,\text{Cos}[5\,(T-\tau)] + \frac{51\,\text{Sin}[5\,(T-\tau)]}{1010}\right) + z\left(-\frac{1}{202}\,\text{Cos}[5\,\tau] - \frac{51\,\text{Sin}[5\,\tau]}{1010}\right)\right)\right) \Big/ \left(e^{-2T} + z^2 - 2\,e^{-T}\,z\,\text{Cos}[5\,T]\right) +$$
$$\left(2\,z\left(-\frac{1}{202}\,\text{Cos}[5\,(T-\tau)] + \frac{5}{101}\,\text{Sin}[5\,(T-\tau)] + z\left(\frac{1}{202}\,\text{Cos}[5\,\tau] + \frac{5}{101}\,\text{Sin}[5\,\tau]\right)\right)\right) \Big/ \left(1 + z^2 - 2\,z\,\text{Cos}[5\,T]\right)$$

7.3.4.1 Computation of Inverse Laplace Transforms

The *Exp(sT)-Rule* of Section 7.3.1 can also be applied to compute the inverse Laplace transforms of image functions of the type

$$F(s) = h(e^{Ts})\,G(s)$$

where $h(z)$ is a rational function of z. Let us consider the example

$$F(s) = \frac{(1-e^{-Ts})^2}{s^2(1-e^{-4Ts})}\,.$$

We have $h(z) = (1-1/z)^2/(1-1/z^4)$. The advanced z-transform of the Laplace image is first computed. Then, by applying the inverse advanced z-transform we obtain $f(t)$ in the form $f(k,\tau) := f(kT+\tau)$.

```
h = (1 - z^-1)^2 / (1 - z^-4) // Together
```

$$\frac{(-1+z)\,z^2}{1+z+z^2+z^3}$$

```
F = h ZLaplace[1 / s^2, s, T, τ, z]
```

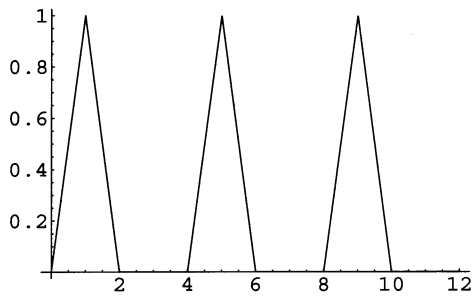
$$\frac{z^3\,(T-\tau+z\,\tau)}{(-1+z)\,(1+z+z^2+z^3)}$$

```
InverseAdvancedZTransform[F, z, f, k, T, τ] /. ExpandGenFac
```

$$\left\{f[k,\,T,\,\tau] \to \frac{T}{4} + \frac{1}{4}\,(-1)^k\,(-T+2\,\tau) + 2\left(\frac{1}{4}\,\tau\,\text{Cos}\left[\frac{k\,\pi}{2}\right] - \left(-\frac{T}{4}+\frac{\tau}{4}\right)\text{Sin}\left[\frac{k\,\pi}{2}\right]\right)\right\}$$

A plot of the time function $f(t)$ is obtained by

```
Plot[EvaluateToFunction[%, 1][t], {t, 0, 12}];
```



■ 7.3.5 Difference Equations with Continuous Variables

7.3.5.1 Forward Difference Equations

The advanced z–transformationis useful for solving initial value problems for linear difference equations with a continuous variable, for example

$$y(t+2T) + 2y(t+T) + 2y(t) = \cos t\,.$$

The initial conditions are given functions on $[0, T]$ and $[T, 2T]$, for example,

$$y(\tau) = \tau, \quad y(\tau + T) = T - \tau \qquad (0 \le \tau \le T).$$

Assuming $y(t) \square\!-\!\triangleright Y(z, \tau)$, the Original Shift Property II implies $y(t+T) \square\!-\!\triangleright z\,Y(z, \tau) - z\,y(\tau)$, $y(t+2T) \square\!-\!\triangleright z^2\,Y(z, \tau) - z^2\,y(\tau) - z\,y(T+\tau)$. Taking into account the specified initial conditions, the image equation

$$(z^2 + 2z + 2)\,Y(z, \tau) = \frac{z(z\cos(\tau) - \cos(T-\tau))}{z^2 - 2z\cos T + 1} + (2z + z^2)\,\tau + z(T-\tau)$$

is obtained. Therefore,

$$Y(z, \tau) = \frac{z(z\cos(\tau) - \cos(T-\tau))}{(z^2 + 2z + 2)(z^2 - 2z\cos T + 1)} + \frac{(2z+z^2)\,\tau + z(T-\tau)}{(z^2 + 2z + 2)}.$$

Let us study the case with $T = 2\pi$. Then

$$Y(z, \tau) = \frac{z(z\cos\tau - \cos(2\pi - \tau))}{(z^2 + 2z + 2)(z^2 - 2z + 1)} + \frac{(2z+z^2)\,\tau + z(2\pi - \tau)}{(z^2 + 2z + 2)}$$

$$= \frac{z(z-1)\cos\tau}{(z^2 + 2z + 2)(z-1)^2} + \frac{(2z+z^2)\,\tau + z(2\pi - \tau)}{(z^2 + 2z + 2)}.$$

First, we use the inverse z-transformation in order to obtain the solution in the form $y(k, \tau) := y(kT + \tau)$, $(k = 0, 1, 2, \ldots;\ 0 \le \tau < T)$.

```
y1 = InverseZTransform[z (z - 1) Cos[τ] / (z^2 + 2 z + 2) (z - 1)^2, z, k] /. ExpandGenFac;

y2 = InverseZTransform[(2 z + z^2) τ + z (2 π - τ) / (z^2 + 2 z + 2), z, k] /. ExpandGenFac;

y = (y1 + y2) // Simplify
```

$$-\frac{1}{5}\cos[\tau]\left(-1 + 2^{k/2}\cos\left[\frac{3k\pi}{4}\right] + 2^{\frac{2+k}{2}}\sin\left[\frac{3k\pi}{4}\right]\right) + 2^{k/2}\left(\tau\cos\left[\frac{3k\pi}{4}\right] + 2\pi\sin\left[\frac{3k\pi}{4}\right]\right)$$

Let us check whether the initial conditions are satisfied.

```
y /. k → 0
```

$$\tau$$

```
(y /. k → 1) // Expand
```

$$2\pi - \tau$$

The advanced z-transform image of the solution is

$$\mathbf{Y = \frac{z\,(z-1)\ \mathrm{Cos}\,[\tau]}{(z^2+2\,z+2)\,(z-1)^2} + \frac{(2\,z+z^2)\ \tau + z\,(2\pi-\tau)}{(z^2+2\,z+2)};}$$

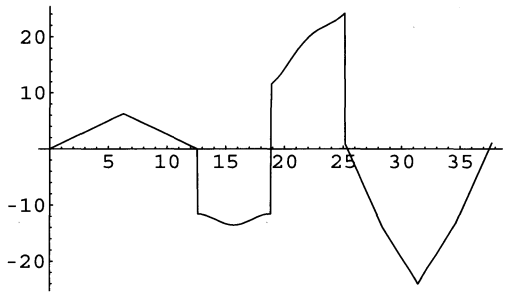
By applying

```
invY = InverseAdvancedZTransform[Y, z, yc, k, 2 π, τ] /. ExpandGenFac;

EvaluateToFunction[invY, 2 π]
```

$$\mathrm{Cos}\left[2\,\pi\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]-\#1\right]\left(\frac{1}{5}+2^{1+\frac{1}{2}\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]}\left(-\frac{1}{10}\,\mathrm{Cos}\left[\frac{3}{4}\,\pi\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]\right]-\frac{1}{5}\,\mathrm{Sin}\left[\frac{3}{4}\,\pi\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]\right]\right)\right)+2^{1+\frac{1}{2}\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]}\left(\pi\,\mathrm{Sin}\left[\frac{3}{4}\,\pi\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]\right]+\frac{1}{2}\,\mathrm{Cos}\left[\frac{3}{4}\,\pi\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]\right]\left(-2\,\pi\,\mathrm{Floor}\left[\frac{\#1}{2\,\pi}\right]+\#1\right)\right)\ \&$$

the time function is obtained in the form of a pure function, ready to be plotted.

```
Plot[%[t], {t, 0, 12 Pi}];
```



The result of the above steps can often be obtained more quickly with

`ForwardDifferenceEquationSolve[`*`diffeqn`*`,{y[τ]->`$y_0(\tau)$`,y[T+τ]->`$y_1(\tau)$`,y[2T+τ]->`$y_2(\tau)$`,...},y[t],t,T,τ,k,opt]`.

```
Clear[y]; diffeqn = y[t + 2 T] + 2 y[t + T] + 2 y[t] == t;

ForwardDifferenceEquationSolve[diffeqn,
  {y[τ] → τ, y[T + τ] → T - τ}, y[t], t, T, τ, k] /. ExpandGenFac
```

$$\left\{\mathrm{y}[\mathrm{k},\ \mathrm{T},\ \tau] \to \frac{\mathrm{k\,T}}{5}+\frac{1}{25}\,(-4\,\mathrm{T}+5\,\tau)+2^{1+\frac{k}{2}}\left(\left(\frac{2\,\mathrm{T}}{25}+\frac{2\,\tau}{5}\right)\mathrm{Cos}\left[\frac{3\,\mathrm{k}\,\pi}{4}\right]-\left(-\frac{14\,\mathrm{T}}{25}+\frac{\tau}{5}\right)\mathrm{Sin}\left[\frac{3\,\mathrm{k}\,\pi}{4}\right]\right)\right\}$$

In complicated problems the difficult step, i.e. the computation of the inverse advanced z-transform, may fail. By setting the option `GoFor->Image[Y,z]` the image function of the solution is returned.

```
Clear[Y];

ForwardDifferenceEquationSolve[diffeqn,
 {y[τ] → τ, y[T + τ] → T - τ}, y[t], t, T, τ, k, GoFor → Image[Y, z]]
```

$$\left\{Y[z,\ T,\ \tau] \to \frac{z\ (2\ T - 2\ T\ z + T\ z^2 - z^2\ \tau + z^3\ \tau)}{(-1 + z)^2\ (2 + 2\ z + z^2)}\right\}$$

7.3.5.2 Backward Difference Equations

An example of a backward difference equation is

$$y(t - 2\,T) - 4\,y(t - T) + 3\,y(t) = t\,.$$

The initial conditions are assumed to be

$$y(t) = 0,\quad \forall\, t < 0.$$

By using the Original Shift Rule I, the image equation becomes

$$(z^{-2} - 4\,z^{-1} + 3)\,Y(z, \tau) = T\,\frac{z}{(z-1)^2} + \frac{z}{z-1}\,\tau\,.$$

From this we obtain the advanced z-transform image function

$$Y(z, \tau) = \frac{1}{3}\,\frac{\tau\,z^4 + (T - \tau)\,z^3}{(z-1)^3\,(z - 1/3)}.$$

By applying the inverse z-transform, we obtain the solution in the form $y(k, \tau) = y(k\,T + \tau)$:

```
InverseZTransform[1/3 (τ z^4 + (T - τ) z^3)/((z - 1)^3 (z - 1/3)), z, k] /. ExpandGenFac
```

$$\frac{1}{3}\left(\frac{3}{4}\ (-1 + k)\ k\,T + \frac{1}{8}\ 3^{-k}\ (-3\,T + 2\,\tau) + \frac{3}{8}\ (T + 2\,\tau) + \frac{3}{4}\ k\ (T + 2\,\tau)\right)$$

In order to obtain a form of the solution that is suitable to be plotted, use

```
Clear [y]; InverseAdvancedZTransform[
  1/3 (τ z^4 + (T - τ) z^3)/((z - 1)^3 (z - 1/3)), z, y, k, T, τ] /. ExpandGenFac
```

$$\left\{y[k,\ T,\ \tau] \to \frac{1}{3}\left(\frac{3}{4}\ (-1 + k)\ k\,T + \frac{1}{8}\ 3^{-k}\ (-3\,T + 2\,\tau) + \frac{3}{8}\ (T + 2\,\tau) + \frac{3}{4}\ k\ (T + 2\,\tau)\right)\right\}$$

Do not forget to assign a numerical value for the sampling period before plotting, for example $T = 1$.

```
Plot[EvaluateToFunction[%, 1][t], {t, 0, 5}];
```

With `BackwardDifferenceEquationSolve[`*diffeqn*`,y[t],t,T,τ,k,opt]` the same result may be obtained.

```
BackwardDifferenceEquationSolve[
 y[t - 2 T] - 4 y[t - T] + 3 y[t] == t, y[t], t, T, τ, k]
```

$$\left\{y[k, T, \tau] \to \frac{1}{8} 3^{-1-k} (-3T + 2\tau) + \frac{1}{8} (T + 2\tau) + \frac{1}{4} (T + 2\tau)\, \text{GenFac}[2, 1, 1][k] + \frac{1}{2} T\, \text{GenFac}[3, 1, 1][k]\right\}$$

Chapter 8
Examples from Electricity

8.1 Transmission Lines

■ 8.1.1 Equations of the Transmission Line

Let us consider a long transmission line modeled by the following circuit (Fig. 8.1).

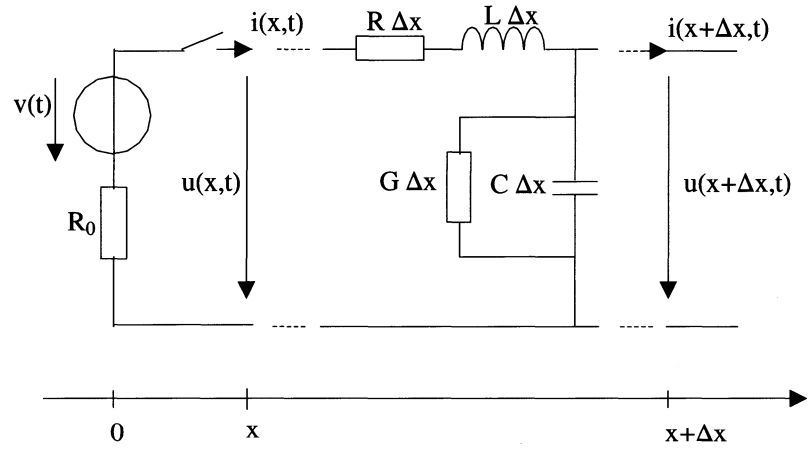


Fig.8.1: Electric transmission line.

The following quantities specify the transmission line and are all defined per unit length of distance: The ohmic resistance R, the inductance L, the leakage conductance G, and the capacitance C. Let $i = i(x, t)$ denote the current and $u = u(x, t)$ the voltage at distance x from the left end of the transmission line and at time t. We consider the piece of the transmission line between x and $x + \Delta x$. Applying Kirchhoff's law, we get the equations

$$R\,\Delta x\; i(x, t) + L\,\Delta x\,\frac{\partial i(x, t)}{\partial t} + u(x + \Delta x,\ t) - u(x, t) = 0,$$

$$i(x, t) = i(x + \Delta x,\ t) + G\,\Delta x\; u(x, t) + C\,\Delta x\,\frac{\partial u(x, t)}{\partial t}.$$

By dividing the above equations by Δx and letting Δx go to zero, we obtain the basic transmission line partial differential equations

$$R\,u(x, t) + L\,\frac{\partial i(x, t)}{\partial t} + \frac{\partial u(x, t)}{\partial x} = 0, \tag{8.1}$$

$$G\,u(x,t) + \frac{\partial i(x,t)}{\partial x} + C\,\frac{\partial u(x,t)}{\partial t} = 0\,. \tag{8.2}$$

In addition to these partial differential equations, we have the boundary conditions

$$u(0,t) = v(t) - R_o\, i(0,t)\,, \tag{8.3}$$

$$u(L,t) = R_L\, i(L,t)\,, \tag{8.4}$$

with (8.4) in the case of a finite length L of the transmission line.

Furthermore, we have the initial conditions

$$i(x,0) = i_0(x) \tag{8.5}$$

$$u(x,0) = u_o(x)\,. \tag{8.6}$$

From (8.1) and (8.2) we find that both the voltage $u(x,t)$ and the current $i(x,t)$ satisfy the so-called telegraph equation

$$LC\,\frac{\partial^2 y}{\partial t^2} + (RC + GL)\,\frac{\partial y}{\partial t} + GR\,y(x,t) = \frac{\partial^2 y}{\partial x^2}\,. \tag{8.7}$$

Equation (8.7) is of the hyperbolic type, if $LC > 0$ and of the parabolic type, if $LC = 0$. Therefore, if L or C are zero, we expect a different type of solution than in the hyperbolic case. Now suppose that the current $i(x,t)$ has the Laplace transform $I(x,s)$ and that the voltage $u(x,t)$ possesses the Laplace transform $U(x,s)$. $\partial i/\partial t$ transforms into $s\,I(x,s) - i_0(x)$, and $\partial u/\partial t$ into $s\,U(x,s) - u_0(x)$. When transforming (8.1) and (8.2) into the image domain and eliminating $I(x,s)$ from the two image equations, we obtain the following differential equation for the image function $U(x,s)$:

$$\frac{\partial^2 U}{\partial x^2} - p(s)^2\,U(x,s) = L\,\frac{\partial i_0}{\partial x} - C(R + s\,L)\,u_0(x)\,, \tag{8.8}$$

$$\mathrm{p(s)} = \sqrt{(R + s\,L)\,(G + C\,s)}\,. \tag{8.9}$$

Equation (8.8) is in fact an ordinary differential equation. The variable s can be considered as a parameter and the partial derivatives can be replaced by ordinary derivatives. We call the expression (8.9) the *image wave propagation coefficient.* Let us rewrite it in a slightly different way. First, in the hyperbolic case, where $LC > 0$, we have

$$p(s) = \sqrt{LC\,[s + (R/L + G/C)/2]^2 - (GL - RC)^2/(4\,LC)}\,.$$

By setting

$$a := \sqrt{LC};\;\; b := \frac{GL + RC}{2\sqrt{LC}};\;\; c := \frac{GL - RC}{2\sqrt{LC}} \tag{8.10}$$

we can write

$$p(s) = \sqrt{(a\,s + b)^2 - c^2}\,. \tag{8.11}$$

In the parabolic case, where $LC = 0$, we write

$$p(s) = \sqrt{a_1 s + b_1} \tag{8.12}$$

with $a_1 := GL + RC$ and $b_1 = RG$.

From (8.1) we see, that if $U(x, s)$ is determined by solving (8.8), the Laplace transform of the current is

$$I(x, s) = \frac{1}{R + sL}\left[-\frac{\partial U}{\partial x} + L\, i_0(x)\right]. \tag{8.13}$$

The Laplace transforms of (8.3) and (8.4) are

$$U(0, s) = V(s) - R_o\, I(0, s)\,; \quad U(L, s) = R_L\, I(L, s). \tag{8.14}$$

They are the boundary conditions when solving (8.8). In the following, we assume that the initial conditions for the voltage and the current are zero. This may be assumed in most applications.

When $u_0(x) = i_0(x) = 0$, (8.8) becomes

$$\frac{\partial^2 U}{\partial x^2} - p(s)^2\, U(x, s) = 0$$

with the general solution

$$U(x, s) = A(s)\, e^{-x\, p(s)} + B(s)\, e^{x\, p(s)}. \tag{8.15}$$

By (8.13), $i_0(x) = 0$, (8.15) and (8.9)

$$I(x, s) = [\,A(s)\, e^{-x\, p(s)} - B(s)\, e^{x\, p(s)}]\,/\,Z(s) \tag{8.16}$$

where

$$Z(s) = \sqrt{\frac{R + Ls}{G + Cs}} = \sqrt{\frac{L}{C}}\,\sqrt{\frac{as + b - c}{as + b + c}} \tag{8.17}$$

is called the *image impedance characteristic of the transmission line.* The second expression follows from (8.10). By (8.15), (8.16) and (8.14), we find the following linear system that determines $A(s)$ and $B(s)$ from the boundary conditions in the case of a finite length L:

$$\left(1 + \frac{R_0}{Z(s)}\right)A(s) + \left(1 - \frac{R_0}{Z(s)}\right)B(s) = V(s), \tag{8.18}$$

$$\left(1 - \frac{R_L}{Z(s)}\right)e^{-p(s)\,L}\,A(s) + \left(1 + \frac{R_L}{Z(s)}\right)e^{p(s)\,L}\,B(s) = 0.$$

If the transmisson line has infinite length, the boundary condition (8.4) is replaced by a boundedness condition that requires that both the voltage and the current remain finite as $x \to \infty$. If $p(s)$ is the function defined by (8.11) or (8.12) with $\mathrm{Re}[p(s)] > 0$, then (8.15) and (8.16) show that we have to set $B(s) = 0$. Consequently, (8.18) gives

$$A(s) = \frac{Z(s)}{Z(s) + R_0} V(s).$$

Finally, in the case of an infinite length, we obtain the following expressions:

$$I(x, s) = \frac{Z(s)}{Z(s) + R_0} e^{-p(s)x} V(s), \tag{8.19}$$

$$I(x, s) = \frac{1}{Z(s) + R_0} e^{-p(s)x} V(s). \tag{8.20}$$

■ 8.1.2 Transmission Line of infinite Length

The basic equations are given by (8.20) and (8.19). Let us now consider some special cases.

8.1.2.1 No Induction and no Leakage Conduction

Assume that $L = 0$ and $G = 0$. Then (8.7) is a parabolic partial differential equation and

$$p(s) = \sqrt{RCs},$$

$$Z(s) = \sqrt{\frac{R}{Cs}}.$$

The equations (8.20) and (8.19) are then

$$U(x, s) = \frac{\mathrm{Exp}\left(-x\sqrt{RC}\sqrt{s}\right)}{1 + R_0\sqrt{C/R}\sqrt{s}} V(s), \tag{8.21}$$

$$I(x, s) = \sqrt{\frac{C}{R}} \frac{\sqrt{s}\ \mathrm{Exp}\left(-x\sqrt{RC}\sqrt{s}\right)}{1 + R_0\sqrt{C/R}\sqrt{s}} V(s). \tag{8.22}$$

If we neglect the internal resistance R_0 of the voltage source, this simplifies to the

Special Case: $R_0 = 0$:

$$U(x, s) = e^{-x\sqrt{RC}\sqrt{s}} V(s), \tag{8.23}$$

$$I(x, s) = \sqrt{\frac{C}{R}}\sqrt{s}\, e^{-x\sqrt{RC}\sqrt{s}} V(s). \tag{8.24}$$

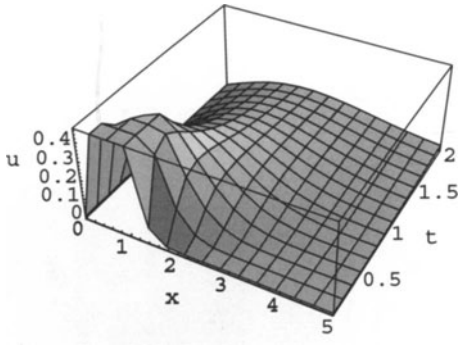
Let us, for example, compute the voltage corresponding to an applied Dirac impulse in which case $V(s) = 1$. Remember to load the Package now (See Section 3.1.2).

```
U = Exp[- x Sqrt[R C] Sqrt[s]];

u0 = InverseLaplaceTransform[U, s, t]
```

$$\frac{e^{-\frac{CRx^2}{4t}}\sqrt{CR}\,x}{2\sqrt{\pi}\,t^{3/2}}$$

This is the solution for the voltage $u(x, t)$ as a function of x and t. In order to get a plot in the (x, t)-plane, we choose, for example, $CR = 1$.

```
u = % /. CR → 1;

Plot3D[u, {x, 0, 5}, {t, 0.1, 2}, AxesLabel → {"x", "t", "u  "}];
```



It clearly shows the propagation of the Dirac impulse along the transmission line for various values of time. By the convolution theorem, the solution for an arbitrary applied voltage $v(t)$, in the case $R_0 = 0$, can be represented in the form of a convolution integral

$$u(t) = \frac{\sqrt{CR}\, x}{2\sqrt{\pi}} \int_0^t e^{-\frac{CRx^2}{4(t-\tau)}} (t-\tau)^{-3/2} v(\tau)\, d\tau . \tag{8.25}$$

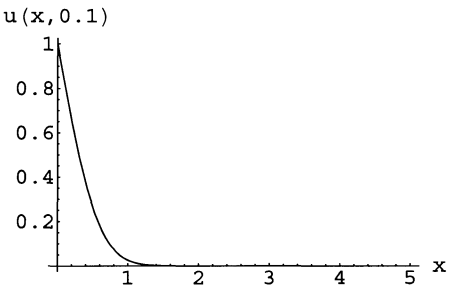
Let us now study the solution, if a unit-step voltage, $V(s) = 1/s$, is applied:

```
U = Exp[- x Sqrt[R C] Sqrt[s]] / s;

u1 = InverseLaplaceTransform[U, s, t]
```

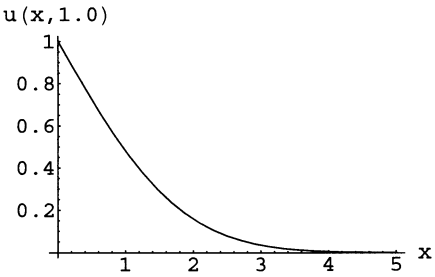
$$\text{Erfc}\left[\frac{\sqrt{CR}\, x}{2\sqrt{t}}\right]$$

Let us see how the signal propagates along the x–axis for different values of t.

```
u1n = u1 /. CR → 1; Plot[u1n /. t → 0.1, {x, 0, 5},
 PlotRange → All, AxesLabel → {"x", "u(x,0.1)"}];
```



```
Plot[u1n /. t → 1.0, {x, 0, 5},
  PlotRange → All, AxesLabel → {"x", "u(x,1.0)"}];
```



Next we consider the case where $v(t) = \cos \omega t$, i.e. $V(s) = s/(s^2 + \omega^2)$.

```
U = Exp[- x Sqrt[R C] Sqrt[s]] s / (s^2 + ω^2);

InverseLaplaceTransform[U, s, t]
```

$$e^{-\frac{\sqrt{CR}\, x \sqrt{\omega}}{\sqrt{2}}} \operatorname{Cos}\left[\frac{\sqrt{CR}\, x \sqrt{\omega}}{\sqrt{2}} - t\,\omega\right] - \frac{\int_0^\infty \frac{e^{-t\,u2}\, u2\, \operatorname{Sin}\left[\sqrt{CR}\, \sqrt{u2}\, x\right]}{u2^2+\omega^2}\, du2}{\pi}$$

The solution is composed of two parts. The first is the stationary or steady-state part of the solution

```
u2st = %[[1]]
```

$$e^{-\frac{\sqrt{CR}\, x \sqrt{\omega}}{\sqrt{2}}} \operatorname{Cos}\left[\frac{\sqrt{CR}\, x \sqrt{\omega}}{\sqrt{2}} - t\,\omega\right]$$

while the second part represents the transitory part

```
u2t = %%[[2]]
```

$$-\frac{\int_0^\infty \frac{e^{-t\,u2}\, u2\, \operatorname{Sin}\left[\sqrt{CR}\, \sqrt{u2}\, x\right]}{u2^2+\omega^2}\, du2}{\pi}$$

This transitory part is a Laplace integral. By Watson's lemma, we can find an asymptotic expansion for large values of t.

```
LI = LaplaceIntegral[-u2 Sin[√(C R) √u2 x] / ((u2^2 + ω^2) π), {u2, 0}][t];

AsymptoticSeries[LI,{t, Infinity, 3}]
```

$$-\frac{3\sqrt{CR}\, x}{4\sqrt{\pi}\, t^{5/2}\, \omega^2} + \frac{5\, CR\sqrt{CR}\, x^3}{16\sqrt{\pi}\, t^{7/2}\, \omega^2} + \text{Landau0}\left[\frac{1}{t^4}\right]$$

This shows how the transitory part diminishes as $t \to \infty$. Finally, we can write for large values of t,

$$u(t) = e^{-\frac{\sqrt{CR}\,x\sqrt{\omega}}{\sqrt{2}}} \operatorname{Cos}\left[\frac{\sqrt{CR}\,x\sqrt{\omega}}{\sqrt{2}} - t\,\omega\right] - \frac{3\sqrt{CR}\,x}{4\sqrt{\pi}\,t^{5/2}\,\omega^2} + \frac{5\,CR\sqrt{CR}\,x^3}{16\sqrt{\pi}\,t^{7/2}\,\omega^2} + \cdots \tag{8.26}$$

Case: $R_0 \neq 0$:

In this general case, but still in absence of induction and leakage conduction, we consider the case of an applied unit-step voltage, i.e. $V(s) = 1/s$.

```
U = Exp[- x Sqrt[R C] Sqrt[s]] / (R0 Sqrt[C/R] s (Sqrt[R/C]/R0 + Sqrt[s]));

u = InverseLaplaceTransform[U, s, t] // Together // PowerExpand
```

$$\operatorname{Erfc}\left[\frac{\sqrt{C}\,\sqrt{R}\,x}{2\sqrt{t}}\right] - e^{\frac{Rt}{CR_0^2} + \frac{Rx}{R_0}} \operatorname{Erfc}\left[\frac{\sqrt{C}\,\sqrt{R}\,x}{2\sqrt{t}} + \frac{\sqrt{R}\,\sqrt{t}}{\sqrt{C}\,R_0}\right]$$

This is the voltage $u(x, t)$ in the transmission line if no induction and no leakage conduction is present. The current is computed by

```
Ic = Exp[- x Sqrt[R C] Sqrt[s]] / (R0 Sqrt[s] (Sqrt[R/C]/R0 + Sqrt[s]));

ic = InverseLaplaceTransform[Ic, s, t] // Together // PowerExpand
```

$$\frac{e^{\frac{Rt}{CR_0^2} + \frac{Rx}{R_0}} \operatorname{Erfc}\left[\frac{\sqrt{C}\,\sqrt{R}\,x}{2\sqrt{t}} + \frac{\sqrt{R}\,\sqrt{t}}{\sqrt{C}\,R_0}\right]}{R_0}$$

Problem 8.1. Show that in the case $L = G = R_0 = 0$ the maximum value of the pulse, produced by an applied Dirac impulse propagates to the right (while diminishing) with velocity $v(t) = \sqrt{2/(RCt)}$.

Problem 8.2. Show that in the case $L = G = R_0 = 0$ a wavefront of constant value and resulting from an applied unit-step voltage travels to the right with velocity $v(t) = \sqrt{1/(RCt)}$.

Problem 8.3. Show that in the case $L = G = R_0 = 0$ the wave produced by an applied voltage $v(t) = \cos\omega t$ travels to the right with the constant velocity $v = \sqrt{2\omega/(RC)}$.

Problem 8.4. Show that in the case $L = G = 0$ and $R_0 \neq 0$, the term resulting from $R_0 \neq 0$ in the solution behaves like

$$\frac{\exp[-\{RCx^2/(4t) + Rt/(R_0^2\,C) + R/R_0\}]}{\sqrt{\pi}\left(\sqrt{RC/(2t)}\,x + 1/R_0\sqrt{Rt/C}\right)}$$

for a fixed t and large values of x, or for a fixed x and large values or small values of t. Hint: Use the asymptotic expansion of the complementary error function, which for large values of the argument is $\operatorname{erfc}(x) \sim \exp(-x^2)/\sqrt{\pi}/x$ as $x \longrightarrow \infty$.

8.1.2.2 No Resistance and no Leakage conduction

Let us assume $R = 0$ and $G = 0$. This is the simplest case. From (8.11) and (8.17) we have $p(s) = a\,s$, and $Z(s) = \sqrt{\frac{L}{C}} = \text{constant}$. Consequently, by (8.20) and (8.21)

$$U(x, s) = \frac{\sqrt{\frac{L}{C}}}{\sqrt{\frac{L}{C}} + R_0}\, e^{-a x s}\, V(s)\,, \tag{8.27}$$

$$I(x, s) = \frac{1}{\sqrt{\frac{L}{C}} + R_0}\, e^{-a x s}\, V(s)\,. \tag{8.28}$$

The shifting property immediately gives the result:

$$u(x, t) = \frac{\sqrt{\frac{L}{C}}}{\sqrt{\frac{L}{C}} + R_0}\, u(t - a\,x)\, v(t - a\,x)\,, \tag{8.29}$$

$$i(x, t) = \frac{1}{\sqrt{\frac{L}{C}} + R_0}\, u(t - a\,x)\, v(t - a\,x) \tag{8.30}$$

with $a = \sqrt{L\,C}$ and $u(t)$ being the unit-step function. Voltage and current are waves travelling to the right with the velocity $c = 1/a$ and with shapes that are not deformed. There is no deformation in the following more general case either.

8.1.2.3 Transmission without Deformation

From (8.10), we see that if $R\,C - G\,L = 0$, i.e. if $\delta := \frac{R}{L} = \frac{G}{C}$, then $c = 0$ and $p(s) = a\,s + b$, while (8.17) gives $Z(s) = \sqrt{L/C} = \text{const}$. Therefore,

$$U(x, s) = \frac{\sqrt{\frac{L}{C}}}{\sqrt{\frac{L}{C}} + R_0}\, e^{-b x}\, e^{-a x s}\, V(s)\,, \tag{8.31}$$

$$I(x, s) = \frac{1}{\sqrt{\frac{L}{C}} + R_0}\, e^{-b x}\, e^{-a x s}\, V(s)\,, \tag{8.32}$$

which, due to the shifting property, yields the solution

$$u(x, t) = \frac{\sqrt{\frac{L}{C}}}{\sqrt{\frac{L}{C}} + R_0}\, e^{-b x}\, u(t - a\,x)\, v(t - a\,x)\,, \tag{8.33}$$

$$i(x, t) = \frac{1}{\sqrt{\frac{L}{C}} + R_0}\, e^{-b x}\, u(t - a\,x)\, v(t - a\,x)\,. \tag{8.34}$$

8.1.2.4 No Leakage Conduction

Let us assume $G = 0$. In this case, by (8.10) we have $b = -c$ and therefore

$$p(s) = a\sqrt{s^2 + (2b/a)s}\ .$$

Also, from (8.17),

$$Z(s) = \sqrt{\frac{L}{C}}\ \frac{\sqrt{s^2 + (2b/a)s}}{s}\ .$$

Special Case $R_0 = 0$:

$$U(x, s) = e^{-x\,a\sqrt{s^2+(2b/a)s}}\ V(s), \tag{8.35}$$

$$I(x, s) = \sqrt{\frac{C}{L}}\ \frac{s}{\sqrt{s^2 + (2b/a)s}}\ e^{-x\,a\sqrt{s^2+(2b/a)s}}\ V(s)\,. \tag{8.36}$$

Let us compute the solutions for the applied unit-step voltage, i.e. $V(s) = 1/s$.

```
U = Exp[- x a Sqrt[s^2 + 2 b / a s]] / s
```

$$\frac{e^{-a\sqrt{\frac{2bs}{a}+s^2}\ x}}{s}$$

```
InverseLaplaceTransform[U, s, t]
```

$$e^{-b\,x}\ \text{UnitStep}[t - a\,x] + b\,x\left(\int_{a\,x}^{t} \frac{e^{-\frac{b\,u3}{a}}\ \text{BesselI}\left[1,\ \frac{b\sqrt{u3^2 - a^2\,x^2}}{a}\right]}{\sqrt{u3^2 - a^2\,x^2}}\ du3\right)\ \text{UnitStep}[t - a\,x]$$

```
Ic = Exp[- x a Sqrt[s^2 + 2 b / a s]] / Sqrt[s^2 + 2 b / a s] Sqrt[C / L]
```

$$\frac{e^{-a\sqrt{\frac{2bs}{a}+s^2}\ x}\ \sqrt{\frac{C}{L}}}{\sqrt{\frac{2bs}{a} + s^2}}$$

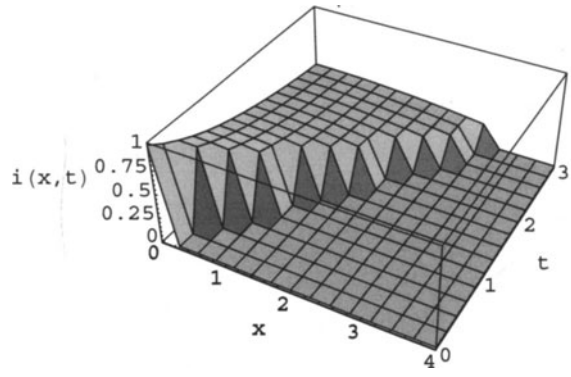
```
InverseLaplaceTransform[Ic, s, t]
```

$$e^{-\frac{b\,t}{a}}\ \sqrt{\frac{C}{L}}\ \text{BesselI}\left[0,\ \frac{b\sqrt{t^2 - a^2\,x^2}}{a}\right]\ \text{UnitStep}[t - a\,x]$$

Let us plot the current.

```
% /. {C → 1, L → 1, a → 1, b → 0.5};
```

```
Plot3D[%, {x, 0, 4}, {t, 0, 3}, AxesLabel → {"x", "t", "i(x,t)  "}];
```



The above figure shows the finite propagation speed $1/\sqrt{LC}$ of the wavefront.

8.1.2.5 The General Case

Let us consider the general case with $R_0 = 0$. From (8.20), (8.21), (8.17) and (8.11), we have

$$U(x, s) = e^{-x\sqrt{(as+b)^2-c^2}}\, V(s), \tag{8.37}$$

$$I(x, s) = \sqrt{\frac{C}{L}}\sqrt{\frac{(as+b)+c}{(as+b)-c}}\, e^{-x\sqrt{(as+b)^2-c^2}}\, V(s). \tag{8.38}$$

By the convolution theorem, we can establish a general formula for $u(x, t)$:

```
InverseLaplaceTransform[
  Exp[-x Sqrt[s^2 - c^2]] // Subst[s, a s +b], s, t] // Expand
```

$$\frac{e^{-\frac{bt}{a}}\,\text{DiracDelta}\left[\frac{t}{a}-x\right]}{a} + \frac{c\,e^{-\frac{bt}{a}}\,x\,\text{BesselI}\left[1,\, c\sqrt{\frac{t^2}{a^2}-x^2}\,\right]\text{UnitStep}\left[\frac{t}{a}-x\right]}{a\sqrt{\frac{t^2}{a^2}-x^2}}$$

```
% /. DiracDeltaRules /. Abs[a] → a
```

$$e^{-bx}\,\text{DiracDelta}[-t+a\,x] + \frac{c\,e^{-\frac{bt}{a}}\,x\,\text{BesselI}\left[1,\, c\sqrt{\frac{t^2}{a^2}-x^2}\,\right]\text{UnitStep}\left[\frac{t}{a}-x\right]}{a\sqrt{\frac{t^2}{a^2}-x^2}}$$

From this result we have

$$u(x, t) = u(t-a\,x)\,v(t-a\,x)\,e^{-b\,x} + c\,x\,u(t-a\,x)\int_{ax}^{t} e^{-b\,(t-\tau)/a}\,\frac{I_1\left(c\sqrt{(t-\tau)^2-a^2\,x^2}\,/a\right)}{\sqrt{(t-\tau)^2-a^2\,x^2}}\,v(\tau)\,d\tau \tag{8.39}$$

where $u(t)$ is the unit-step function. The integral in (8.39) usually cannot be evaluated in closed manner. Let us consider in more detail the case where the applied voltage is a cosine function: $v(t) = \cos \omega t$. From (8.37) and (8.38), we have

$$U(x, s) = e^{-x\sqrt{(as+b)^2 - c^2}} \frac{s}{s^2 + \omega^2}, \tag{8.40}$$

$$I(x, s) = \sqrt{\frac{C}{L}} \sqrt{\frac{(as + b) + c}{(as + b) - c}}\, e^{-x\sqrt{(as+b)^2 - c^2}} \frac{s}{s^2 + \omega^2}. \tag{8.41}$$

This can also be written as

$$U(x, s) = e^{-x\sqrt{(s+\alpha)(s+\beta)}} \frac{s}{s^2 + \omega^2}, \tag{8.42}$$

$$I(x, s) = \sqrt{\frac{C}{L}} \sqrt{\frac{as + b + c}{as + b - c}}\, e^{-x\sqrt{(s+\alpha)(s+\beta)}} \frac{s}{s^2 + \omega^2}, \tag{8.43}$$

where $\alpha = (b - c)/a$ and $\beta = (b + c)/a$. Even in this special case, it is hard to find a closed expression for the inverse Laplace transform of (8.42), and even more so for (8.43). Let us find an asymptotic expression for (8.42) that would give a good approximation for large values of t. $U(x, s)$ has simple poles at $s = i\omega$ and $s = -i\omega$. Moreover, it has two branch points at $s = -\alpha$ and $s = -\beta$, where $-\alpha$ is located to the right of $-\beta$. We consider the integration contour of Fig. 8.2.

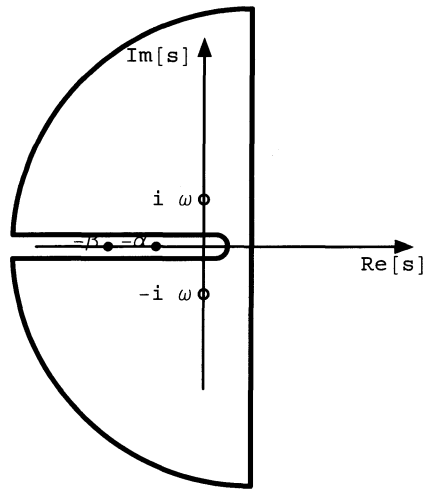


Figure 8.2: Contour integration path

The complex inversion formula yields

$$u(x, t) = \frac{1}{2\pi i} \int_{x_0 - j\infty}^{x_0 + i\infty} e^{ts} \frac{s}{s^2 + \omega^2} e^{-x\sqrt{(s+\alpha)(s+\beta)}}\, ds \quad (x_0 > 0). \tag{8.44}$$

For $t > c$, the contributions to the contour integral (with the contour of Fig. 8.2) from the two left quarter-circles tend to zero as their radius goes to infinity. The integral over the vertical

integration path is then equal to the Hankel-type loop integral plus the sum of the residues at ωi and $-\omega$i:

$$u(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{(\alpha+)} e^{ts}\,\frac{s}{s^2+\omega^2}\,e^{-x\sqrt{(s+\alpha)(s+\beta)}}\,ds + \operatorname*{Res}_{i\omega,-i\omega}\left\{e^{ts}\,\frac{s}{s^2+\omega^2}\,e^{-x\sqrt{(s+\alpha)(s+\beta)}}\right\}.$$

The Hankel-type loop integral is the first term of the second summand. The integration path starts at $-\infty$, runs along and below the real axis to the right, turns around α in the positive sense, and runs back above the real axis to $-\infty$. The transitory part is given by the asymptotic expansion of the Hankel-type loop integral at s = $-\alpha$.

```
F = s Exp[-x Sqrt[(s+α)(s+β)]]/(s^2 + ω^2);

DefPositive[c]; DefPositive[α]; DefPositive[β]; DefPositive[ω];

ResidueLaplace[F,{I ω,-I ω},s,t]
```

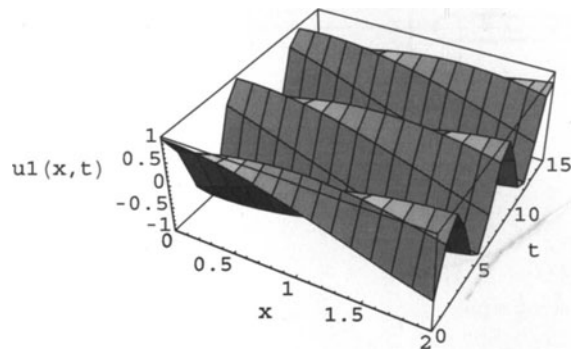
$$\frac{1}{2}\,e^{-x\sqrt{(\alpha-i\omega)\,(\beta-i\omega)}-it\omega} + \frac{1}{2}\,e^{-x\sqrt{(\alpha+i\omega)\,(\beta+i\omega)}+it\omega}$$

```
% /. ExpandUnderSqrt /. CartesianSqrt /.
{Sign[α ω + β ω] -> 1, Sign[-α ω - β ω] -> -1} /. EulerRule /. (-α ω -
β ω)^2 -> (α ω + β ω)^2;

u1=Expand[%]
```

$$e^{-\frac{x\sqrt{\alpha\beta-\omega^2+\sqrt{(\alpha\omega+\beta\omega)^2+(\alpha\beta-\omega^2)^2}}}{\sqrt{2}}}\,\mathrm{Cos}\left[t\omega - \frac{x\sqrt{-\alpha\beta+\omega^2+\sqrt{(\alpha\omega+\beta\omega)^2+(\alpha\beta-\omega^2)^2}}}{\sqrt{2}}\right]$$

Let us take a look at this function which, for a fixed x, is the steady-state part of the voltage.

```
Plot3D[u1 /. {α → 0.02, β → 0.1, ω → 1}, {x, 0, 2},
  {t, 0, 15}, AxesLabel → {"x", "t", "u1(x,t)  "}];
```



The transitory part of the voltage $u(x, t)$ can be obtained by asymptotic expansion of the loop integral of the Hankel type. Only the asymptotic expansion about the branch point $-\alpha$ (the right-most) needs to be taken.

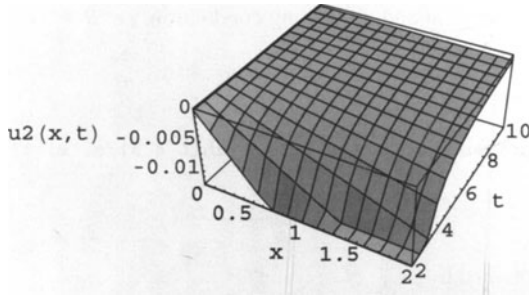
```
F1 = s Exp[-x Sqrt[(s+α)(s+β)]]/(s^2 + ω^2);

u2 = AsymptoticSeries[HankelIntegral[F1,{s,-α}][t],{t, Infinity, 2}]
```

$$e^{-t\alpha}\left(-\frac{x\,\alpha\sqrt{-\alpha+\beta}}{2\sqrt{\pi}\,t^{3/2}\,(\alpha^2+\omega^2)}+\right.$$

$$\left.\frac{3\left(\frac{2x\alpha^2\sqrt{-\alpha+\beta}}{(\alpha^2+\omega^2)^2}+\frac{-x\sqrt{-\alpha+\beta}-\frac{1}{3}\alpha\left(-\frac{3x}{2\sqrt{-\alpha+\beta}}-\frac{1}{2}x^3(-\alpha+\beta)^{3/2}\right)}{\alpha^2+\omega^2}\right)}{4\sqrt{\pi}\,t^{5/2}}+\text{LandauO}\left[\frac{1}{t^{7/2}}\right]\right)$$

Let us plot the transitory part.

```
Plot3D[u2 /. RemoveLandauO /. {α → 0.02, β → 0.1, ω → 1},
  {x, 0, 2}, {t, 2, 10}, AxesLabel → {"x", "t", "u2(x,t)      "}];
```



For values of t that are not too small, the global response to the sinusoidal input is represented by the sum $u(x, t) = u_1(x, t) + u_2(x, t)$.

■ 8.1.3 Transmission Lines of Finite Length

We start with the equations (8.18) and (8.19).

$$\left(1+\frac{R_0}{Z(s)}\right)A(s)+\left(1-\frac{R_0}{Z(s)}\right)B(s)=V(s),$$

$$\left(1-\frac{R_L}{Z(s)}\right)e^{-p(s)L}A(s)+\left(1+\frac{R_L}{Z(s)}\right)e^{p(s)L}B(s)=0.$$

When solving this system, we get fairly complicated expressions. In order to avoid this, we consider the case with no internal resistance of the source and no charge.

8.1.3.1 No Voltage Source Resistance and no Load

Let us assume $R_0 = 0$ and $R_L = 0$. We immediately obtain

$$A(s) + B(s) = V(s),$$

$$e^{-p(s)L} A(s) + e^{p(s)L} B(s) = 0,$$

which gives

$$A(s) = \frac{e^{L\,p(s)}}{2\sinh[L\,p(s)]}\,;\; B(s) = -\frac{e^{-L\,p(s)}}{2\sinh[L\,p(s)]}\,.$$

Finally, from (8.15) and (8.16),

$$U(x, s) = \frac{\sinh[(L-x)\,p(s)]}{\sinh[L\,p(s)]}\,V(s)\,, \tag{8.45}$$

$$I(x, s) = \frac{\cosh[(L-x)\,p(s)]}{Z(s)\,\sinh[L\,p(s)]}\,V(s)\,, \tag{8.46}$$

with $p(s)$ given by (8.11) and $Z(s)$ by (8.17).

a) Let us consider the case with no damping and no leaking conduction, i.e. $R = G = 0$. This implies $b = c = 0$, $p(s) = a\,s$ and $Z(s) = \sqrt{L/C}$.

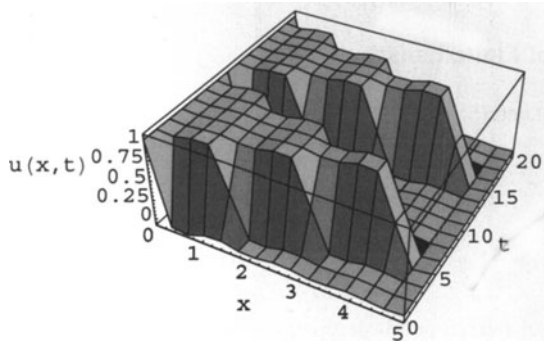
a) The applied voltage is the unit–step $u(t)$:

```
U = InverseLaplaceTransform[Sinh[a (L-x) s]/Sinh[L a s]/s, s, t]
```

$$\frac{L - x}{L} + \text{Summation}\left[-\frac{2\,\text{Cos}\left[\frac{k1\,\pi\,t}{a\,L}\right]\text{Sin}\left[\frac{k1\,\pi\,x}{L}\right]}{k1\,\pi}, \{k1, 1, \infty\}\right]$$

A visual representation of this is obtained by

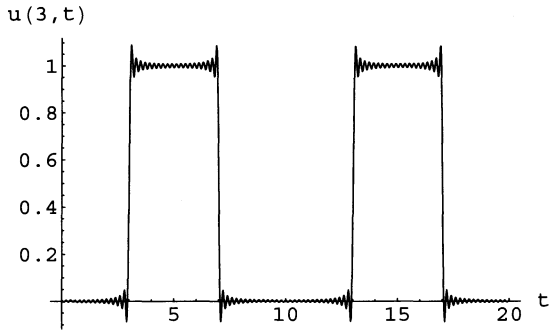
```
Uaprox = U /. TruncateSummation[20];

Plot3D[Uaprox /. {L → 5, a → 1}, {x, 0, 5},
  {t, 0, 20}, AxesLabel → {"x", "t", "u(x,t)  "}];
```



At some fixed point x, we have for example

```
Uaprox = U /. TruncateSummation[50]  /. {L → 5, a → 1};
```

```
Plot[Uaprox /. {x → 3}, {t, 0, 20}, AxesLabel → {"t", "u(3,t)  "}];
```



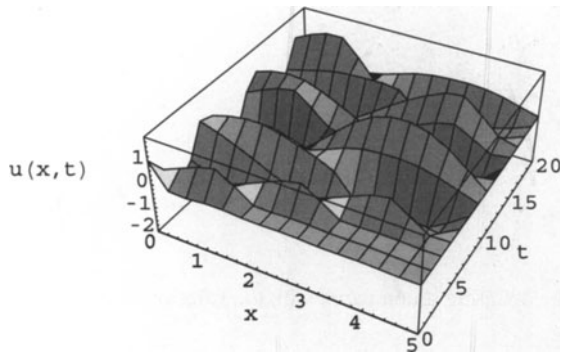
b) The applied voltage is a cosine:

```
U = InverseLaplaceTransform[s Sinh[a (L-x) s]/Sinh[L a s]/(s^2+ω^2),
s, t]
```

$$-\frac{2\,\text{Cos}[t\,\omega]\,\text{Sin}[a\,L\,\omega]\,\text{Sin}[a\,(L-x)\,\omega]}{-1+\text{Cos}[2\,a\,L\,\omega]}+$$

$$\text{Summation}\Big[-\frac{2\,k2\,\pi\,(k2^2\,\pi^2-a^2\,L^2\,\omega^2)\,\text{Cos}[\frac{k2\,\pi\,t}{a\,L}]\,\text{Sin}[\frac{k2\,\pi\,x}{L}]}{a^4\,L^4\,\left(-\frac{k2^2\,\pi^2}{a^2\,L^2}+\omega^2\right)^2},\ \{k2,\ 1,\ \infty\}\Big]$$

```
Uaprox = U /. TruncateSummation[20]  /. {L → 5, a → 1, ω → 1};

Plot3D[Uaprox , {x, 0, 5}, {t, 0, 20}, AxesLabel → {"x", "t", "u(x,t)  "}];
```



c) Let us now consider a transmission line with no deformation, but with damping and satisfying the condition $RC = GL$. In this case $c = 0$, $p(s) = as + b$, and $Z(s) = \sqrt{L/C}$. Equations (8.45) and (8.46) then give

$$U(x, s) = \frac{\sinh[(L-x)(as+b)]}{\sinh[L(as+b)]}\,V(s), \tag{8.47}$$

$$I(x, s) = \frac{\cosh[(L-x)(as+b)]}{Z(s)\,\sinh[L(as+b)]}\,V(s). \tag{8.48}$$

In order to invert these expressions, we consider

$$U_1(x, s) = \frac{\sinh[(L-x)\,s]}{\sinh[L\,s]}\; V((s-b)/a)\,, \tag{8.49}$$

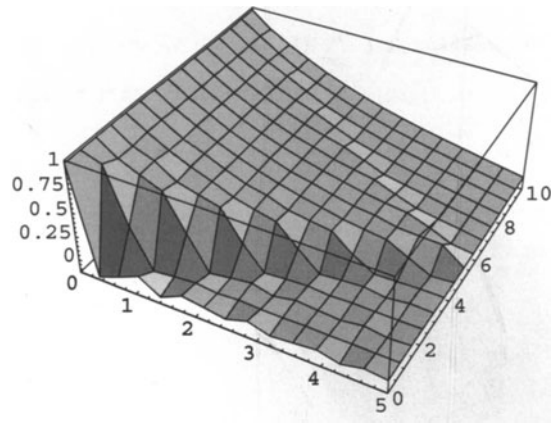
$$I_1(x, s) = \frac{\cosh[(L-x)\,s]}{Z(s)\,\sinh[L\,s]}\; V((s-b)/a)\,. \tag{8.50}$$

In the case of a unit-step input $V(s) = 1/s$, we use the substitution rule

```
u = InverseLaplaceTransform[(Sinh[(L-x) s]/Sinh[L s] a /(s-
b))//Subst[s, a s + b], s, t] //Expand
```

$$\text{Csch}[b\,L]\,\text{Sinh}[b\,(L-x)] + \frac{1}{a}\left(e^{-\frac{b\,t}{a}}\,\text{Summation}\left[\left(2\,(-1)^{k3}\left(-\frac{(-1)^{k3}\,a\,k3\,\pi\,\text{Cos}\left[\frac{k3\,\pi\,t}{a\,L}\right]\text{Sin}\left[\frac{k3\,\pi\,x}{L}\right]}{L} - (-1)^{k3}\,a\,b\,\text{Sin}\left[\frac{k3\,\pi\,t}{a\,L}\right]\text{Sin}\left[\frac{k3\,\pi\,x}{L}\right]\right)\right) \Big/ \left(L\left(b^2 + \frac{k3^2\,\pi^2}{L^2}\right)\right),\ \{k3,\ 1,\ \infty\}\right]\right)$$

```
u /. {a → 1, L → 5, b → 0.4} /. TruncateSummation[10];

Plot3D[%, {x, 0, 5}, {t, 0, 10}];
```



This solution distinctly shows the damping if compared with the solution of case a).

d) We consider now the parabolic case where the induction L or C are zero. According to (8.12) we have $p(s) = \sqrt{a_1\,s + b_1}$ and

$$U(x, s) = \frac{\sinh\left[(L-x)\,\sqrt{a_1\,s + b_1}\right]}{\sinh\left[L\,\sqrt{a_1\,s + b_1}\right]}\; V(s)\,, \tag{8.51}$$

$$I(x, s) = \frac{\cosh\left[(L-x)\,\sqrt{a_1\,s + b_1}\right]}{Z(s)\,\sinh\left[L\,\sqrt{a_1\,s + b_1}\right]}\; V(s)\,, \tag{8.52}$$

where

$Z(s) = \sqrt{R/G}$ if $L = C = 0$,

$Z(s) = R \,/\, \sqrt{a_1 s + b_1}$ if $L = 0$,

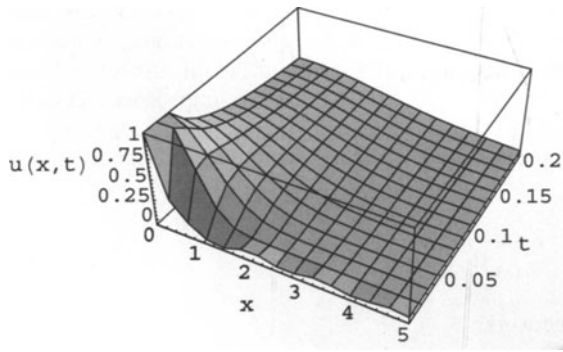
$Z(s) = \sqrt{a_1 s + b_1}\,/G$ if $C = 0$.

Let us compute the current for an applied Dirac impulse, if $C = 0$ but $L \neq 0$.

```
u = G InverseLaplaceTransform[(Sinh[(L-x) Sqrt[s]]/Sinh[L Sqrt[s]]/
Sqrt[s])//Subst[s,a1 s + b1], s, t]
```

$$\frac{1}{a1}\left(e^{-\frac{b1\,t}{a1}}\,G\left(\frac{1}{\sqrt{\pi}\sqrt{\frac{t}{a1}}}-\frac{x}{L\sqrt{\pi}\sqrt{\frac{t}{a1}}}+\frac{1}{2\sqrt{\pi}\left(\frac{t}{a1}\right)^{3/2}}\,\text{Summation}\left[-\frac{4\,t\,\text{Sin}\left[\frac{k5\,\pi\,x}{L}\right]}{a1\,k5\,\pi}+\frac{4\,e^{-\frac{k5^2\,\pi^2\,t}{a1\,L^2}}\sqrt{\frac{k5^2}{L^2}}\,\sqrt{\pi}\,\text{Erfi}\left[\frac{\sqrt{\frac{k5^2}{L^2}}\,\pi}{\sqrt{\frac{a1}{t}}}\right]\text{Sin}\left[\frac{k5\,\pi\,x}{L}\right]}{k5\left(\frac{a1}{t}\right)^{3/2}},\{k5,1,\infty\}\right]\right)\right)$$

```
u /. {a1 → 0.1, b1 → 0.02, G → 0.1, L → 5};

% /. Infinity → 20;

Plot3D[%, {x, 0, 5}, {t, 0.001, 0.2}, AxesLabel → {"x", " t", "u(x,t)  "}];
```



8.2 Electrical Networks

■ 8.2.1 Cascade Connection of Four-Terminal Networks

Let us consider a four-terminal network Q (Fig.8.3) defined by its chain or transmission matrix, denoted by the same symbol $Q(s)$:

$$Q(s) = \begin{pmatrix} \mu(s) & Z(s) \\ G(s) & \nu(s) \end{pmatrix}.$$

Then

$$\begin{pmatrix} U_1(s) \\ J_1(s) \end{pmatrix} = \begin{pmatrix} \mu(s) & Z(s) \\ G(s) & \nu(s) \end{pmatrix} \begin{pmatrix} U_2(s) \\ J_2(s) \end{pmatrix},$$

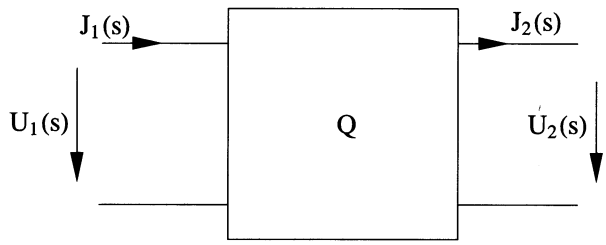


Fig.8.3: A four-terminal network.

where $U_1(s)$, $J_1(s)$ are the applied input image voltage and image current (the Laplace transforms of the voltage and the current) and $U_2(s)$, $J_2(s)$ the output image voltage and image current. The entries of the transmission matrix are Laplace image functions, too. The quantities $\mu(s)$ and $\nu(s)$ are dimensionless functions while $Z(s)$ is an image impedance and $G(s)$ an image conductance. If the output terminals of a four-terminal network Q_1 are connected to the input terminals of a second four-terminal network Q_2, the resulting four-terminal network has the transmission matrix

$$Q_1(s)\,Q_2(s) = \begin{pmatrix} \mu_1(s) & Z_1(s) \\ G_1(s) & \nu_1(s) \end{pmatrix} \begin{pmatrix} \mu_2(s) & Z_2(s) \\ G_2(s) & \nu_2(s) \end{pmatrix}.$$

By the theory of four-terminal networks

$$\det \begin{pmatrix} \mu(s) & Z(s) \\ G(s) & \nu(s) \end{pmatrix} = 1 \tag{8.53}$$

must hold.

This implies

$$\begin{pmatrix} U_2(s) \\ J_2(s) \end{pmatrix} = \begin{pmatrix} \nu(s) & -Z(s) \\ -G(s) & \mu(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ J_1(s) \end{pmatrix}.$$

Let us consider now a network of $n+1$ identical four-terminals in cascade or series connection (Fig.8.4). We then have

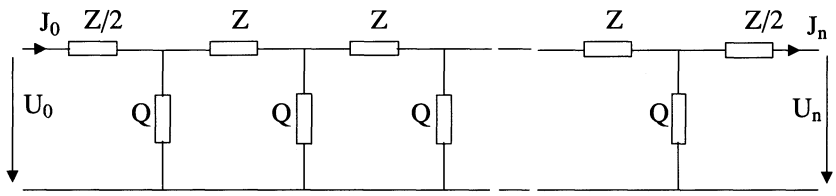


Fig.8.4: $n+1$ four-terminals in cascade.

$$\begin{pmatrix} U_0(s) \\ J_0(s) \end{pmatrix} = \begin{pmatrix} \mu(s) & Z(s) \\ G(s) & \nu(s) \end{pmatrix}^n \begin{pmatrix} U_n(s) \\ J_n(s) \end{pmatrix}$$

$$\begin{pmatrix} U_n(s) \\ J_n(s) \end{pmatrix} = \begin{pmatrix} \nu(s) & -Z(s) \\ -G(s) & \mu(s) \end{pmatrix}^n \begin{pmatrix} U_0(s) \\ J_0(s) \end{pmatrix}.$$

Since we have the recurrence relations

$$\begin{pmatrix} U_k(s) \\ J_k(s) \end{pmatrix} = \begin{pmatrix} \mu(s) & Z(s) \\ G(s) & \nu(s) \end{pmatrix} \begin{pmatrix} U_{k+1}(s) \\ J_{k+1}(s) \end{pmatrix} \quad (k = 0, 1, 2, \ldots, n),$$

the coupled system of difference equations

$$\begin{aligned} U_k(s) &= \mu(s)\, U_{k+1}(s) + Z(s)\, J_{k+1}(s), \\ J_k(s) &= G(s)\, U_{k+1}(s) + \nu(s)\, J_{k+1}(s) \end{aligned}$$

is obtained. This system is solved by the method of z-transformation with respect to the index k. Let

$$U_k \circ\!-\!\bullet\, U(z), \quad J_k \circ\!-\!\bullet\, J(z)$$

where the variable s is omitted in the image functions since it only takes the role of a parameter. The z-image system then is

$$\begin{aligned} U(z) &= \mu\,[z\,U(z) - z\,U_0] + Z\,[z\,J(z) - z\,J_0], \\ J(z) &= G\,[z\,U(z) - z\,U_0] + \nu\,[z\,J(z) - z\,J_0] \end{aligned}$$

or

$$\begin{aligned} (\mu z - 1)\, U(z) + Z z\; J(z) &= z\,(\mu\, U_0 + Z\, J_0) \\ z\, G\, U(z) + (\nu z - 1)\, J(z) &= z(G\, U_0 + \nu\, J_0). \end{aligned}$$

Solving this linear algebraic system yields

$$U(z) = \frac{z\,(z-\mu)}{z^2 - (\mu+\nu)\,z + 1}\, U_0 - \frac{z\,Z}{z^2 - (\mu+\nu)\,z + 1}\, J_0,$$

$$J(z) = -\frac{z\,G}{z^2 - (\mu+\nu)\,z + 1}\, U_0 + \frac{z\,(z-\nu)}{z^2 - (\mu+\nu)\,z + 1}\, J_0,$$

where $\mu\,\nu - G\,Z = 1$ is used. By setting

$$z^2 - (\mu+\nu)\,z + 1 = (z-\alpha_1)\,(z-\alpha_2)$$

we have $\alpha_1 + \alpha_2 = \mu + \nu$ and $\alpha_1 \alpha_2 = 1$. This enables us to set $\alpha_1 = e^{\tau}$, $\alpha_2 = e^{-\tau}$ and therefore $\cosh \tau = (\mu + \nu)/2$. By these substitutions the above expressions become

$$U(z) = \frac{z(z - \frac{\mu+\nu}{2}) - z\frac{\mu-\nu}{2}}{z^2 - (\mu+\nu)z + 1} U_0 - \frac{zZ}{z^2 - (\mu+\nu)z + 1} J_0 ,$$

$$J(z) = -\frac{zG}{z^2 - (\mu+\nu)z + 1} U_0 + \frac{z(z - \frac{\nu+\mu}{2}) - z\frac{\nu-\mu}{2}}{z^2 - (\mu+\nu)z + 1} J_0 .$$

Hence

$$U(z) = \left[\frac{z(z - \cos\tau)}{z^2 - 2z\cosh\tau + 1} - \frac{\mu - \nu}{2} \frac{z}{z^2 - 2z\cosh\tau + 1} \right] U_0 - \frac{zZ}{z^2 - 2z\cosh\tau + 1} J_0 ,$$

$$J(z) = -\frac{zG}{z^2 - 2z\cosh\tau + 1} U_0 + \left[\frac{z(z - \cos\tau)}{z^2 - 2z\cosh\tau + 1} + \frac{\mu - \nu}{2} \frac{z}{z^2 - 2z\cosh\tau + 1} \right] J_0 .$$

The inverse z-transform is now readily found by using the correspondences

$$\frac{z(z - \cos\tau)}{z^2 - 2z\cosh\tau + 1} \blacksquare - \circ \cos(\tau k) ,$$

$$\frac{z\sinh\tau}{z^2 - 2z\cosh\tau + 1} \blacksquare - \circ \sin(\tau k) .$$

For $k = 0, 1, 2, \ldots n$, we obtain

$$U_k(s) = \left[\cosh(\tau k) - \frac{\mu(s) - \nu(s)}{2} \frac{\sinh(\tau k)}{\sinh\tau}\right] U_0(s) - Z(s) \frac{\sinh(\tau k)}{\sinh\tau} J_0(s) , \tag{8.54}$$

$$J_k(s) = -G(s) \frac{\sinh(\tau k)}{\sinh\tau} U_0(s) + \left[\cosh(\tau k) + \frac{\mu(s) - \nu(s)}{2} \frac{\sinh(\tau k)}{\sinh\tau}\right] J_0(s) \tag{8.55}$$

where again the image variable s was inserted. Also

$$\cosh\tau = \frac{\mu(s) + \nu(s)}{2} .$$

The last result can also be written as

$$\begin{pmatrix} U_k(s) \\ J_k(s) \end{pmatrix} = \left\{ \frac{\sinh(\tau k)}{\sinh\tau} \begin{pmatrix} -(\mu(s) - \nu(s))/2 & -Z(s) \\ -G(s) & (\mu(s) - \nu(s))/2 \end{pmatrix} + \cosh(\tau k) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} U_0(s) \\ J_0(s) \end{pmatrix}$$

which gives

$$\begin{pmatrix} \nu(s) & -Z(s) \\ -G(s) & \mu(s) \end{pmatrix}^k$$
$$= \frac{\sinh(\tau k)}{\sinh \tau} \begin{pmatrix} -(\mu(s)-\nu(s))/2 & -Z(s) \\ -G(s) & (\mu(s)-\nu(s))/2 \end{pmatrix} + \cosh(\tau k) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The replacements $\nu(s) \to \mu(s)$, $\mu(s) \to \nu(s)$, $-Z(s) \to Z(s)$, $-G(s) \to G(s)$ yield

$$\begin{pmatrix} \mu(s) & Z(s) \\ G(s) & \nu(s) \end{pmatrix}^k = \frac{\sinh(\tau k)}{\sinh \tau} \begin{pmatrix} (\mu(s)-\nu(s))/2 & Z(s) \\ G(s) & -(\mu(s)-\nu(s))/2 \end{pmatrix} + \cosh(\tau k) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The last equation implies

$$U_0(s) = \left[\cosh(\tau n) + \frac{\mu(s)-\nu(s)}{2} \frac{\sinh(\tau n)}{\sinh \tau}\right] U_n(s) + Z(s) \frac{\sinh(\tau n)}{\sinh \tau} J_n(s),$$

$$J_0(s) = G(s) \frac{\sinh(\tau n)}{\sinh \tau} U_n(s) + \left[\cosh(\tau n) - \frac{\mu(s)-\nu(s)}{2} \frac{\sinh(\tau n)}{\sinh \tau}\right] J_n(s).$$

These results can be written more nicely. The Chebyshev Polynomials of the first and the second kind are defined as

$$T_n(x) := \cos[n \arccos x] = \cosh[n \operatorname{arccosh} x],$$
$$\hat{T}_n(x) := \frac{\sin[(n+1)\arccos x]}{\sin[\arccos x]} = \frac{\sinh[(n+1)\operatorname{arccosh} x]}{\sinh[\operatorname{arccosh} x]}.$$

The trigonometric terms in the above expressions can then be replaced by

$$\frac{\sinh(\tau n)}{\sinh \tau} = \hat{T}_{n-1}\left(\frac{\mu(s)+\nu(s)}{2}\right), \quad \cosh(\tau n) = T_n\left(\frac{\mu(s)+\nu(s)}{2}\right).$$

8.2.1.1 Ladder Networks and Filters

A special case of a four-terminal cascade connection is a ladder network (Fig.8.5). One element of the ladder is the four-terminal network shown in Fig.8.6. We have therefore

$$Z(s)/2\, J_1(s) + Q(s)\,[J_1(s) - J_2(s)] - U_1(s) = 0,$$
$$-Q(s)\,[J_1(s) - J_2(s)] + Z(s)/2\, J_2(s) + U_2(s) = 0.$$

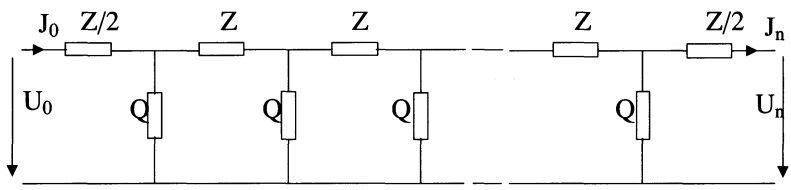


Fig.8.5: A ladder network

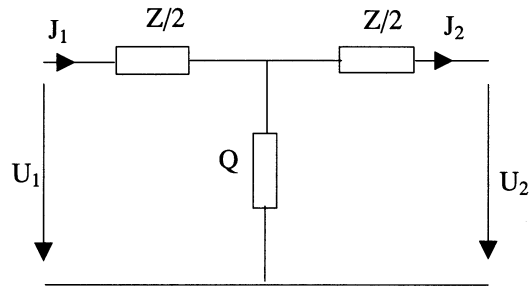


Fig.8.6: One element of the above ladder network.

By solving for $U_1(s)$, $J_1(s)$ as functions of $U_2(s)$, $J_2(s)$, the transmission matrix

$$\begin{pmatrix} U_1(s) \\ J_1(s) \end{pmatrix} = \begin{pmatrix} 1 + Z(s)/2\,Q(s) & Z(s)\,[1 + Z(s)/4\,Q(s)] \\ 1/Q(s) & 1 + Z(s)/2\,Q(s) \end{pmatrix} \begin{pmatrix} U_2(s) \\ J_2(s) \end{pmatrix}$$

is obtained. Therefore, a comparison of the entries of the general transmission matrix in Section 8.2.1 yields

$$\mu(s) = 1 + \frac{Z(s)}{2\,Q(s)},\quad Z(s) \to Z(s)\left(1 + \frac{Z(s)}{4\,Q(s)}\right),$$

$$G(s) = \frac{1}{Q(s)},\quad \nu(s) = 1 + \frac{Z(s)}{2\,Q(s)},$$

and also

$$\cosh\tau = 1 + \frac{Z}{2\,Q},\quad \sinh\tau = \frac{\sqrt{Z\,Q + Z^2/4}}{Q},$$

$$\frac{\mu(s) - \nu(s)}{2} = 0.$$

The equations (8.54) and (8.55) can be re-written as

$$U_k(s) = \cosh(\tau\,k)\,U_0(s) - Z(s)\left(1 + \frac{Z(s)}{4\,Q(s)}\right)\frac{\sinh(\tau\,k)}{\sinh\tau}\,J_0(s), \tag{8.56}$$

$$J_k(s) = -\frac{1}{Q(s)}\,\frac{\sinh(\tau\,k)}{\sinh\tau}\,U_0(s) + \cosh(\tau\,k)\,J_0(s). \tag{8.57}$$

A) No charge at the right terminal of the ladder network

In this case the right-boundary condition is $J_n(s) = 0$. From (8.57), with $k = n$, we get

$$J_n(s) = 0 = -\frac{1}{Q(s)}\,\frac{\sinh(\tau\,n)}{\sinh\tau}\,U_0(s) + \cosh(\tau\,n)\,J_0(s).$$

We obtain $J_0(s)$, which, substituted into (8.56) yields

$$U_k(s) = \left\{\cosh(\tau\,k) - \frac{Z\,Q + Z^2/4}{Q^2}\,\frac{\sinh(\tau\,k)}{\sinh^2\tau}\,\frac{\sinh(\tau\,n)}{\cosh(\tau\,n)}\right\}U_0(s)$$

or, by use of the above expression for $\sinh \tau$,

$$U_k(s) = \left\{\cosh(\tau k) - \frac{\sinh(\tau k)\sinh(\tau n)}{\cosh(\tau n)}\right\} U_0(s).$$

Finally, we have

$$U_k(s) = \frac{\cosh(\tau(n-k))}{\cosh(\tau n)} U_0(s). \tag{8.58}$$

By using the Chebyshev Polynomials T_k, this can be written as

$$U_k(s) = \frac{T_{n-k}(1+Z/2Q)}{T_n(1+Z/2Q)} U_0(s) \quad (k = 0, 1, 2, \ldots, n).$$

The transfer function between the applied voltage at the left terminal and the voltage at the uncharged right terminal of the ladder network then is

$$G(s) = \frac{U_n(s)}{U_0(s)} = \frac{1}{T_n(1+Z/2Q)}. \tag{8.59}$$

A filter is obtained by choosing $Z(s) = Ls$, i.e. an inductance, and $Q(s) = 1/Cs$, i.e. a capacitance. The transfer function of this LC-ladder then becomes

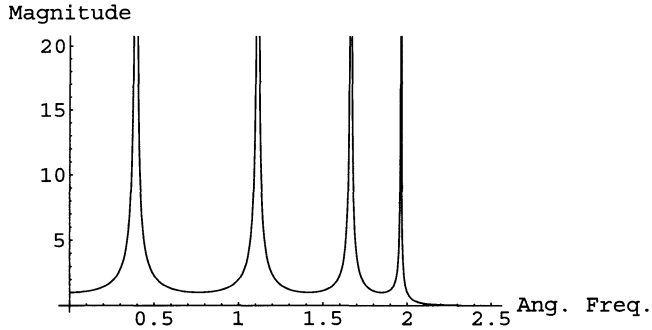
$$G(s) = \frac{U_n(s)}{U_0(s)} = \frac{1}{T_n(1+\frac{LC}{2}s^2)}.$$

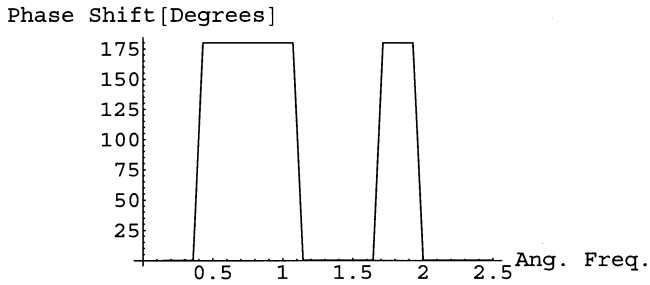
Let us study its frequency-response characteristics for $n = 4$, for example. Assuming $LC = 1$, we have

```
G = 1 / ChebyshevT[4, 1 + s^2 / 2] // ExpandDenominator
```

$$\frac{1}{1 + 8\,s^2 + 10\,s^4 + 4\,s^6 + \frac{s^8}{2}}$$

```
FrequencyResponsePlot[G, {s, 0, 2.5}];
```





B) The Right terminal of the ladder network is closed (short circuited)

This gives the boundary condition $U_n(s) = 0$. Viewed from the left terminal, the ladder becomes a two-port terminal. From (8.56) we obtain

$$U_n(s) = 0 = \cosh(\tau\, n)\, U_0(s) \;-\; Z(s)\left(1 + \frac{Z(s)}{4\,Q(s)}\right)\frac{\sinh(\tau\, n)}{\sinh\tau}\, J_0(s).$$

Equation (8.57) then becomes

$$J_k(s) = \left\{ -\frac{1}{Q(s)}\,\frac{\sinh(\tau\, k)}{\sinh\tau}\, U_0(s) \;+\; \cosh(\tau\, k)\,\frac{Q\,\sinh\tau\,\cosh(\tau\, n)}{(Z\,Q + Z^2/4)\sinh(\tau\, n)} \right\} U_0(s)$$

$$= \frac{1}{Q\sinh\tau}\left\{ -\sinh(\tau\, k) \;+\; \cosh(\tau\, k)\,\frac{\cosh(\tau\, n)}{\sinh(\tau\, n)} \right\} U_0(s)$$

or,

$$J_k(s) = \frac{1}{Q\sinh\tau}\,\frac{\cosh(\tau(n-k))}{\sinh(\tau\, n)}\, U_0(s) \quad (k = 0,\, 1,\, 2,\, \ldots,\, n). \tag{8.60}$$

By use of the Chebyshev Polynomials of the first and the second kind, we can also write

$$J_k(s) = \frac{1}{Q\sinh^2\tau}\,\frac{\cosh(\tau(n-k))}{\sinh(\tau\, n)/\sinh\tau}\, U_0(s)$$

$$= \frac{Q}{Z\,Q + Z^2/4}\,\frac{T_{n-k}(1 + Z/2\,Q)}{\hat{T}_{n-1}(1 + Z/2\,Q)}\, U_0(s).$$

The image admittance of the two-port then is

$$G_n(s) = \frac{J_n(s)}{U_0(s)} \;=\; \frac{Q(s)}{Z(s)\,[\,Q(s) + Z(s)/4\,]}\,\frac{1}{\hat{T}_{n-1}(1 + Z/2\,Q)}. \tag{8.61}$$

For $Z(s) = L\,s$ and $Q(s) = 1/C\,s$ the image admittance of the two-port cascade LC-filter becomes

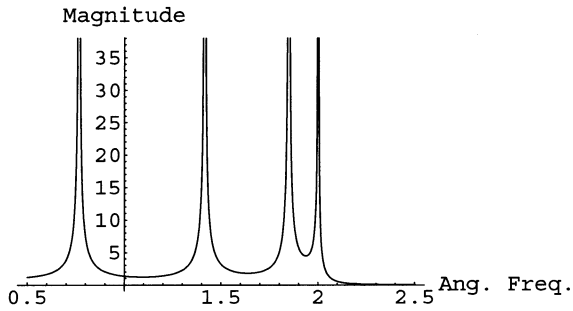
$$G_n(s) = \frac{J_n(s)}{U_0(s)} \;=\; \frac{1}{L\,s\,[\,1 + L\,C\,s^2/4\,]}\,\frac{1}{\hat{T}_{n-1}(1 + L\,C\,s^2/2)}.$$

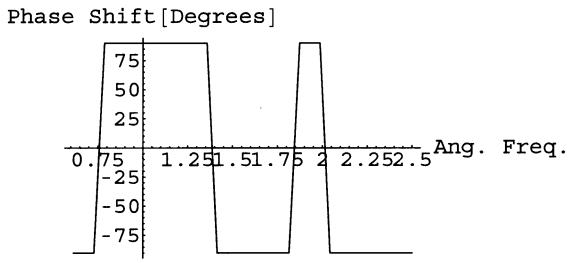
Let us study its frequency-response characteristics ($L = C = 1$) for $n = 4$, for example. Assuming $L = C = 1$, we get

```
G4 = 1 / ((ChebyshevU[3, 1 + s^2 / 2] // ExpandDenominator) (s (1 + s^2 / 4)))
```

$$\frac{1}{s\left(1+\frac{s^2}{4}\right)\left(-4\left(1+\frac{s^2}{2}\right)+8\left(1+\frac{s^2}{2}\right)^3\right)}$$

```
FrequencyResponsePlot[G4, {s, 0.5, 2.5}];
```





8.2.1.2 A Variable Low-High-Pass Filter

For another example of an electrical network, let us consider a tone control filter (bass and trebles) of James and Baxandall. The network of the filter is shown in Fig.8.7. The resistors $(1-\lambda)R_2$, λR_2 and $(1-\mu)R_6$, μR_6 in fact represent two potentiometers where the parameters λ and μ may be set between 0 and 1. The physical values are:

```
PhysicalValues = {R0 → 38 10^3, R1 → 10^5,
   R2 → 10^6, R3 → 10^4, R4 → 180 10^3, R6 → 470 10^3, RL → 10^6,
   C1 → 4.7 10^-10, C2 → 4.7 10^-9, C3 → 3.3 10^-10, C4 → 3.3 10^-9};
```

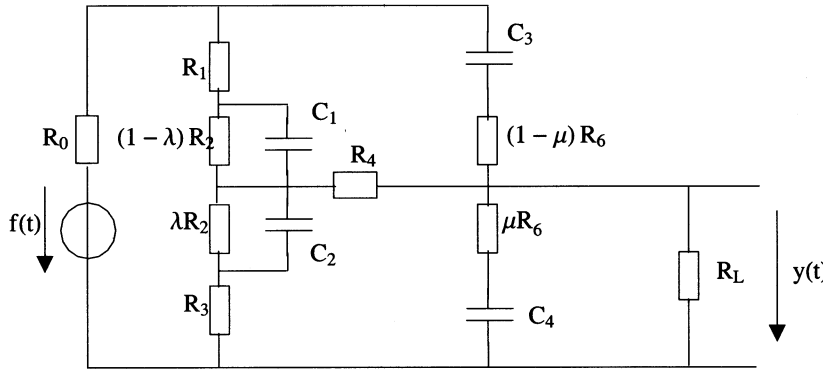


Fig.8.7: The James–Baxandall Filter

The Laplace image network has the structure shown in Fig.8.8 with the image impedances

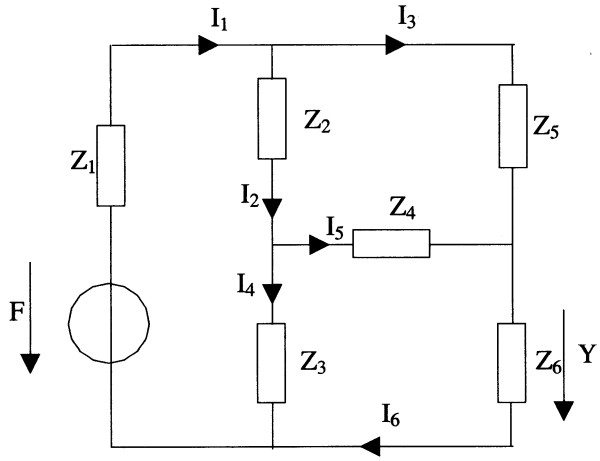


Fig.8.8: Image network of the James–Baxandall Filter.

$$\texttt{Impedances} = \left\{\texttt{Z1} \to \texttt{R0},\ \texttt{Z2} \to \texttt{R1} + \frac{(1-\lambda)\ \texttt{R2}}{(1-\lambda)\ \texttt{R2 C1 s} + 1},\ \texttt{Z3} \to \texttt{R3} + \frac{\lambda\ \texttt{R2}}{\lambda\ \texttt{R2 C2 s} + 1},\right.$$
$$\left.\texttt{Z4} \to \texttt{R4},\ \texttt{Z5} \to \frac{1}{\texttt{s C3}} + (1-\mu)\ \texttt{R6},\ \texttt{Z6} \to \frac{\texttt{RL}\ (1 + \mu\ \texttt{R6 C4 s})}{1 + (\mu\ \texttt{R6} + \texttt{RL})\ \texttt{C4 s}}\right\};$$

The governing equations of the network in Fig.8.7 are

```
Clear[F]; Clear[Y]

eqns = {-F + I1 Z1 + I2 Z2 + I4 Z3 == 0,
   -I2 Z2 + I3 Z5 - I5 Z4 == 0, -I4 Z3 + I5 Z4 + I6 Z6 == 0, Y == I6 Z6,
   I1 == I2 + I3, I2 == I4 + I5, I1 == I4 + I6, I6 == I3 + I5};

Solve[eqns, Y, {I1, I2, I3, I4, I5, I6}]

{{Y → (F Z3 (-Z2 Z4 - Z3 (Z2 + Z4 + Z5)) Z6) /
     ((-Z2 Z3 - (Z1 + Z2 + Z3) Z4) (Z3 Z5 - Z2 Z6) -
       (-Z2 Z4 - Z3 (Z2 + Z4 + Z5)) (-Z1 Z3 - (Z1 + Z2 + Z3) Z6))}}

Y = First[Y /. %];
```

```
Y = Y /. Impedances /. PhysicalValues // Together;

Yn = Collect[Numerator[Y], {s}];

Yd = Collect[Denominator[Y], {s}];
```

The transfer function of the network becomes

```
G = Yn / Yd / F // Simplify;
```

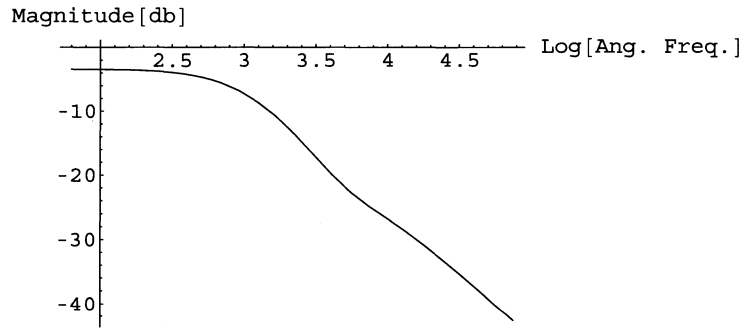
Let us take a look at the transfer function for $\lambda = 1$ and $\mu = 0$.

```
G10 = G /. {λ → 1, μ → 0} // Chop
```

$$(-1.54561\times10^{16} - 5.35476\times10^{12}\,s - 8.20955\times10^{8}\,s^2 - 28164.8\,s^3)\,/\,(-2.2863\times10^{16} - 3.63746\times10^{13}\,s - 1.3923\times10^{10}\,s^2 - 1.66485\times10^{6}\,s^3 - 50.8814\,s^4)$$

The Bode plot shows that the network acts as low-pass filter which suppresses the trebles.

```
BodePlot[G10, {s, 20 Pi, 24000 Pi }];
```

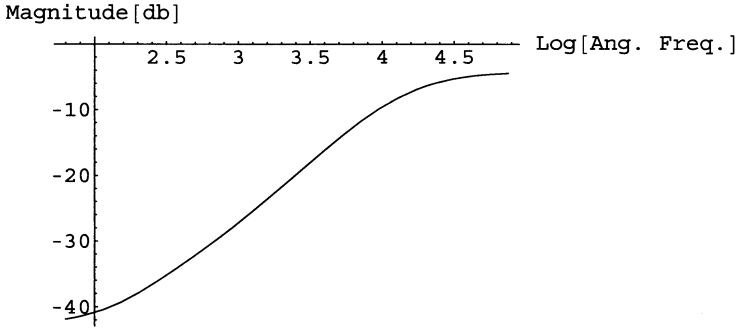


Now we set the parameters $\lambda = 0$ and $\mu = 1$.

```
G01 = G /. {λ → 0, μ → 1} // Chop
```

$$(-1.51515\times10^{12} - 1.43142\times10^{10}\,s - 2.43338\times10^{7}\,s^2 - 9190.32\,s^3 - 0.356321\,s^4)\,/\,(-2.06973\times10^{14} - 5.53209\times10^{11}\,s - 2.63134\times10^{8}\,s^2 - 24193.7\,s^3 - 0.580941\,s^4)$$

```
BodePlot[G01, {s, 20 Pi, 24000 Pi }];
```

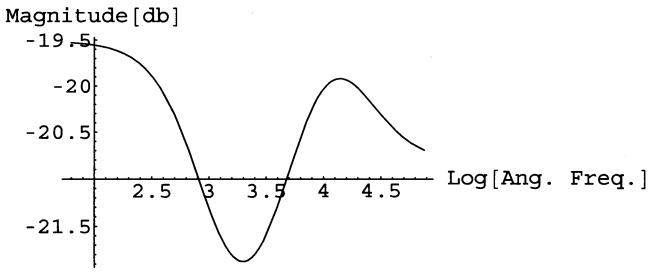


We see that the network now acts as a high-pass filter suppressing the bass. By choosing the parameters $\lambda = \mu = 0.15$, we obtain

```
G11 = G /. {λ → 0.15, μ → 0.15} // Chop
```

$$(-3.87879\times10^{14} - 7.66913\times10^{11}\,s - 5.40486\times10^{8}\,s^2 - 171186.\,s^3 - 24.2538\,s^4 - 0.00129899\,s^5 - 2.28796\times10^{-8}\,s^6) / (-3.6672\times10^{15} - 8.59904\times10^{12}\,s - 6.22357\times10^{9}\,s^2 - 1.89473\times10^{6}\,s^3 - 250.96\,s^4 - 0.0135875\,s^5 - 2.51488\times10^{-7}\,s^6)$$

```
BodePlot[G11, {s, 20 Pi, 24000 Pi }];
```

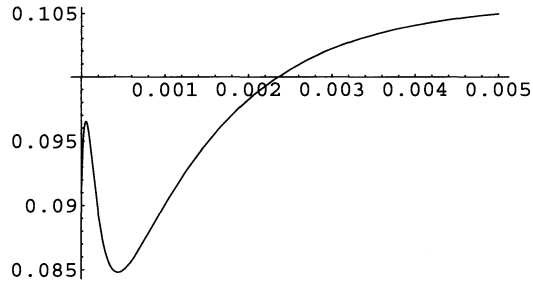


This shows that bass and trebles pass through equally. Let us compute the unit-step response in the latter case.

```
InverseLaplaceTransform[G11 / s, s, t]
```

$$0.10577 + 9.09564\times10^{-16}\,e^{-22695.\,t} - 0.0277952\,e^{-18717.\,t} + 0.0435961\,e^{-6576.99\,t} + 0.00319003\,e^{-2788.19\,t} - 1.19017\times10^{-15}\,e^{-2503.13\,t} - 0.0337838\,e^{-747.853\,t}$$

```
Plot[%, {t, 0, 5 10^-3}];
```

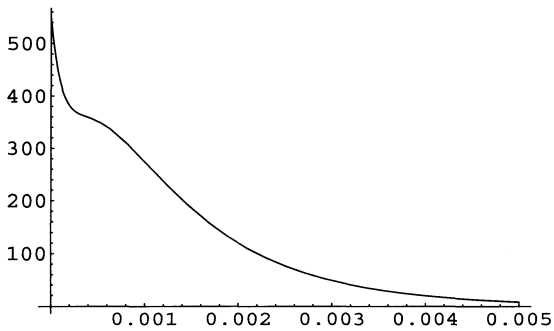


Finally, we compute the impulse response of the low-pass configuration.

```
InverseLaplaceTransform[G10, s, t]
```

$$6.07071 \times 10^{-11}\, e^{-21489.4\,t} + 498.103\, e^{-7093.91\,t} - 696.536\, e^{-3222.05\,t} + 751.971\, e^{-914.816\,t}$$

```
Plot[%, {t, 0, 5 10^-3}];
```



Chapter 9
Examples from Control Engineering

9.1 Control of an Inverted Pendulum

■ 9.1.1 The System

Stabilizing an inverted pendulum (Fig. 9.1) is an example that is widespread in control education. A cart moving on a track without friction carries an inverted pendulum which is attached by a friction-free joint. Suppose that the cart is moved along the track by controlling the input voltage of a DC motor that applies a force on the cart proportional to the voltage. The following variables and constants of the system are used: $x = x(t)$ translation of the cart from the center of the track; $\alpha = \alpha(t)$ angle of the inverted pendulum with the normal of the track; $f = f(t)$ force applied to the cart; M mass of the cart; m mass of the pendulum; l center of mass of the pendulum rod (half of full length) g gravitational acceleration. Let us start by modeling the open-loop system. The force $f(t)$ acting on the cart is chosen as the input of the system. First we shall use Lagrange's method to obtain the model of the system, then the nonlinear equations will be linearized.

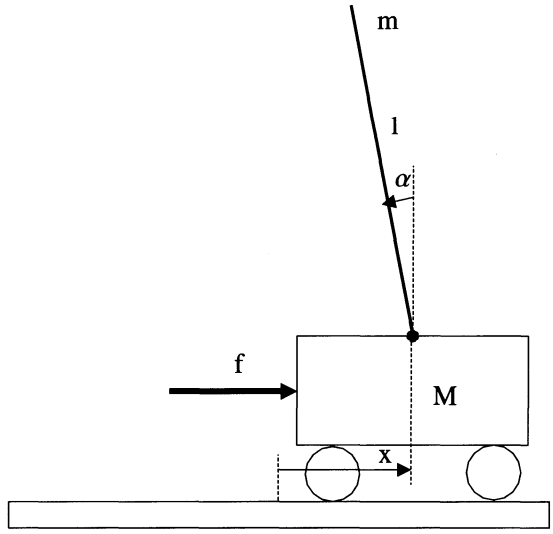


Fig. 9.1: Inverted pendulum

■ 9.1.2 Modeling

Following Lagrange's method, see for example [Goldstein (1980)], we have to compute the total kinetic and potential energy in terms of the generalized coordinates x, α and f. The Lagrangian is the difference between these two quantities.

9.1.2.1 Computing of the Lagrangian

Potential and kinetic energy of the cart:

```
V_M = 0; T_M = Simplify[ M (∂_t x[t])^2 / 2];
```

Coordinates of the center of mass of the pendulum:

```
x_p = x[t] - l Sin[α[t]];  y_p = l Cos[α[t]];
```

Moment of inertia of the pendulum with respect to the center of mass, potential and kinetic energy of the pendulum:

```
J = 1 / 3 m l^2; V_p = m g y_p;
T_p = Simplify[m / 2 ((∂_t x_p)^2 + (∂_t y_p)^2) + J / 2 (∂_t α[t])^2];
```

Total potential and kinetic energy with the Lagrangian of the system:

```
V = V_M + V_p; T = T_M + T_p; L = T - V
```

$$-g\,l\,m\,\text{Cos}[\alpha[t]] + \frac{1}{2} M\,x'[t]^2 + \frac{1}{6} m\,(3\,x'[t]^2 - 6\,l\,\text{Cos}[\alpha[t]]\,x'[t]\,\alpha'[t] + 4\,l^2\,\alpha'[t]^2)$$

9.1.2.2 Equations of Motion

The equations of motion are the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \quad (k = 1, \ldots n),$$

where L is the Lagrangian of the system, the q_k's are the generalized coordinates, the Q_k's are the generalized forces associated with each coodinate and n is the degree of freedom of the system. In our case we have $n = 2$, $q_1 = x$, $Q_1 = f$ (the input of the system), $q_2 = \alpha$, $Q_2 = 0$. Therefore the two Lagrange equations become:

```
eqn1 = Simplify[∂_t (∂_x'[t] L) - ∂_x[t] L == f[t]]
```

$$l\,m\,\text{Sin}[\alpha[t]]\,\alpha'[t]^2 + (m + M)\,x''[t] == f[t] + l\,m\,\text{Cos}[\alpha[t]]\,\alpha''[t]$$

```
eqn2 = Simplify[∂_t (∂_α'[t] L) - ∂_α[t] L == 0]
```

$$\frac{1}{3} l\,m\,(-3\,g\,\text{Sin}[\alpha[t]] - 3\,\text{Cos}[\alpha[t]]\,x''[t] + 4\,l\,\alpha''[t]) == 0$$

We have two coupled nonlinear differential equations. By assuming small angles α of deviations and small angular velocities, we can linearize the system and get:

$$(m + M)\frac{d^2x}{dt^2} - l\,m\frac{d^2\alpha}{dt^2} = f(t) + w_1(t),$$

$$4\,l\frac{d^2\alpha}{dt^2} - 3\frac{d^2x}{dt^2} - 3\,g\,\alpha(t) = w_2(t),$$

where we introduced a perturbation force $w_1(t)$ acting on the cart and $w_2(t)$ acting on the pendulum. This is a second order system of dimension 2 (see Section 5.3). In matrix form, the previous equations are

$$\begin{pmatrix} m+M & -l\,m \\ -3 & 4\,l \end{pmatrix}\begin{pmatrix} x'' \\ \alpha'' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -3\,g \end{pmatrix}\begin{pmatrix} x(t) \\ \alpha(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}. \tag{9.1}$$

```
A2 = {{m + M, -l m}, {-3, 4 l}}; A1 = NullMatrix[2];
A0 = {{0, 0}, {0, -3 g}}; B0 = {{1, 0}, {0, 0}};
B1 = NullMatrix[2];
```

With the above matrices we have the following matrix differential system of the second order (Section 5.3.1.1).

```
sys = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}];
```

The perturbation matrix is

```
H = IdentityMatrix[2];
```

The transfer matrix between the input vector and the output vector is

```
TransferMatrices[sys, H, s] // First // MatrixForm
```

$$\begin{pmatrix} \frac{3\,g-4\,l\,s^2}{s^2\,(3\,g\,m+3\,g\,M-l\,m\,s^2-4\,l\,M\,s^2)} & 0 \\ \frac{3}{-3\,g\,m-3\,g\,M+l\,m\,s^2+4\,l\,M\,s^2} & 0 \end{pmatrix}$$

and the transfer matrix between the perturbation vector and the output vector becomes

```
TransferMatrices[sys, H, s] // Last // MatrixForm
```

$$\begin{pmatrix} \frac{3\,g-4\,l\,s^2}{s^2\,(3\,g\,m+3\,g\,M-l\,m\,s^2-4\,l\,M\,s^2)} & \frac{l\,m}{-3\,g\,m-3\,g\,M+l\,m\,s^2+4\,l\,M\,s^2} \\ \frac{3}{-3\,g\,m-3\,g\,M+l\,m\,s^2+4\,l\,M\,s^2} & \frac{m+M}{-3\,g\,m-3\,g\,M+l\,m\,s^2+4\,l\,M\,s^2} \end{pmatrix}$$

The unstable denominator polynomials of the transfer matrices of the open-loop system evidently reflects the system's instability.

9.1.2.3 The Feedback System I

Let us consider a full state feedback

$$f(t) = -c_1\,x(t) - c_2\,x'(t) - c_3\,\alpha(t) - c_4\,\alpha'(t)$$

or, in matrix form

$$\begin{pmatrix} f(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -c_1 & -c_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \alpha(t) \end{pmatrix} + \begin{pmatrix} -c_2 & -c_4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ \alpha' \end{pmatrix}.$$

The two feedback matrices are

```
K0 = {{-c1, -c3}, {0, 0}}; K1 = {{-c2, -c4}, {0, 0}};
```

assuming that we can measure the instantaneous position of the cart, its velocity, the instantaneous angle of the pendulum and its angular velocity. The task is to find values for the *ci*'s such that the zero position of the cart and the upright position (zero angle) of the pendulum become asymptotically stable positions. Substituting the feedback function into the matrix differential equations (9.1) yields

$$\begin{pmatrix} m+M & -l\,m \\ -3 & 4\,l \end{pmatrix} \begin{pmatrix} x'' \\ \alpha'' \end{pmatrix} + \begin{pmatrix} c2 & c4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ \alpha' \end{pmatrix} + \begin{pmatrix} c1 & c3 \\ 0 & -3\,g \end{pmatrix} \begin{pmatrix} x(t) \\ \alpha(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.$$

Let us compute the closed-loop characteristic function between the perturbation and the output of the feedback system (Section 5.3.1.1):

```
ClosedLoopTransferMatrix[sys, H, {K0, K1}, s] // Short
```

$$\left\{\left\{\frac{3\,g - 4\,l\,s^2}{3\,c1\,g + \text{<<15>>}}, \frac{-c3 - c4\,s + l\,m\,s^2}{\text{<<1>>}}\right\}, \left\{\frac{3\ \text{<<1>>}}{\text{<<1>>}}, \text{<<1>>}\right\}\right\}$$

The full feedback controller configuration is defined by

```
sysfb = FullFeedbackCC[sys, {K0, K1}, s];
```

We need the closed-loop characteristic function, too.

```
cf = ClosedLoopCharacteristicFunction[sysfb]
```

```
-3 c1 g - 3 c2 g s + (3 c3 + 4 c1 l - 3 g m - 3 g M) s^2 +
 (3 c4 + 4 c2 l) s^3 + (l m + 4 l M) s^4
```

In order that the feedback system be asymptotically stable, all zeros of the closed-loop characteristic function must have strictly negative real parts. A necessary condition is that all coefficients have the same sign. This implies the necessary conditions

$$c1 < 0;\ \ c2 < 0;\ \ 3\,g\,(m+M) < 3\,c3 + 4\,l\,c1;\ \ -4\,l\,c2 < 3\,c4$$

Therefore c4 must be positive. Sufficient conditions are found by the `HurwitzTest`. This will give a list of elements that all should have the value True.

```
HurwitzTest[cf, s]
```

```
{-3 c1 g > 0, -3 c2 g > 0, 3 c4 + 4 c2 l > 0, 3 c3 + 4 c1 l - 3 g m - 3 g M > 0,
 -27 c2 c3 c4 g + 27 c1 c4^2 g - 36 c2^2 c3 g l + 36 c1 c2 c4 g l +
   27 c2 c4 g^2 m + 27 c2^2 g^2 l m + 27 c2 c4 g^2 M > 0, l m + 4 l M > 0}
```

We now choose a fixed system by assuming some numerical quantities.

```
physicalvalues = {M → 0.48, m → 0.2, g → 9.81, l → 0.335};
```

We also try some values for c1, c2 and c4.

```
tryvalues = {c4 → 1, c2 → -0.4, c1 → -0.5};
```

```
HurwitzTest[cf /. physicalvalues /. tryvalues, s]
```

```
{True, -20.6824 + 3 c3 > 0, -787.677 + 87.0186 c3 > 0}
```

Inspection shows that with $c3 = 10$ for example, the sufficient stability conditions are met.

```
StabilityQ[cf /. physicalvalues /. tryvalues /. c3 → 10, s]
```

```
True
```

Let us now simulate the system by solving the above coupled system of differential equations. We assume that we have a perturbation of the zero state in form of a Dirac impulse $\delta(t)/10$ (a small slap given to the pendulum).

$$\begin{pmatrix} m+M & -l\,m \\ -3 & 4\,l \end{pmatrix}\begin{pmatrix} x'' \\ \alpha'' \end{pmatrix} + \begin{pmatrix} c2 & c4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x' \\ \alpha' \end{pmatrix} + \begin{pmatrix} c1 & c3 \\ 0 & -3\,g \end{pmatrix}\begin{pmatrix} x(t) \\ \alpha(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \delta(t)/10 \end{pmatrix}.$$

The matrix differential operator becomes

```
MDOp = {{(m + M) p^2 + c2 p + c1, -l m p^2 + c4 p + c3},
      {-3 p^2, 4 l p^2 - 3 g}} /. physicalvalues /. tryvalues /. c3 → 10;

MatrixForm[%]
```

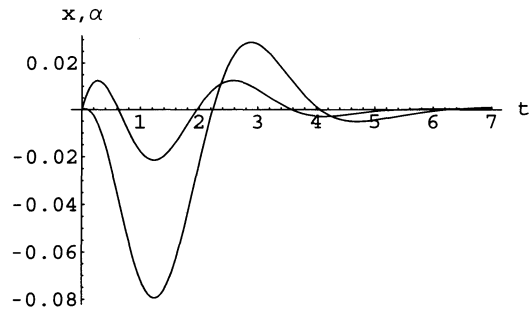
$$\begin{pmatrix} -0.5 - 0.4\,p + 0.68\,p^2 & 10 + p - 0.067\,p^2 \\ -3\,p^2 & -29.43 + 1.34\,p^2 \end{pmatrix}$$

```
LinearDOpSolve[
  MDOp[{x, α}] == {0, 0.1 DiracDelta[t]}, p, {x[t], α[t]}, t] // Chop
```

```
{x[t] → 2 e^(-0.750375 t) (-0.000823329 Cos[1.50276 t] - 0.105299 Sin[1.50276 t]) +
    2 e^(-0.984348 t) (0.000823329 Cos[2.52485 t] + 0.0646172 Sin[2.52485 t]),
 α[t] → 2 e^(-0.750375 t) (-0.0208223 Cos[1.50276 t] - 0.0187017 Sin[1.50276 t]) +
    2 e^(-0.984348 t) (0.0208223 Cos[2.52485 t] + 0.0320217 Sin[2.52485 t])}
```

It is seen that the cart responds to the slap in the positive angle direction first with an elongation in the negative x-direction. After some oscillations, pendulum and cart gain the stable zero position.

```
Plot[Evaluate[{x[t], α[t]} /. %], {t, 0, 7}, AxesLabel → {"t", "x,α"}];
```



9.1.2.4 The Feedback System II

If the upright position and the zero position of the cart are stabilized, the whole system is not very spectacular to observe. Also, taking 4 measurements for the feedback is not economical. One would want to see some movements, too. This can be achieved by stabilizing the pendulum's upright position, but not the zero position of the cart. We just want that the cart comes to a stop. In order to do this, let us again take the matrix transfer function of the system (9.1), with no perturbation term this trial. We obtain two transfer functions: $F(s)$ to $X(s)$ and $F(s)$ to $A(s)$.

```
TransferMatrices[sys, NullMatrix[2], s] // First
```

$$\left\{\left\{\frac{3g-4ls^2}{s^2(3gm+3gM-lms^2-4lMs^2)},\ 0\right\},\ \left\{\frac{3}{-3gm-3gM+lms^2+4lMs^2},\ 0\right\}\right\}$$

We therefore have,

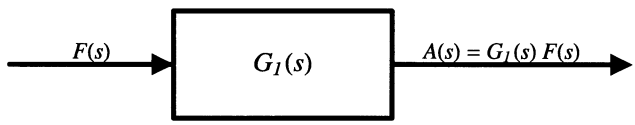
$$X(s) = \frac{-3g+4ls^2}{s^2[-3g(m+M)+l(m+4M)s^2]}F(s)$$

$$A(s) = \frac{3}{-3g(m+M)+l(m+4M)s^2}F(s).$$

Let us introduce the following positive constants

```
const =
  {a → Sqrt[3 g (m + M) / (l (m + 4 M))], b → 4 / (m + 4 M), c → 3 / (l (m + 4 M))};
```

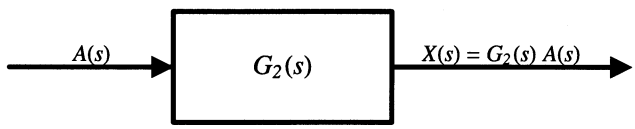
We now have two subsystems:



with the transfer function

$$G_1(s) = \frac{c}{(s^2 - a^2)}$$

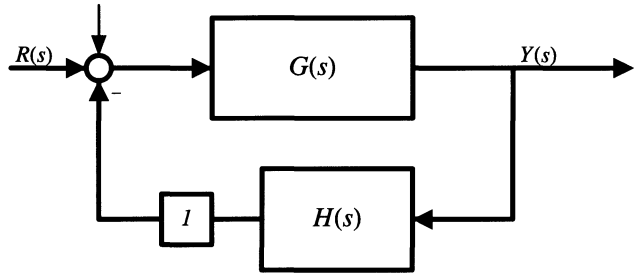
from F(s) to A(s), and



with the transfer function

$$G_2(s) = \frac{b}{c}\,\frac{s^2 - cg/b}{s^2}$$

from $A(s)$ to $X(s)$. Observe that $G_1(s)\,G_2(s)$ gives the first component of the above matrix transfer function. We will stabilize the first subsystem by an appropriate feedback, i.e. we consider the following controller configuration:



For the feedback transfer function we choose a lead network which can physically be realized by a simple electrical network,

$$H(s) = k\,\frac{s+\mu}{s+\nu} \quad (\nu > \mu > 0).$$

For example, choose $\nu = 4\,\mu$. The closed-loop transfer function of this controller configuration with negative feedback (see Section 5.1)

```
sys1 = StandardCC[-1, {c / (s^2 - a^2)}, {k (s + μ) / (s + 4 μ)}, s]
```

$$\text{StandardCC}\left[-1,\ \left\{\frac{c}{-a^2+s^2}\right\},\ \left\{\frac{k\,(s+\mu)}{s+4\,\mu}\right\},\ s\right]$$

then becomes

```
G3 = ClosedLoopTransferFunction[sys1]
```

$$\frac{c\,(s+4\,\mu)}{c\,k\,(s+\mu) - a^2\,(s+4\,\mu) + s^2\,(s+4\,\mu)}$$

The closed-loop characteristic function which is the denominator of $G_3(s)$ is obtained by

```
ClosedLoopCharacteristicFunction[sys1];

cf1 = Collect[%, {1, s}]
```

$$(-a^2 + c\,k)\,s + s^3 - 4\,a^2\,\mu + c\,k\,\mu + 4\,s^2\,\mu$$

A necessary condition for stability is that all four coefficients be positive, implying

$$c\,k - a^2 > 0, \quad (c\,k - 4\,a^2)\,\mu > 0.$$

The second condition implies the first one. Let us assume the given physical quantities for the system parameters:

```
phyval = Union[const /. physicalvalues, {g → 9.81}];

cf1nu = cf1 /. phyval
```

$$(-28.1785 + 4.22416\,k)\,s + s^3 - 112.714\,\mu + 4.22416\,k\,\mu + 4\,s^2\,\mu$$

```
HurwitzTest[cf1nu, s]
```

```
{True, -28.1785 + 4.22416 k > 0, 4 μ > 0,
 -112.714 μ + 4.22416 k μ > 0, 0. μ^2 - 1428.37 k μ^2 + 53.5306 k^2 μ^2 > 0}
```

By inspection we see that for any positive value of μ it is sufficient to obtain a stable polynomial if $k > 112.714 / 4.22416 = 26.68$. Let us take $k = 30$ and $\mu = 10$, for example.

```
StabilityQ[cf1nu /. {k → 30, μ → 10}, s]

True
```

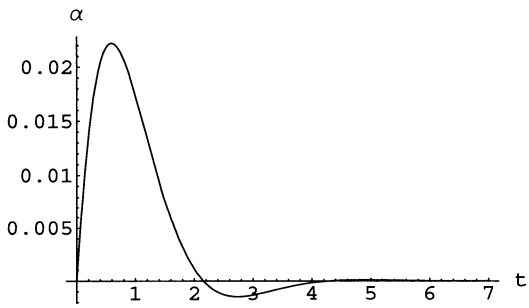
With these parameter values, the closed-transfer function becomes

```
G3 = G3 /. phyval /. {k → 30, μ → 10}
```

$$\frac{4.22416\ (40 + s)}{126.725\ (10 + s) - 28.1785\ (40 + s) + s^2\ (40 + s)}$$

Let us do a test and compute the response of the pendulum, if we give the cart a slight slap, modeled by a Dirac impusle $\delta(t)/50$.

```
InverseLaplaceTransform[G3 / 50, s, t] // Chop

0.000162815 e^-37.4698 t +
 2 e^-1.26511 t (-0.0000814074 Cos[1.46242 t] + 0.0309 Sin[1.46242 t])

Plot[%, {t, 0, 7}, AxesLabel → {"t", "α"}];
```



The upright position of the pendulum is stabilized. What is the motion of the cart during the controlled recovery of the pendulum positions. The cart will move away as a result of the slap and finally fall off the track (we didn't consider damping). An additional control is necessary to keep the cart on track. We want the cart to move but then come to a stop. By inspection of the transfer functon $G_3(s)\,G_2(s)$ with

$$G_2(s) = \frac{b}{c}\,\frac{s^2 - c\,g/b}{s^2},$$

```
G2 = b / c (s^2 - c g / b) / s^2 /. phyval; G2 G3
```

$$\frac{1.88679\ (40 + s)\ (-21.9627 + s^2)}{s^2\ (126.725\ (10 + s) - 28.1785\ (40 + s) + s^2\ (40 + s))}$$

we see that the denominator contains a stable polynomial of degree 3 belonging to G_3, but also having a double pole at the origin. This double pole produces a pulling away of the cart that is proportional to the time t. By introducing a controller block with the transfer function

$$G_4(s) = \frac{\lambda s}{s + \lambda}$$

which is physically easy to realize, we remove the double pole and now get the transfer function from $F(s)$ to $X(s)$ as

```
G5 = G3 G2 λ s / (s + λ)
```

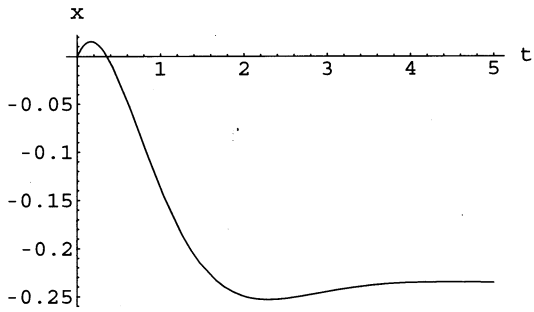
$$\frac{1.88679\ (40+s)\ (-21.9627+s^2)\ \lambda}{s\ (126.725\ (10+s)-28.1785\ (40+s)+s^2\ (40+s))\ (s+\lambda)}$$

By giving the pendulum, for example, a slap $\delta(t)/50$, the cart will do the following motion:

```
InverseLaplaceTransform[(G5 /. λ → 5) / 50, s, t] // Chop
```

$$-0.236614 + 0.000413049\, e^{-37.4698\,t} - 0.00767943\, e^{-5.\,t} + 2\, e^{-1.26511\,t}\ (0.12194\ \text{Cos}[1.46242\,t] + 0.162161\ \text{Sin}[1.46242\,t])$$

```
Plot[%, {t, 0, 5}, AxesLabel → {"t", "x"}];
```



The cart moves and comes to a stop. The pendulum's upright position is still stabilized. Its movement is as follows:

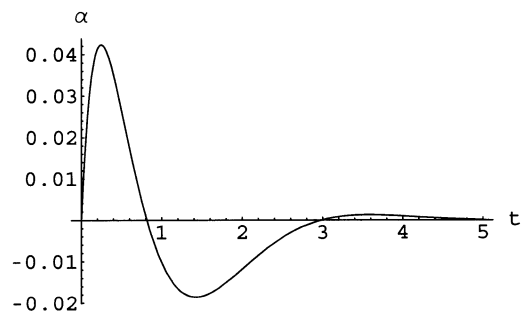
```
G3 λ s / (s + λ)
```

$$\frac{4.22416\ s\ (40+s)\ \lambda}{(126.725\ (10+s)-28.1785\ (40+s)+s^2\ (40+s))\ (s+\lambda)}$$

```
InverseLaplaceTransform[(% /. λ → 5) / 50, s, t] // Chop
```

$$0.000939432\, e^{-37.4698\,t} - 0.141513\, e^{-5.\,t} + 2\, e^{-1.26511\,t}\ (0.0702867\ \text{Cos}[1.46242\,t] - 0.024653\ \text{Sin}[1.46242\,t])$$

```
Plot[%, {t, 0, 5}, AxesLabel → {"t", "α"}];
```



9.2 Controling a Seesaw-Pendulum

This example is inspired by the article [Jirstrand and Gunnarsson (1999)] from which we have also taken the physical values for the pendulum in the preceeding example.

■ 9.2.1 The System

The system considered consists of a seesaw, two carts called C_1 and C_2, two parallel tracks, an inverted pendulum mounted on cart C_2, and a weight carried by cart C_1. Each cart on its track is driven by a DC motor controlled by an input voltage in a way such that the applied force is proportional to the voltage. We neglect any frictional forces (Fig. 9.2) and we introduce the following system variables and constants: $x_1 = x_1(t)$ translation of cart C_1 from center of track; $x_2 = x_2(t)$ translation of cart C_2 from center of track; $\theta = \theta(t)$ angle between the seesaw and the vertical; $\alpha = \alpha(t)$ angle between the inverted pendulum and the normal of the track; $f_1 = f_1(t)$ force applied to cart C_1 (input 1); $f_2 = f_2(t)$ force applied to cart C_2 (input 2); J moment of inertia of seesaw with tracks with respect to its center of mass; m_s mass of seesaw with tracks; c height of center of mass of the seesaw from pivot point; h height of tracks from pivot point; m_1 mass of cart C_1 with weight (on the back track); m_2 mass of cart C_2 (pendulum cart on the front track); m_p mass of pendulum; l_p center of mass of pendulum rod (half of full length); J_p moment of inertia of pendulum with respect to its center of mass ($= m_p l_p^2 / 3$); g gravitational acceleration. We take the following numerical values for the physical quantities.

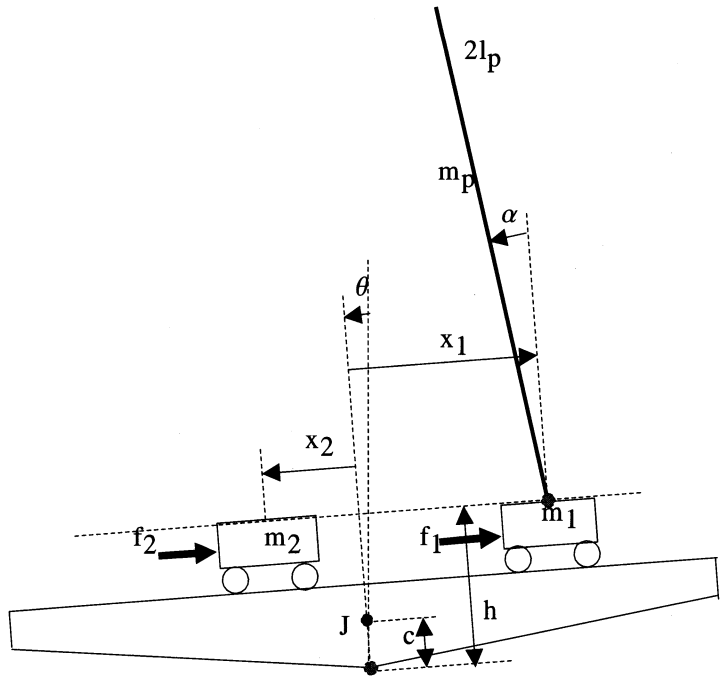


Figure 9.2: Seesaw Pendulum

```
physicalvalues = {J → 1.6, m_s → 6.6, c → 0.06, h → 0.115,
   m_1 → 0.48 + 0.38, m_2 → 0.48, m_p → 0.2, l_p → 0.335, g → 9.81};
```

■ 9.2.2 Modeling

Following Lagrange's method, see for example [Goldstein (1980)], we have to compute the total kinetic and potential energy in terms of the generalized coordinates x_1, x_2, θ, α. The forces f_1 and f_2 will be introduced as generalized forces.

9.2.2.1 Computing the Lagrangian

Coordinates of the center of mass of the seesaw:

```
x_s = - c Sin[θ[t]]; y_s = c Cos[θ[t]];
```

Potential and kinetic energies of the seesaw:

```
V_s = m_s g y_s;

T_s = Simplify[J / 2 (∂_t θ[t])^2];
```

Coordinates of center of track:

```
x_ct = - h Sin[θ[t]]; y_ct = h Cos[θ[t]];
```

Coordinates of cart C_1:

```
x_m1 = x_ct + x_1[t] Cos[θ[t]]; y_m1 = y_ct + x_1[t] Sin[θ[t]];
```

Potential and kinetic energies of cart C_1:

```
V_m1 = m_1 g y_m1;
T_m1 = Simplify[m_1 / 2 ((∂_t x_m1)^2 + (∂_t y_m1)^2)];
```

Coordinates of cart C_2:

```
x_m2 = x_ct + x_2[t] Cos[θ[t]]; y_m2 = y_ct + x_2[t] Sin[θ[t]];
```

Potential and kinetic energies of cart C_2:

```
V_m2 = m_2 g y_m2;
T_m2 = Simplify[m_2 / 2 ((∂_t x_m2)^2 + (∂_t y_m2)^2)];
```

Coordinates of center of mass of the pendulum:

```
x_p = x_m2 - l_p Sin[α[t] + θ[t]];

y_p = y_m2 + l_p Cos[α[t] + θ[t]];
```

Potential and kinetic energies of the pendulum:

```
V_p = m_p g y_p;

T_p = Simplify[m_p / 2 ((∂_t x_p)^2 + (∂_t y_p)^2 + l_p^2 / 3 (∂_t (α[t] + θ[t]))^2) ];
```

Total potential and kinetic energy:

```
Vtot = Vs + Vm1 + Vm2 + Vp;

Ttot = Ts + Tm1 + Tm2 + Tp;
```

Lagrangian of the system:

```
L = Simplify[Ttot - Vtot];
```

If Lagrange's equations are computed from this Lagrangian, large nonlinear expressions will result. At any case, as we can only treat linear differential equations, we need to linearize these equations. Instead of doing so, we retain from the Lagrangian only the terms up to the second order. If Lagrange's equations are then computed from this "quadratic" Lagrangian, one obtains directly the linearized differential equations. Let us do so and retain of the above Lagrangian only the terms up to the second order about the origin of the 8-dimensional state space.

```
Lred = ReduceLagrangian[L, {{x1[t], 0}, {x2[t], 0}, {θ[t], 0},
   {α[t], 0}, {x1'[t], 0}, {x2'[t], 0}, {θ'[t], 0}, {α'[t], 0}}]
```

$$\frac{1}{2}\left(-2\,g\,h\,m_1 - 2\,g\,h\,m_2 - 2\,g\,(h+l_p)\,m_p - 2\,c\,g\,m_s\right) + \frac{1}{2}\,g\,l_p\,m_p\,\alpha[t]^2 +$$
$$g\,l_p\,m_p\,\alpha[t]\,\theta[t] + \frac{1}{4}\left(2\,g\,h\,m_1 + 2\,g\,h\,m_2 - 2\,g\,(-h-l_p)\,m_p + 2\,c\,g\,m_s\right)\theta[t]^2 -$$
$$g\,m_1\,\theta[t]\,x_1[t] + \frac{1}{2}\left(-2\,g\,m_2 - 2\,g\,m_p\right)\theta[t]\,x_2[t] +$$
$$\frac{2}{3}\,l_p^2\,m_p\,\alpha'[t]^2 + \frac{1}{2}\left(\frac{2\,l_p^2}{3} + 2\,l_p\,(h+l_p)\right)m_p\,\alpha'[t]\,\theta'[t] +$$
$$\frac{1}{4}\left(2\,J + 2\,h^2\,m_1 + 2\,h^2\,m_2 + \left(\frac{2\,l_p^2}{3} + 2\,(h+l_p)^2\right)m_p\right)\theta'[t]^2 -$$
$$h\,m_1\,\theta'[t]\,x_1'[t] + \frac{1}{2}\,m_1\,x_1'[t]^2 - l_p\,m_p\,\alpha'[t]\,x_2'[t] +$$
$$\frac{1}{2}\left(-2\,h\,m_2 - 2\,(h+l_p)\,m_p\right)\theta'[t]\,x_2'[t] + \frac{1}{4}\left(2\,m_2 + 2\,m_p\right)x_2'[t]^2$$

9.2.2.2 Equations of Motion

The equations of motion are Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \quad (k = 1, \ \ldots n),$$

where L is the Lagrangian of the system, the q_k's are the generalized coordinates, the Q_k's the generalized forces, associated with each coordinate, and n the degree of freedom of the system. In our case we have $n = 4, q_1 = x_1, q_2 = x_2, q_3 = \theta, q_4 = \alpha, Q_1 = f_1, Q_2 = f_2, Q_3 = 0, Q_4 = 0$. Therefore the four linearized Lagrange's equations are:

```
eqn1 = Simplify[∂t (∂x1'[t] Lred) - ∂x1[t] Lred == f1[t]]

m1 (g θ[t] - h θ''[t] + x1''[t]) == f1[t]

eqn2 = Simplify[∂t (∂x2'[t] Lred) - ∂x2[t] Lred == f2[t]]

m2 (g θ[t] - h θ''[t] + x2''[t]) +
  mp (g θ[t] - h θ''[t] - lp (α''[t] + θ''[t]) + x2''[t]) == f2[t]
```

```
eqn3 = Simplify[∂t (∂θ'[t] Lred) - ∂θ[t] Lred == 0]
```

$$\frac{1}{3}\,(4\,l_p^2\,m_p\,(\alpha''[t]+\theta''[t]) -$$
$$3\,l_p\,m_p\,(g\,\alpha[t]+g\,\theta[t]-h\,\alpha''[t]-2\,h\,\theta''[t]+x_2''[t]) -$$
$$3\,(g\,h\,m_p\,\theta[t]+c\,g\,m_s\,\theta[t]-g\,m_p\,x_2[t]-J\,\theta''[t]-h^2\,m_p\,\theta''[t]+$$
$$m_1\,(g\,h\,\theta[t]-g\,x_1[t]+h\,(-h\,\theta''[t]+x_1''[t]))+h\,m_p\,x_2''[t]+$$
$$m_2\,(g\,h\,\theta[t]-g\,x_2[t]+h\,(-h\,\theta''[t]+x_2''[t])))) == 0$$

```
eqn4 = Simplify[∂t (∂α'[t] Lred) - ∂α[t] Lred == 0]
```

$$\frac{1}{3}\,l_p\,m_p\,(-3\,g\,\alpha[t]-3\,g\,\theta[t]+4\,l_p\,\alpha''[t]+3\,h\,\theta''[t]+4\,l_p\,\theta''[t]-3\,x_2''[t]) == 0$$

The linearized model of the seesaw-pendulum is described by

$$\mathrm{A2}\begin{pmatrix}\ddot{x}_1(t)\\ \ddot{x}_2(t)\\ \ddot{\theta}(t)\\ \ddot{\alpha}(t)\end{pmatrix}+\mathrm{A0}\begin{pmatrix}x_1(t)\\ x_2(t)\\ \theta(t)\\ \alpha(t)\end{pmatrix}=\begin{pmatrix}f_1(t)\\ f_2(t)\\ 0\\ 0\end{pmatrix},$$

where the symmetric 4×4-matrices are

```
A2 = {{m1, 0, -m1 h, 0},
   {0, m2 + mp, -h (m2 + mp) - lp mp, -mp lp},
   {-m1 h, -h (m2 + mp) - lp mp,
    J + h^2 (m1 + m2 + mp) + 2 h lp mp + 4 / 3 mp lp^2, h lp mp + 4 / 3 mp lp^2},
   {0, -mp lp, mp lp (h + 4 / 3 lp), 4 / 3 mp lp^2}};

A0 = {{0, 0, m1 g, 0},
   {0, 0, (m2 + mp) g, 0},
   {m1 g, (m2 + mp) g, h g (m1 + m2 + mp) + mp lp g + ms c g, -mp lp g},
   {0, 0, -mp lp g, -mp lp g}};
```

The matrix forms of these two matrices are found in the notebook92.nb. With the assumed physical values we get

```
A2n = A2 /. physicalvalues;

A0n = A0 /. physicalvalues;

MatrixForm[A2n]
```

$$\begin{pmatrix} 0.86 & 0 & -0.0989 & 0 \\ 0 & 0.68 & -0.1452 & -0.067 \\ -0.0989 & -0.1452 & 1.6657 & 0.0376317 \\ 0 & -0.067 & 0.0376317 & 0.0299267 \end{pmatrix}$$

```
MatrixForm[A0n]
```

$$\begin{pmatrix} 0 & 0 & 8.4366 & 0 \\ 0 & 0 & 6.6708 & 0 \\ 8.4366 & 6.6708 & 6.27938 & -0.65727 \\ 0 & 0 & -0.65727 & -0.65727 \end{pmatrix}$$

Again, we use the commands explained in Chapter 5, especially in Section 5.3. We have a second order differential system defined by

```
A1n = NullMatrix[4];
B0n = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
B1n = NullMatrix[4];
sys = MatrixDifferentialSystem[{A0n, A1n, A2n}, {B0n, B1n}];
```

We also introduce a perturbation that acts on the two cart positions and on the two angles, i.e. we consider the system

$$\mathrm{A2}\begin{pmatrix}\ddot{x}_1(t)\\ \ddot{x}_2(t)\\ \ddot{\theta}(t)\\ \ddot{\alpha}(t)\end{pmatrix} + \mathrm{A0}\begin{pmatrix}x_1(t)\\ x_2(t)\\ \theta(t)\\ \alpha(t)\end{pmatrix} = \begin{pmatrix}f_1(t)\\ f_2(t)\\ 0\\ 0\end{pmatrix} + W\begin{pmatrix}w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\end{pmatrix}$$

where W is the 4×4-identity matrix. Next the transfer function matrices of the open-loop system are computed. First the transfer matrix between the input vector and the output vector

```
W = IdentityMatrix[4];

G1 = TransferMatrices[sys, W, s] // First // Short
```

$$\left\{\left\{\frac{\text{<<1>>}}{\text{<<1>>}}, \frac{\text{<<1>>}}{\text{<<5>>} + \text{<<20>>}\ s^8}, 0, 0\right\}, \text{<<2>>}, \left\{\frac{\text{<<1>>}}{\text{<<1>>}}, \frac{\text{<<1>>}}{\text{<<1>>}}, 0, 0\right\}\right\}$$

Second the tranfer matrix between the perturbation vector and the output

```
G2 = TransferMatrices[sys, W, s] // Last // Chop;
```

9.2.2.3 Simulation

The different signs of the coefficients and the lack of all odd powers in the denominator polynomial clearly exhibits the highly unstable nature of the zero position in the state space. If we apply a small perturbation to the first cart, for example, i.e. $w_1(t) = \delta(t)/10$, the pendulum's motion will be the following:

```
(G2. {1 / 10, 0, 0, 0}) [[4]]
```

$$\frac{0.133814\ s^4 - 0.00156867\ s^6}{10\ (56.9652\ s^2 - 6.27605\ s^4 - 0.472978\ s^6 + 0.0218249\ s^8)}$$

```
Factor[%];

InverseLaplaceTransform[%, s, t] // Chop
```

$$-0.00118542\ e^{-5.34252\,t} + 0.00165179\ e^{-2.59338\,t} - 0.00165179\ e^{2.59338\,t} + 0.00118542\ e^{5.34252\,t} - 0.00306081\ \mathrm{Sin}[3.68737\,t]$$

```
Plot[%, {t, 0, 1.5}, AxesLabel → {"t", "α"}];
```

i.e. the pendulum falls down to the left, while the seesaw capsizes to the right. This is shown by

```
(G2 . {1 / 10, 0, 0, 0}) [[3]] // Factor;

InverseLaplaceTransform[%, s, t] // Chop
```

$$0.0000151159\ e^{-5.34252\,t} + 0.00526875\ e^{-2.59338\,t} - 0.00526875\ e^{2.59338\,t} - 0.0000151159\ e^{5.34252\,t} + 0.0094042\ \mathrm{Sin}[3.68737\,t]$$

```
Plot[%, {t, 0, 1.5}, AxesLabel → {"t", "θ"}];
```

■ 9.2.3 Controling the System

9.2.3.1 Determination of the Feedback

In order to stabilize the zero position we use a full state feedback, i.e. ,

$$\begin{pmatrix} f_1(t) \\ f_2(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \theta(t) \\ \alpha(t) \end{pmatrix} + \begin{pmatrix} b11 & b12 & b13 & b14 \\ b21 & b22 & b23 & b24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\theta}(t) \\ \dot{\alpha}(t) \end{pmatrix}$$

$$= \mathbf{K_0} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \theta(t) \\ \alpha(t) \end{pmatrix} + \mathbf{K_1} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\theta}(t) \\ \dot{\alpha}(t) \end{pmatrix}.$$

```
K0 =
  {{a11, a12, a13, a14}, {a21, a22, a23, a24}, {0, 0, 0, 0}, {0, 0, 0, 0}};
K1 = {{b11, b12, b13, b14}, {b21, b22, b23, b24}, {0, 0, 0, 0}, {0, 0, 0, 0}};
```

By substituting this into the right-hand side of the above linearized equation of motion we get

$$\mathrm{A2} \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{\theta}(t) \\ \ddot{\alpha}(t) \end{pmatrix} - \begin{pmatrix} b11 & b12 & b13 & b14 \\ b21 & b22 & b23 & b24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\theta}(t) \\ \dot{\alpha}(t) \end{pmatrix} + \mathrm{C0} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \theta(t) \\ \alpha(t) \end{pmatrix} = W \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{pmatrix}$$

where the matrix C0 is

$$\mathrm{C0} = \mathrm{A0} - \begin{pmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We again compute the characteristic function of the feedback system. The coefficients of the characteristic function will depend on the 16 feedback parameters. These feedback parameters have to be chosen such that the characteristic function (a polynomial of degree 8) becomes stable. The choice of the full state feedback is certainly not an economical solution. It needs eight measurement devices (sensors). The following procedure also shows that there is plenty of freedom in the choice of the 16 parameters. For space saving reasons the determination of the characteristic function and the 16 feedback parameters are done in the notebook92.nb. An example of 16 feedback parameters such that the corresponding characteristic function becomes a stable polynomial is the following (now, go to the notebook92, from which we obtain):

```
feedbp
```

```
{b11 → 14.9516, b12 → 4.14323, b13 → 16.6713, b14 → 27.5024,
 a11 → 58.4382, a12 → 43.4642, a13 → 111.097, a14 → 86.3534, a21 → -5,
 a22 → -5, a23 → -5, a24 → -5, b21 → -6, b22 → -6, b23 → -6, b24 → -6}
```

The characteristic function (polynomial) (also a result from notebook92) then becomes

```
chF /. feedbp
```

```
145.267 + 321.263 s + 350.945 s^2 + 224.36 s^3 +
 92.0575 s^4 + 24.6622 s^5 + 4.25586 s^6 + 0.436498 s^7 + 0.0218249 s^8
```

```
StabilityQ[%, s]
```

```
True
```

9.2.3.2 Simulation

The two feedback matrices with the computed coefficients are:

```
K0n = K0 /. feedbp; K1n = K1 /. feedbp;
```

The closed-loop transfer matrix becomes

```
GcL = ClosedLoopTransferMatrix[sys, W, {K0n, K1n}, s];
```

Let us simulate by giving a perturbation, modeled by $w_1 = \delta(t)/10$, to cart C1, for example,

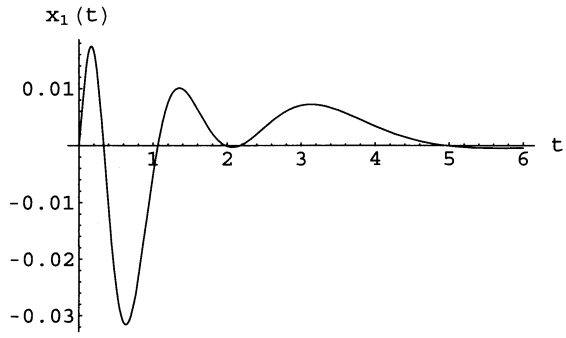
```
resp = GcL.{1 / 10, 0, 0, 0};
```

The motion of chart C_1 then is:

```
InverseLaplaceTransform[resp[[1]], s, t] // Chop
```

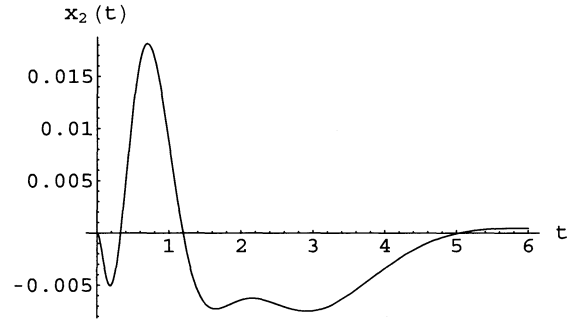
```
2 e^(-1. t) (-0.0974942 Cos[1. t] - 0.0234134 Sin[1. t]) +
 2 e^(-2. t) (-0.356837 Cos[2. t] + 0.575891 Sin[2. t]) +
 2 e^(-3. t) (0.72745 Cos[2. t] - 1.10678 Sin[2. t]) +
 2 e^(-4. t) (-0.273119 Cos[4. t] + 0.355613 Sin[4. t])
```

```
Plot[%, {t, 0, 6}, AxesLabel → {"t", "x1 (t)"}, PlotRange → All];
```



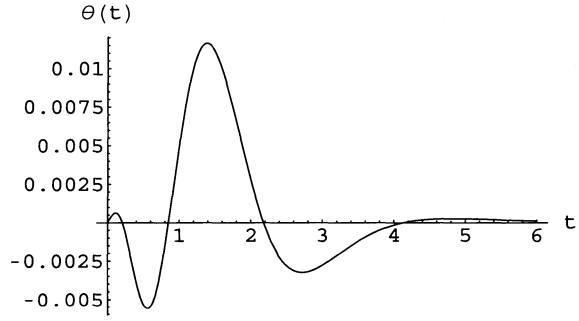
The motion of chart C_2 is

```
InverseLaplaceTransform[resp[[2]], s, t] // Chop;

Plot[%, {t, 0, 6}, AxesLabel → {"t", "x2 (t)"}, PlotRange → All];
```



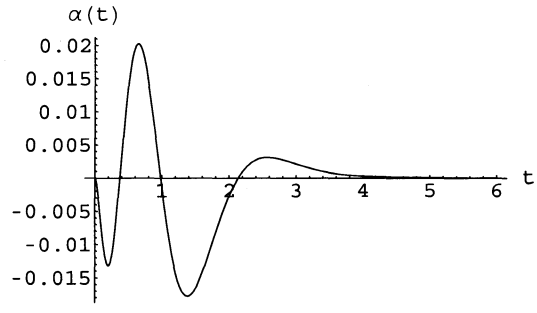
The motion of the seesaw is

```
InverseLaplaceTransform[resp[[3]], s, t] // Chop;

Plot[%, {t, 0, 6}, AxesLabel → {"t", "θ(t)"}, PlotRange → All];
```



The motion of the pendulum finally is

```
InverseLaplaceTransform[resp[[4]], s, t] // Chop;

Plot[%, {t, 0, 6}, AxesLabel → {"t", "α(t)"}, PlotRange → All];
```



9.3 Control of a DC Motor

References for the following example are [Cannon (1967)], [Goette (1999)], [Kuo B.C (1995)].

■ 9.3.1 The System

Let us consider a DC motor with permanent magnets driving a load. The load may have a varying magnitude. One aim is to stabilize the angular velocity of the rotor shaft. As a second example, let us consider the problem of an electro-mechanical position control. In this example the DC motor drives a load which has a dominant vibration mode. A typical example of this kind of problem is the computer-disk-head control of a disk drive.

■ 9.3.2 Modeling

9.3.2.1 Modeling of a DC Motor with Permanent Magnets

A DC motor with permanent magnets for the stator has the advantage that it naturally produces an essential linear model of the motor. Two physical laws govern an electrical motor: Lorentz's law and Faraday's law of induction. Lorentz's law says the following: An electrical charge q moving with constant velocity $\mathbf{v}$ in an electro-magnetic field ($\mathbf{E}$, $\mathbf{B}$) experiences a force $\mathbf{F}$ on itself given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

If there is no electrical field $\mathbf{E}$, and if the moving charge is a current I in a conductor, then (Fig. 9.3) $q\,\mathbf{v} = I\,\mathbf{dl}$ and the Lorentz law becomes

$$d\mathbf{F} = I(\,d\mathbf{l} \times \mathbf{B}).$$

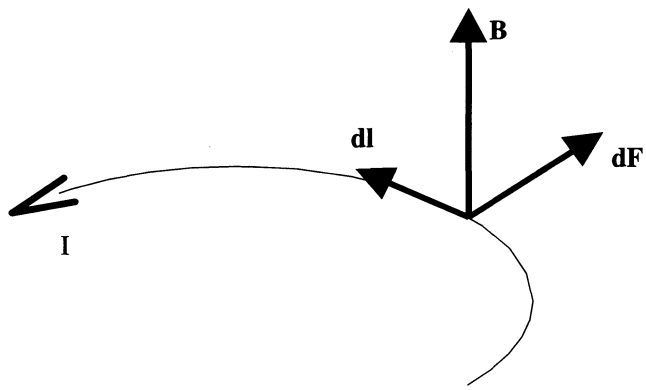


Fig. 9.3: Lorentz Force

Next let us consider a single conductor loop of the rotor armature (Fig. 9.4). The constant magnetic induction $\mathbf{B}$ is produced by a pair of permanent magnets not drawn in Fig.9.4. Also,

the commutator with its brush springs is not drawn. We wish to compute the torque generated by the Lorentz force. Looking from the front side the situation presents itself as shown in Fig. 9.5. Applying Lorentz's law and by taking into account the commutator device, the torque T produced by one conductor loop becommes

$$T(\theta) = \begin{cases} 2BIl\,r\sin\theta, & 0 \le \theta \le \pi \\ 2B(-I)l\,r\sin\theta, & \pi \le \theta \le 2\pi \end{cases} = 2BIl\,r\,|\sin\theta|\,.$$

The applied mean torque over one revolution of one conductor loop then is

$$T_a = \frac{1}{2\pi}\int_0^{2\pi} T(\theta)\,d\theta = \frac{2BIl\,r}{2\pi}\int_0^{2\pi} |\sin\theta|\,d\theta = \frac{4BIl\,r}{\pi}.$$

If there are N conductor loops evenly distributed over the circumference of the rotor, the applied total torque becomes

$$T_a = K_m I, \quad K_m = \frac{4NBIl\,r}{\pi}. \tag{9.2}$$

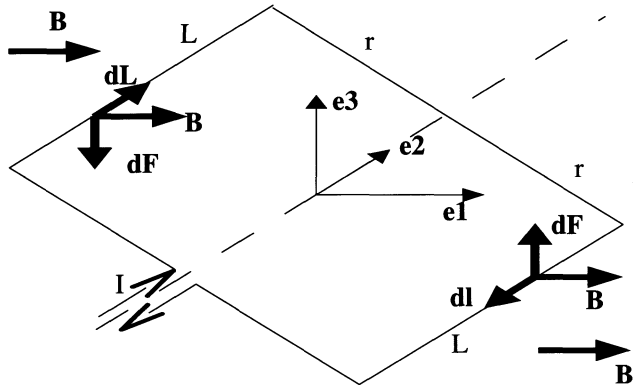


Fig. 9.4: A single conductor Loop.

The motor constant K_m is proportional to the number of windings N of the rotor, the applied magnetic induction B of the stator field and the rectangular section air $l\,r$ of the rotor. The electrically produced torque is proportional to the current I of the rotor armature. The torque will have the effect that the rotor starts to revolve and Faraday's law of induction comes into play. It says that if

$$\Phi = \int\int_S \mathbf{B}\cdot\mathbf{n}\,dS$$

is the magnetic flux through the open surface S bounded by the circuit C (Fig.9.6), the back e.m.f. $e(t)$ generated by a time variation of the flux is given by

$$e(t) = \frac{d\Phi}{dt} = \frac{d}{dt}\int\int_S \mathbf{B}\cdot\mathbf{n}\,dS\,.$$

If C is one conductor loop of the rotor armature as above the magnetic flux is given by (Fig. 9.5)

$$\Phi(\theta) = \begin{cases} 2B\,l\,r\cos\theta, & 0 \le \theta \le \pi \\ -2B\,l\,r\cos\theta, & \pi \le \theta \le 2\pi. \end{cases}$$

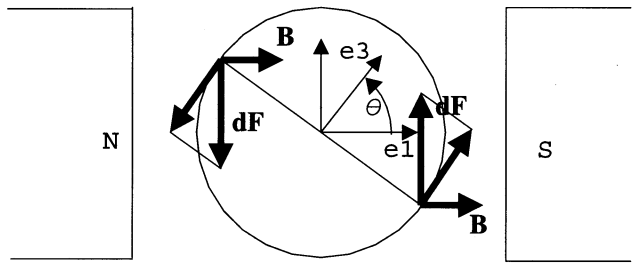


Fig. 9.5: Permanent magnets with rotor.

The generated back e.m.f. is the voltage produced in the conductor loop due to its rotation. It acts against the exterior applied voltage. We have

$$e(t,\theta) = \frac{d\Phi}{dt} = \begin{cases} -2B\,l\,r\dot{\theta}\sin\theta, & 0 \le \theta \le \pi \\ 2B\,l\,r\,\dot{\theta}\sin\theta, & \pi \le \theta \le 2\pi \end{cases} = -2B\,l\,r\frac{d\theta}{dt}\,|\sin\theta|$$

The mean back e.m.f. over one revolution is

$$e_b(t) = \frac{1}{2\pi}\int_0^{2\pi} e(t,\theta)\,d\theta = \frac{d\theta}{dt}\,\frac{-2BIl\,r}{2\pi}\int_0^{2\pi}|\sin\theta|\,d\theta = -\frac{4BIl\,r}{\pi}\frac{d\theta}{dt}.$$

Finally the total back e.m.f. of N evenly distributed conductor loops over the circumference of the rotor is given by

$$e_b(t) = -K_g\frac{d\theta}{dt}, \quad K_g = \frac{4NBIl\,r}{\pi}. \tag{9.3}$$

We see that the theoretic generator constant K_g is the same as the theoretic motor constant K_m. Clearly, in the physical world this is only an approximation.

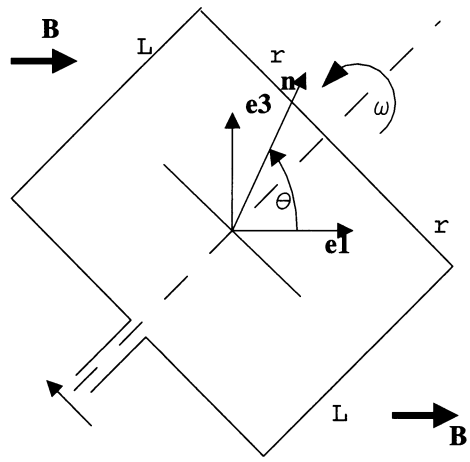


Fig. 9.6: One conductor loop for the back e.m.f.

In order to complete the dynamic description of the DC motor, we have to take into account the dynamic behavior of the electrical armature circuit and the rotor with the loads. The armature circuit consists of an externally applied voltage $v(t)$ that produces the armature current I flowing in the rotor windings (Fig. 9.7). The rotor windings are represented by a self-induction L and a resistance R due to the copper losses. The voltage source $e_b(t)$ on the right-hand side represents the produced back e.m.f. due to the rotation of the rotor windings.

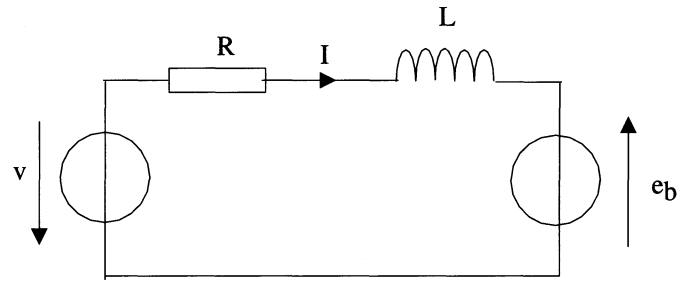


Fig. 9.7: Armature circuit

The rotor and the loads (Fig. 9.8) describe the mechanical part of the system. The fly-wheel with moment of inertia J takes care of the inertia of the rotor and of the driven load. The torque T_f takes care of the various frictional forces, such as friction in the bearings and in the air. T_f is assumed proportional to the angular velocity of the rotor shaft: $T_f = b\,\omega = b\,d\theta/dt$. Finally, T_L is the torque of the load, and T_a is the torque which is generated electrically by the motor. Due to Newton's law, the mechanical system is governed by the differential equation

$$J\frac{d^2\theta}{dt^2} = T_a - T_f - T_L = K_m I(t) - b\frac{d\theta}{dt} - T_L(t).$$

Due to Kirchhoff's law, the armature circuit of the rotor is governed by

$$R\,I(t) + L\frac{dI}{dt} = v(t) + e_b(t), \quad e_b(t) = -K_g\frac{d\theta}{dt}.$$

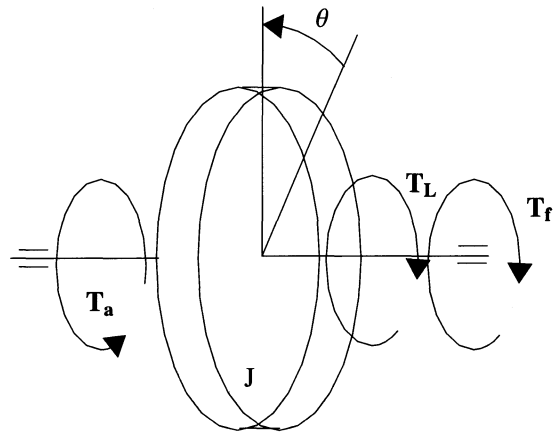


Figure 9.8: Rotor and load

Therefore, the DC motor with its loads is modeled by the third order system of differential equations

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} - K_m I(t) = -T_L(t), \tag{9.4}$$

$$L\frac{dI}{dt} + RI(t) + K_g\frac{d\theta}{dt} = v(t). \tag{9.5}$$

For problems regarding the angular velocity, we prefer the second order system

$$J\frac{d\omega}{dt} + b\,\omega(t) - K_m I(t) = -T_L(t), \tag{9.6}$$

$$L\frac{dI}{dt} + RI(t) + K_g\,\omega(t) = v(t). \tag{9.7}$$

Upon introducing the Laplace transforms

$$\theta(t) \circ\!-\!\bullet\, \theta(s) \quad I(t) \circ\!-\!\bullet\, I(s) \quad v(t) \circ\!-\!\bullet\, V(s),$$
$$T_L(t) \circ\!-\!\bullet\, T_L(s) \quad \omega(t) \circ\!-\!\bullet\, \Omega(s),$$

the equations (9.6) and (9.7) can be written as

$$I(s) = \frac{1}{Ls+R}\,[V(s) - K_g\,\Omega(s)],$$

$$\Omega(s) = \frac{1}{Js+b}\left\{-T_L(s) + \frac{K_m}{Ls+R}\,[V(s) - K_g\,\Omega(s)]\right\}.$$

From the last equation we get

$$\Omega(s) = \frac{1}{(Js+b)(Ls+R)+K_m K_g}\,\{K_m V(s) - (Ls+R)\,T_L(s)\},$$

which in turn can be written as

$$\Omega(s) = G(s)\,[V(s) + H(s)\,T_L(s)] \tag{9.8}$$

with the two transfer functions

$$G(s) = \frac{K_m}{(Js+b)(Ls+R)+K_m K_g}, \quad H(s) = -\frac{Ls+R}{K_m}. \tag{9.9}$$

Often the DC motor is driven by a power amplifier that acts as a source of energy. We assume that in the linear range the amplifier has a constant gain λ which is the slope of the linear range of the amplifier. Thus,

$$v(t) = \lambda\,(v_{\text{in}}(t) - k\,I(t)) \circ\!-\!\bullet\, V(s) = \lambda\,(V_{\text{in}}(s) - k\,I(s)). \tag{9.10}$$

The equations (9.6), (9.7) and (9.10) result in the following block diagram of the amplifier-DC motor system (Fig.9.9). Equation (9.9) however yields a block diagram as in a Feedforward controller configuration (see Section 5.1 of Chapter 5) without the controller and without the feedback path.

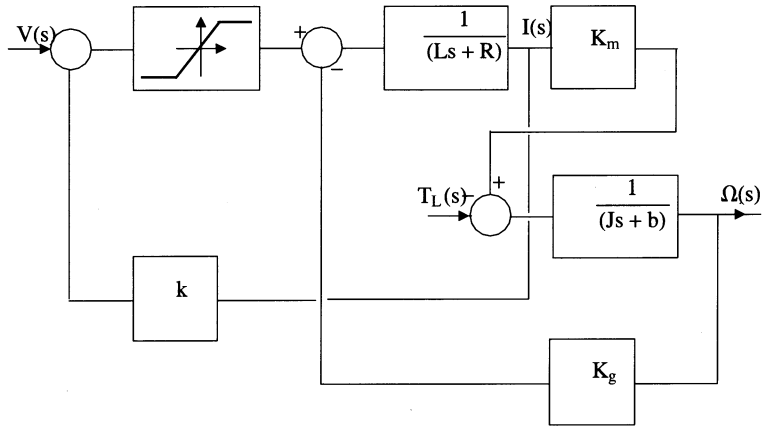


Fig.9.9: Amplifier /DC motor system.

The above system of differential equations can also be written in the standard form of a fixed analog system

$$\frac{d\mathbf{x}}{\mathrm{dt}} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{w}(t),$$

$$y(t) = \mathbf{C}\,\mathbf{x}(t),$$

with the state vector $\mathbf{x}$, the control vector $\mathbf{u}$, the perturbation vector $\mathbf{w}(t)$ and the constant matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$:

$$\mathbf{x}(t) = \begin{pmatrix} \omega(t) \\ I(t) \end{pmatrix}, \quad \mathbf{u}(t) = \begin{pmatrix} v(t) \\ 0 \end{pmatrix}, \quad \mathbf{w}(t) = \begin{pmatrix} 0 \\ T_L(t) \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} -b/J & K_m/J \\ -K_g/J & -R/J \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1/L & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & -1/J \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

If the DC motor is driven by a power amplifier in the linear range of the amplifier, we have

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\,\mathbf{x}(t),$$

$$\mathbf{K} = \begin{pmatrix} 0 & \lambda k \\ 0 & 0 \end{pmatrix}, \quad \mathbf{r}(t) = \begin{pmatrix} \lambda\, v_{\mathrm{in}}(t) \\ 0 \end{pmatrix},$$

i.e. the armature current is amplified by a negative feedback that is applied to the input voltage $v(t)$. The quantity $\lambda\, v_{\mathrm{in}}(t)$ is the specified reference voltage.

9.3.2.2 Simulations

We choose numerical quantities for the system parameters as below. These values are not intended to be real characteristic values of a DC motor configuration, but rather to demonstrate clearly the dynamic behavior. We first work within the frame of state-variable analysis.

```
physicalvalues = {Km → 20, Kg → 20, J → 1, L → 1, R → 50, b → 1, λ → 1};
```

```
Am = {{-b / J, Km / J}, {-Kg / L, -R / L}};
Bm = {{0, -1 / J}, {1 / L, 0}};
Em = {{0, -1 / J}, {0, 0}};
Cn = {{1, 0}, {0, 0}}; Km = {{0, λ k}, {0, 0}};
Dn = NullMatrix[2]; Hn = NullMatrix[2];

An = Am /. physicalvalues; Bn = Bm /. physicalvalues;
En = Em /. physicalvalues; Kn = Km /. physicalvalues;
```

Referring to Section 5.2 of Chapter 5, the system is defined by

```
sys = StateControlSystem[An, Bn, Cn, Dn];
```

We consider first the control system without load, i.e. without perturbation but with the feedback by a power amplifier. The closed-loop transfer matrix between the reference and the output, i.e. between the applied reference voltage and the angular velocity of the motor shaft is obtained by

```
ClosedLoopTransferMatrix[sys, Kn, s]
```

$$\left\{\left\{\frac{20}{450 + k + 51\,s + k\,s + s^2}, \frac{-50 - k - s}{450 + k + 51\,s + k\,s + s^2}\right\}, \{0, 0\}\right\}$$

The closed-loop characteristic function for this state feedback controller configuration is directly obtained by

```
ClosedLoopCharacteristicFunction[StateFeedbackCC[sys, Kn, s]]
```

$$450 + k + (51 + k)\,s + s^2$$

Even with a negative feedback gain, but $k > -51$, a stable system is obtained. The reason is that the motor itself, by its produced back e.m.f, already is a stable system. However the added negative feedback with the power amplifier will greatly enhance the stability. Let us chose $k = 49$, for example.

```
Kn = Kn /. k → 49;

G = ClosedLoopTransferMatrix[sys, Kn, s]
```

$$\left\{\left\{\frac{20}{499 + 100\,s + s^2}, \frac{-99 - s}{499 + 100\,s + s^2}\right\}, \{0, 0\}\right\}$$

With a unit-step voltage of magnitude 500, for example, for the applied reference voltage, the following angular velocity output will be obtained:

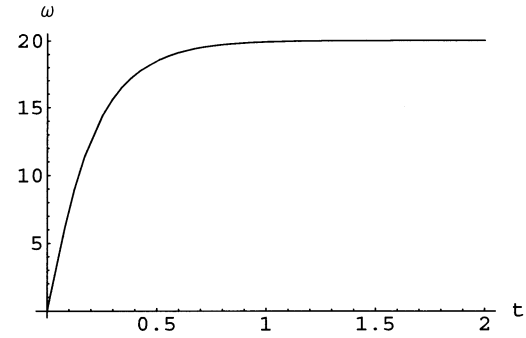
```
Ω = G.{{500 / s}, {0}}
```

$$\left\{\left\{\frac{10000}{s\,(499 + 100\,s + s^2)}\right\}, \{0\}\right\}$$

```
ω = InverseLaplaceTransform[Ω[[1, 1]], s, t]
```

$$\frac{10000}{499} - \frac{5000\,e^{-\left(50-\sqrt{2001}\right)t}}{-2001 + 50\sqrt{2001}} + \frac{5000\,e^{-\left(50+\sqrt{2001}\right)t}}{2001 + 50\sqrt{2001}}$$

```
Plot[ω, {t, 0, 2}, PlotRange → All, AxesLabel → {"t", "ω"}];
```



Note that the steady-state angular velocity is $500\,G(0) = 500 \times 20/499$.

Let us consider the loaded motor. The load is introduced via the perturbation vector. The closed-loop transfer matrices with respect to the reference voltage and the load are obtained by the following list:

```
G1 = ClosedLoopTransferMatrices[sys, En, Hn, Kn, s]
```

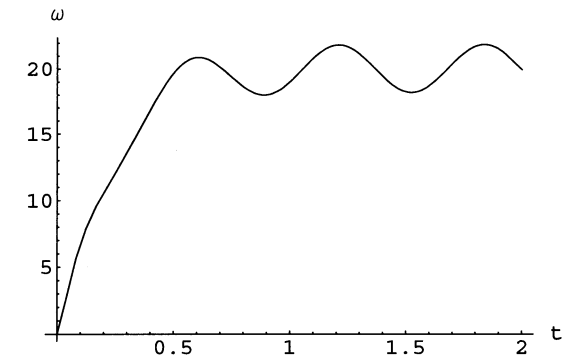
$$\left\{\left\{\left\{\frac{20}{499 + 100\,s + s^2},\ \frac{-99 - s}{499 + 100\,s + s^2}\right\},\ \{0,\ 0\}\right\},\right.$$
$$\left.\left\{\{0,\ \frac{-99 - s}{499 + 100\,s + s^2}\},\ \{0,\ 0\}\right\}\right\}$$

The resulting angular velocity with an applied voltage of $v(t) = 500\,u(t)$ and a periodically changing load of $T_L(t) = 20 \sin 10\,t$, for example, then is

```
Ω = G1[[1]].{{500 / s}, {0}} + G1[[2]].{{0}, {200 / (s^2 + 100)}} // Together
```

$$\left\{\left\{\frac{200\,(5000 - 99\,s + 49\,s^2)}{s\,(100 + s^2)\,(499 + 100\,s + s^2)}\right\},\ \{0\}\right\}$$

```
ω = InverseLaplaceTransform[Ω[[1, 1]], s, t];

Plot[ω, {t, 0, 2}, PlotRange → All, AxesLabel → {"t", "ω"}];
```



■ 9.3.3 Control of the Angular Velocity

We first compute the transfer matrix of the open-loop system (we now cease to use an amplifier in order to feed back the armature current). The transfer matrix between the input vector and the output vector is obtained by (see Section 5.2.2.2)

```
sys = StateControlSystem[Am, Bm, Cn, Dn];

Gmo = TransferMatrices[sys, Em, Hn, s][[1]]
```

$$\left\{\left\{\frac{K_m}{b\,R + b\,L\,s + J\,R\,s + J\,L\,s^2 + K_g\,K_m},\ -\frac{R + L\,s}{b\,R + b\,L\,s + J\,R\,s + J\,L\,s^2 + K_g\,K_m}\right\},\ \{0,\ 0\}\right\}$$

The transfer function between the input voltage $v(t)$ and the output angular velocitiy $\omega(t)$ is given by entry (1,1) (compare equations (9.9):

```
Gi = Gmo[[1, 1]]
```

$$\frac{K_m}{b\,R + b\,L\,s + J\,R\,s + J\,L\,s^2 + K_g\,K_m}$$

The transfer matrix between the perturbation vector and the output vector is obtained by (see Section 5..2.2.2)

```
Gmp = TransferMatrices[sys, Em, Hn, s][[2]]
```

$$\left\{\left\{0,\ -\frac{R + L\,s}{b\,R + b\,L\,s + J\,R\,s + J\,L\,s^2 + K_g\,K_m}\right\},\ \{0,\ 0\}\right\}$$

Hence, the transfer function between the load $T_L(t)$ and the output angular velocity $\omega(t)$ is then given by the entry (1,2):

```
Gw = Gmp[[1, 2]]
```

$$-\frac{R + L\,s}{b\,R + b\,L\,s + J\,R\,s + J\,L\,s^2 + K_g\,K_m}$$

9.3.3.1 P-I-Regulator

Now the aim is to obtain a specified angular velocity ω_0 of the rotor shaft that is as independent as possible of the applied load. In order to obtain this, we introduce the feedback

$$v(t) = [\omega_0 - \omega(t)] + \mu \int_0^t [\omega_0 - \omega(\tau)]\, d\tau$$

or, in the image domain,

$$V(s) = \left[\frac{\omega_0}{s} - \Omega(s)\right] + \frac{\mu}{s}\left[\frac{\omega_0}{s} - \Omega(s)\right].$$

This amounts to choosing a cascade controller configuration with perturbation (PerturbationCascadeCC), see Section 5.1. We have (see equations (9.9))

$$C(s) = 1 + \frac{\mu}{s},$$

$$H(s) = -\frac{1}{K_m}(L s + R),$$

$$G(s) = \frac{K_m}{(b R + K_g K_m) + (b L + J R)\, s + J L s^2}.$$

The closed-loop transfer functions between reference and angular velocity, and load and angular velocity then become

```
GclT = ClosedLoopTransferFunction[
  PerturbationCascadeCC[-1, {1 + μ / s}, {-(L s + R) / Km }, {Gi}, s]]
```

$$\left\{\frac{(s+\mu)\,K_m}{s\,(b+J\,s)\,(R+L\,s)+(s+\mu+s\,K_g)\,K_m},\ -\frac{s\,(R+L\,s)}{s\,(b+J\,s)\,(R+L\,s)+(s+\mu+s\,K_g)\,K_m}\right\}$$

The two characteristic functions are obtained by

```
Gcf = ClosedLoopCharacteristicFunction[
  PerturbationCascadeCC[-1, {1 + μ / s}, {-(L s + R) / Km }, {Gi}, s]]
```

$$\{s\,(b+J\,s)\,(R+L\,s)+(s+\mu+s\,K_g)\,K_m,\ s\,(b+J\,s)\,(R+L\,s)+(s+\mu+s\,K_g)\,K_m\}$$

which are identical.

```
cf = Collect[%[[1]] // Expand, {1, s}]
```

$$(b\,L+J\,R)\,s^2+J\,L\,s^3+\mu\,K_m+s\,(b\,R+K_m+K_g\,K_m)$$

By introducing the physical values

```
cfn = cf /. physicalvalues
```

$$470\,s+51\,s^2+s^3+20\,\mu$$

the Hurwitz test will yield the following stability conditions:

```
HurwitzTest[cfn, s]
```

$$\{\text{True},\ 20\,\mu>0,\ 479400\,\mu-400\,\mu^2>0\}$$

We choose μ= 600 and perform the test

```
HurwitzTest[cfn /. μ → 600, s]
```

$$\{\text{True}\}$$

The two transfer functions now become

```
GclTn = GclT /. physicalvalues /. μ → 600
```

$$\left\{\frac{20\,(600+s)}{s\,(1+s)\,(50+s)+20\,(600+21\,s)},\ -\frac{s\,(50+s)}{s\,(1+s)\,(50+s)+20\,(600+21\,s)}\right\}$$

The response of the closed-loop system to a reference angular velocity $\omega_0 = 25\,u(t)$ and a load $w(t) = T_L(t) = 20\,u(t)$ then becomes

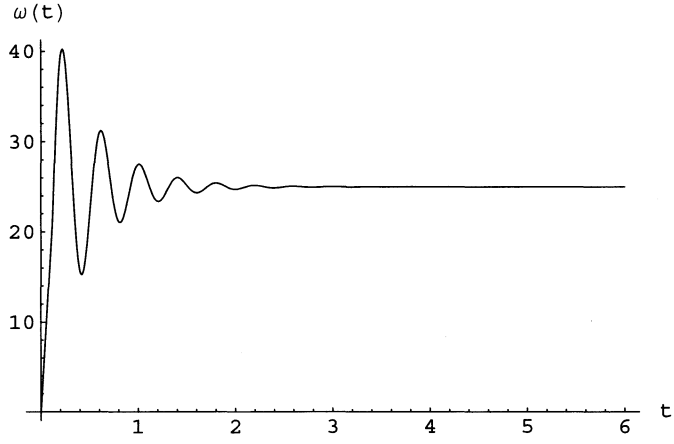
```
Together[GclTn[[1]] 25 / s + GclTn[[2]] 20 / s]
```

$$-\frac{20\,(-15000+25\,s+s^2)}{s\,(12000+470\,s+51\,s^2+s^3)}$$

```
InverseLaplaceTransform[%, s, t] // Chop
```

```
25. - 2.73651 e^-46.4435 t +
 2 e^-2.27827 t (-11.1317 Cos[15.9119 t] - 6.21596 Sin[15.9119 t])
```

```
Plot[%, {t, 0, 6}, AxesLabel → {"t", "ω(t)"}, PlotRange → All];
```



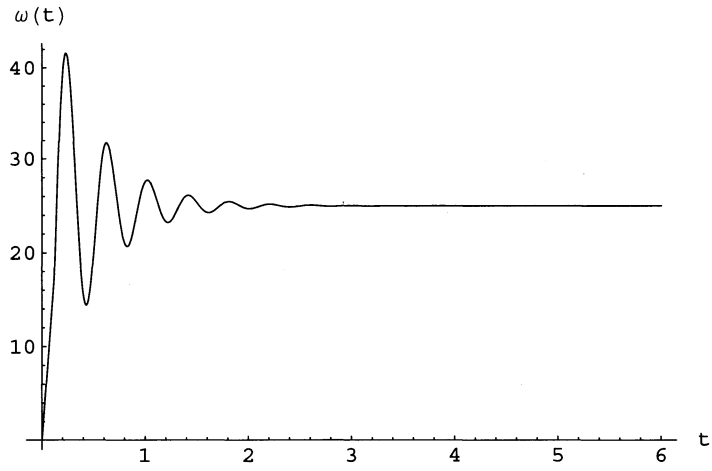
Now we augment the load to $w(t) = T_L(t) = 100\,u(t)$:

```
Together[GclTn[[1]] 25 / s + GclTn[[2]] 100 / s];
```

```
InverseLaplaceTransform[%, s, t] // Chop
```

```
25. - 2.86562 e^-46.4435 t +
 2 e^-2.27827 t (-11.0672 Cos[15.9119 t] - 8.90898 Sin[15.9119 t])
```

```
Plot[%, {t, 0, 6}, AxesLabel → {"t", "ω(t)"}, PlotRange → All];
```



This shows that the reference angular velocity is maintained at the same level for different loads.

■ 9.3.4 Electro-Mechanical Position Control

9.3.4.1 Modeling

A typical problem of an electro-mechanical position control is a DC motor driving a load that has a dominant vibration mode. We already mentioned the computer-disk-head control of a disk drive as a typical example of this kind. Most of these problems can be modeled by a DC motor coupled by an elastic shaft to another flywheel.

Assume that the moment of inertia of the rotor of the motor is J_1 and the one of the fly-wheel J_2. Assume further that the viscous friction constant of the motor is b_1 and that of the fly-wheel b_2. The rotational spring constant of the shaft that couples the motor with the fly-wheel is denoted by k. The angular position of the rotor is denoted by θ_1 and that of the fly-wheel by θ_2 (see Figure 9.10). Then, the applied torque to the motor by the fly-wheel is given by $T_L = k(\theta_1 - \theta_2)$. This must be substituted in (9.4) so that we arrive at the following system of differential equations of order 5.

$$\frac{d^2\theta_1}{dt^2} + \frac{b_1}{J_1}\frac{d\theta_1}{dt} - \frac{K_m}{J_1} I(t) = -\frac{k}{J_1}\,[\theta_1(t) - \theta_2(t)]\,, \tag{9.11}$$

$$\frac{d^2\theta_2}{dt^2} + \frac{b_2}{J_2}\frac{d\theta_2}{dt} = \frac{k}{J_2}\,[\theta_1(t) - \theta_2(t)]\,, \tag{9.12}$$

$$\frac{dI}{dt} + \frac{R}{L} I(t) + \frac{K_g}{L}\frac{d\theta_1}{dt} = \frac{1}{L} v(t)\,. \tag{9.13}$$

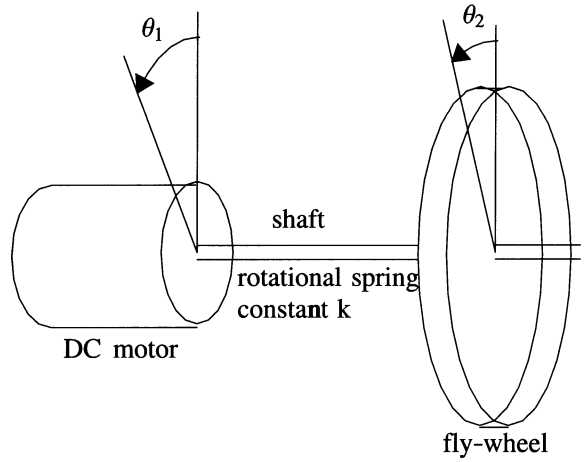


Fig. 9.10: Position control: dc motor with fly-wheel

Consider θ_2 as the output variable to be controled. The input variable is the applied voltage $v(t)$ (we do not feed back the armature current as we did in Section 9.3.2.3). In matrix form the above system of differential equations reads

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\frac{d^2}{dt^2}\begin{pmatrix}\theta_1\\\theta_2\\I\end{pmatrix}+\begin{pmatrix}b_1/J_1&0&0\\0&b_2/J_2&0\\K_g/L&0&1\end{pmatrix}\frac{d}{dt}\begin{pmatrix}\theta_1\\\theta_2\\I\end{pmatrix}$$

$$+\begin{pmatrix}k/J_1&-k/J_1&-K_m/J_1\\-k/J_2&k/J_2&0\\0&0&R/L\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\\I\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1/L\end{pmatrix}\begin{pmatrix}0\\0\\v(t)\end{pmatrix},$$

```
A2 = {{1, 0, 0}, {0, 1, 0}, {0, 0, 0}};
A1 = {{b1 / J1, 0, 0}, {0, b2 / J2, 0}, {Kg / L, 0, 1}};
A0 = {{k / J1, -k / J1, -Km / J1}, {-k / J2, k / J2, 0}, {0, 0, R / L}};
B0 = {{0, 0, 0}, {0, 0, 0}, {0, 0, 1 / L}};
B1 = NullMatrix[3];
```

We take the following physical values for the parameters (these values are chosen in order to amplify the vibration effects and are not to be considered as real system parameter values).

```
physicalvalues2 = {Km → 20, Kg → 20, J1 → 1,
   λ → 1, L → 1, R → 50, b1 → 1, J2 → 2, b2 -> 0.1, k → 20};
```

We do not introduce a special perturbation term, i.e. the perturbation matrix is the null matrix.

```
Hn = NullMatrix[3];
```

9.3.4.2 Simulations

We compute the transfer function matrix, referring to Section 5.3 again. The second order differential system is defined by

```
sys = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}]
```

$$\text{MatrixDifferentialSystem}\left[\left\{\left\{\frac{k}{J1},-\frac{k}{J1},-\frac{K_m}{J1}\right\},\left\{-\frac{k}{J2},\frac{k}{J2},0\right\},\left\{0,0,\frac{R}{L}\right\}\right\},\right.$$
$$\left\{\left\{\frac{b1}{J1},0,0\right\},\left\{0,\frac{b2}{J2},0\right\},\left\{\frac{K_g}{L},0,1\right\}\right\},\{\{1,0,0\},\{0,1,0\},\{0,0,0\}\}\Big\},$$
$$\left.\left\{\{\{0,0,0\},\{0,0,0\},\left\{0,0,\frac{1}{L}\right\}\right\},\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}\right\}\right]$$

The transfer matrix between the input and the output then is

```
G = TransferMatrices[sys, Hn, s][[1]];
```

We need the entry (2,3) of this matrix.

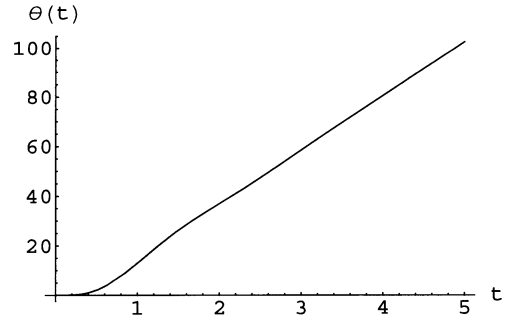
```
G23n = Factor[G[[2, 3]] /. physicalvalues2]
```

$$\frac{200.}{s\,(8.62503+s)\,(39.8292+s)\,(13.2449+2.59579\,s+s^2)}$$

For a unit-step input with magnitude 500 for the voltage we get

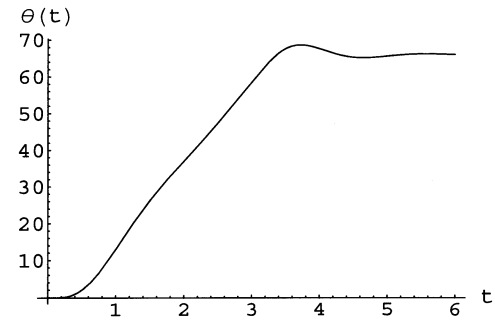
```
InverseLaplaceTransform[500 G23n / s , s, t] // Chop;
```

```
Plot[%, {t, 0, 5}, AxesLabel → {"t", "θ(t)"}, PlotRange → All];
```



To show the overshoot and the vibration of the system, we now apply an input voltage in the form of a rectangular pulse, for example $v(t) = 500[u(t) - u(t-3)]$.

```
v = 500 (UnitStep[t] - UnitStep[t - 3]);
V = LaplaceTransform[v, t, s];

InverseLaplaceTransform[Expand[V G23n ] , s, t] // Chop;

Plot[%, {t, 0, 6}, AxesLabel → {"t", "θ(t)"}, PlotRange → All];
```



9.3.4.3 Controling a specified angular position

We try a P-D-regulator, i.e. a feedback of the form

$$v(t) = \lambda \left\{ [\alpha_0\, u(t) - \theta_2(t)] + \lambda_1 \frac{d}{dt} [\alpha_0\, u(t) - \theta_2(t)] \right\}.$$

The derivative of the unit-step function has to be taken in the distributional sense, i.e.,

$$v(t) = \lambda \{ [\alpha_0\, u(t) - \theta_2(t)] + \lambda_1 [\alpha_0\, \delta(t) - \theta_2'(t)] \}.$$

This is obtained with a cascade controller configuration with the controller

$$H(s) = \lambda + \lambda_1\, s\,.$$

```
H = λ + λ1 s;
```

The closed-loop transfer function then becomes

```
Gc = ClosedLoopTransferFunction[CascadeCC[-1, {H}, {G23n}, s]]
```

$$(200.\,\lambda + 200.\,s\,\lambda 1)\,/\,(4550.\,s + 1533.5\,s^2 + 482.55\,s^3 + 51.05\,s^4 + s^5 + 200.\,\lambda + 200.\,s\,\lambda 1)$$

The characteristic function of this feedback system is

```
CF = ClosedLoopCharacteristicFunction[CascadeCC[-1, {H}, {G23n}, s]]
```

$$1533.5\,s^2 + 482.55\,s^3 + 51.05\,s^4 + s^5 + 200.\,\lambda + s\,(4550. + 200.\,\lambda 1)$$

We choose $\lambda 1 = 1$.

```
CF = CF /. λ1 → 1
```

$$4750.\,s + 1533.5\,s^2 + 482.55\,s^3 + 51.05\,s^4 + s^5 + 200.\,\lambda$$

```
HurwitzTest[CF, s]
```

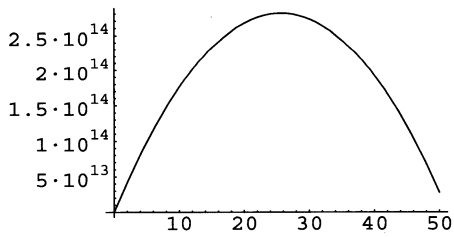
$$\{\text{True},\ 200.\,\lambda > 0,\ 2.30459\times 10^7 + 10210.\,\lambda > 0,\ 2.18936\times 10^{13}\,\lambda - 4.2649\times 10^{11}\,\lambda^2 - 8.\times 10^6\,\lambda^3 > 0\}$$

Inspection shows that the last inequality is the crucial one. Let take a closer look:

```
First[%[[4]]]
```

$$2.18936\times 10^{13}\,\lambda - 4.2649\times 10^{11}\,\lambda^2 - 8.\times 10^6\,\lambda^3$$

```
Plot[%, {λ, 0, 50}];
```



So, let us take $\lambda = 25$ and perform a test

```
HurwitzTest[CF /. λ → 25, s]
```

```
{True}
```

```
Gst = Gc /. {λ → 25, λ1 → 1}
```

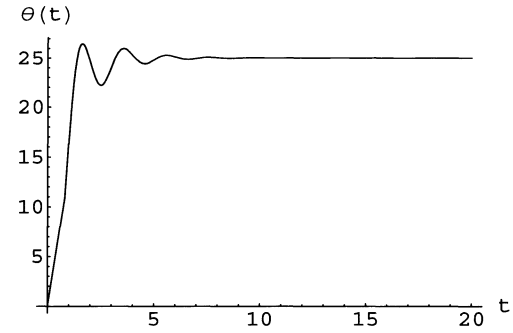
$$\frac{5000. + 200.\,s}{5000. + 4750.\,s + 1533.5\,s^2 + 482.55\,s^3 + 51.05\,s^4 + s^5}$$

We specify an angle of 25, for example,

```
InverseLaplaceTransform[Together[Gst * 25 / s], s, t] // Chop
```

$$25. + 0.00100127\,e^{-39.8276\,t} + 0.645266\,e^{-8.42403\,t} - 28.9415\,e^{-1.4809\,t} + 2\,e^{-0.658745\,t}\,(1.64763\,\text{Cos}[3.10312\,t] - 5.67386\,\text{Sin}[3.10312\,t])$$

```
Plot[%, {t, 0, 20}, AxesLabel → {"t", "θ(t)"}, PlotRange → All];
```



We see several oscillations about the specified angle. This is because we specified a very elastic shaft and a large moment of inertia of the second fly-wheel in order to put in evidence the vibrational mode of the system.

9.4 A Magnetic-Ball-Suspension-System

See also [Kuo B.C. (1995)]. We now consider a system consisting of an electromagnet and a steel ball, as shown in Fig.9.11. The objective is to control the position of the steel ball (output) by adjusting the current through the input voltage $e(t)$. The mathematical model is

$$m\,\frac{d^2y}{dt^2} = mg - k\,\frac{i(t)^2}{y(t)},$$

$$e(t) = R\,i(t) + L\,\frac{di}{dt},$$

where $e(t)$ is the input voltage, $y(t)$ the ball position, $i(t)$ the winding current, R the winding resistance, L the winding inductance, m the mass of the steel ball, and g the gravitational acceleration. We propose to stabilize the position of the ball by use of a sampled-data system. Let (e_0, i_0, y_0) be the equilibrium of the system satisfying

$$mg = k\,\frac{i_0^2}{y_0}, \quad e_0 = R\,i_0.$$

We linearize the system about the equilibrium by setting

$$e(t) = e_0 + \Delta e(t), \quad i(t) = i_0 + \Delta i(t), \quad y(t) = y_0 + \Delta y(t):$$

$$k\,\frac{i(t)^2}{y(t)} = k\,\frac{[i_0 + \Delta i(t)]^2}{y_0 + \Delta y(t)}$$

$$= k\,\frac{{i_0}^2}{y_0}\,\frac{1 + 2\,\Delta i(t)/i_0 + \cdots}{1 + \Delta y(t)/y_0}$$

$$= mg\,[1 + 2\,\Delta i(t)/i_0]\,[1 - \Delta y(t)/y_0]$$

$$= mg\,[1 - \Delta y(t)/y_0 + 2\,\Delta i(t)/i_0 + \cdots],$$

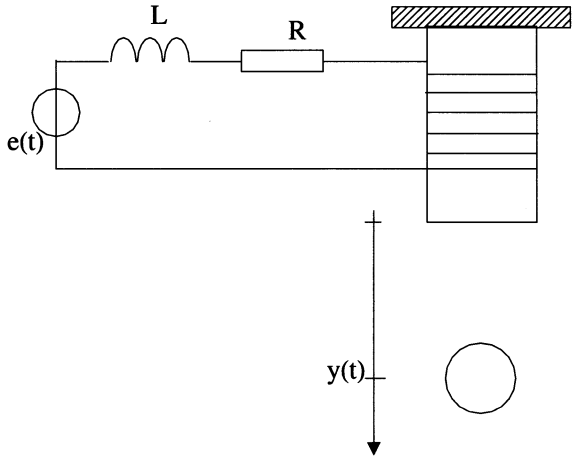


Fig9.11: An electromagnetic-ball-suspension-system

$$m\,\frac{d^2 y}{dt^2} = m\,y_0\,\frac{d^2(\Delta y / y_0)}{dt^2} = m\,g\,\{1 - 1 + \Delta y(t)/y_0 - 2\,\Delta i(t)/i_0\},$$

$$e(t) = e_0 + \Delta e(t) = R\,i_0\,[1 + \Delta i(t)/i_0] + i_0\,L\,\frac{d(\Delta i/i_0)}{dt}$$

$$= e_0\,[1 + \Delta i(t)/i_0] + e_0\,\frac{L}{R}\,\frac{d(\Delta i/i_0)}{dt}.$$

Finally we get a system of linearized differential equations

$$\frac{d^2(\Delta y/y_0)}{dt^2} = \frac{g}{y_0}\,\{\Delta y(t)/y_0 - 2\,\Delta i(t)/i_0\},$$

$$\frac{\Delta e(t)}{e_0} = \Delta i(t)/i_0 + \frac{L}{R}\,\frac{d(\Delta i/i_0)}{dt}.$$

By introducing the Laplace transforms

$$\frac{\Delta e(t)}{e_0} \circ\!-\!\bullet\, E(s), \quad \frac{\Delta i(t)}{i_0} \circ\!-\!\bullet\, I(s), \quad \frac{\Delta y(t)}{y_0} \circ\!-\!\bullet\, Y(s),$$

the image system becomes

$$s^2\,Y(s) = \frac{g}{y_0}\,\{Y(s) - 2\,I(s)\}, \quad E(s) = I(s) + \frac{L}{R}\,s\,I(s).$$

Hence

$$Y(s) = -\frac{2\,g\,R}{y_0\,L}\,\frac{1}{(s + R/L)\left(s + \sqrt{g/y_0}\right)\left(s - \sqrt{g/y_0}\right)}\,E(s)$$

$$=: G(s)\,E(s).$$

We propose the sampled-data system shown in Fig. 9.12. The output $y(t)$ is sampled with a sampling period T. A digital controller is introduced. It is described by its z-transfer function $D(z)$, followed by a D/A converter and a zero-hold device.

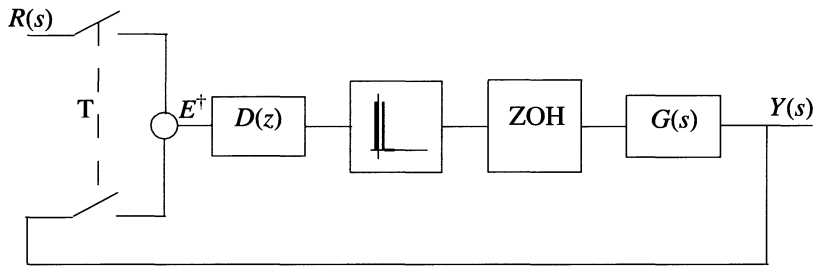


Fig.9.12: Sampled-data feedback system for the magnetic-ball-suspension-system.

We refer to the theory of Section 5.6. The input of the digital controller $D(z)$ is the sequence $e(kT)/e_0 - y(kT)/y_0$. With the z-transforms $y(kT)/y_0 \circ\!-\!\diamond\, Y^{\dagger}(z)$, and the equivalent z-transfer function

$$M(z) := \frac{z-1}{z} Z\left[\frac{G(s)}{s}\right]$$

$$= \frac{z-1}{z} Z\left[-\frac{2gR}{y_0 L} \frac{1}{s(s+R/L)\left(s+\sqrt{g/y_0}\right)\left(s-\sqrt{g/y_0}\right)}\right](z),$$

the closed-loop z-transfer function is

$$\frac{D(z)M(z)}{1+D(z)M(z)}.$$

By setting

$$D(z) = \frac{D_n(z)}{D_d(z)}, \quad M(z) = \frac{M_n(z)}{M_d(z)}$$

with polynomials in z on the right-hand side, the closed-loop characteristic function becomes

$$P(z) = D_n(z)M_n(z) + D_d(z)M_d(z).$$

For the discrete controller, we set

$$D(z) = \frac{d_m z^m + \cdots + d_0}{z^m + e_{m-1} z^{m-1} + \cdots + e_0}$$

with $2m+1$ control coefficients. If the degree of the denominator of $M(z)$ is μ, the degree of the polynomial $P(z)$ will be $\mu+m$, resulting in a linear system of order $\mu+m$. Thus $\mu+m = 2m+1$. Hence $m = \mu - 1 = 3 - 1 = 2$.

Let us now compute $M(z)$. To avoid large expressions, we use the following physical values:

```
physicalvalues = {g → 9.81, R → 10, L → 1/10, y0 → 0.05};

K = -2 g R / y0 / L /. physicalvalues;

a = R / L /. physicalvalues;

b = √(g / y0) /. physicalvalues;
```

```
G = K / ((s + a) (s - b) (s + b))
```

$$-\frac{39240.}{(-14.0071 + s)\,(14.0071 + s)\,(100 + s)}$$

We choose a sampling period of $T = 0.04$

```
T = 0.05;

M = (z - 1) / z LaplaceToZImage[G / s, s, T, z] // Together // Simplify
```

$$(5.38605 \times 10^{-17}\,(-6.38625 \times 10^{15} + z)\,(0.0563995 + z)\,(1.79298 + z)) /\\ ((-2.01447 + z)\,(-0.496408 + z)\,(-0.00673795 + z))$$

The controller must be

```
Df = (d2 z^2 + d1 z + d0) / (z^2 + e1 z + e0)
```

$$\frac{d0 + d1\,z + d2\,z^2}{e0 + e1\,z + z^2}$$

The characteristic function then becomes

```
P = Collect[Numerator[M] Numerator[Df] + Denominator[M] Denominator[Df] //
    Expand, {1, z}] // Chop
```

$$-0.034783\,d0 - 0.00673795\,e0 +\\ (-0.636125\,d0 - 0.034783\,d1 + 1.01692\,e0 - 0.00673795\,e1)\,z +\\ (-0.00673795 - 0.343967\,d0 - 0.636125\,d1 -\\ \quad 0.034783\,d2 - 2.51762\,e0 + 1.01692\,e1)\,z^2 +\\ (1.01692 - 0.343967\,d1 - 0.636125\,d2 + e0 - 2.51762\,e1)\,z^3 +\\ (-2.51762 - 0.343967\,d2 + e1)\,z^4 + z^5$$

All roots of this polynomial of degree 6 must lie inside the unit-circle. The simplest way to achieve this is to set all coefficients of the polynomial except the highest one to zero. We end up with the equation $5.96952\,z^9 = 0$, i.e. all roots are zero.

```
CLL = CoefficientList[P, z]
```

$$\{-0.034783\,d0 - 0.00673795\,e0,\\ -0.636125\,d0 - 0.034783\,d1 + 1.01692\,e0 - 0.00673795\,e1,\\ -0.00673795 - 0.343967\,d0 - 0.636125\,d1 - 0.034783\,d2 - 2.51762\,e0 +\\ \quad 1.01692\,e1,\ 1.01692 - 0.343967\,d1 - 0.636125\,d2 + e0 - 2.51762\,e1,\\ -2.51762 - 0.343967\,d2 + e1,\ 1\}$$

```
eqns = Table[CLL[[j]] == 0, {j, Length[CLL] - 1}]
```

$$\{-0.034783\,d0 - 0.00673795\,e0 == 0,\\ -0.636125\,d0 - 0.034783\,d1 + 1.01692\,e0 - 0.00673795\,e1 == 0,\\ -0.00673795 - 0.343967\,d0 - 0.636125\,d1 - 0.034783\,d2 - 2.51762\,e0 +\\ \quad 1.01692\,e1 == 0,\ 1.01692 - 0.343967\,d1 - 0.636125\,d2 + e0 - 2.51762\,e1 == 0,\\ -2.51762 - 0.343967\,d2 + e1 == 0\}$$

```
Solve[eqns, {d0, d1, d2, e0, e1}] // Chop
```

$$\{\{d0 \to -0.0121593,\ d1 \to 1.83102,\ d2 \to -3.92019,\ e0 \to 0.0627695,\ e1 \to 1.1692\}\}$$

The controller is

```
Df = Df /. First[%]
```

$$\frac{-0.0121593 + 1.83102\,z - 3.92019\,z^2}{0.0627695 + 1.1692\,z + z^2}$$

and the closed-loop transfer function

```
Df M / (1 + Df M) // Together // Expand // Chop
```

$$\frac{0.000422938}{z^5} - \frac{0.0559534}{z^4} - \frac{1.02422}{z^3} + \frac{1.86392}{z^2} + \frac{1.34841}{z}$$

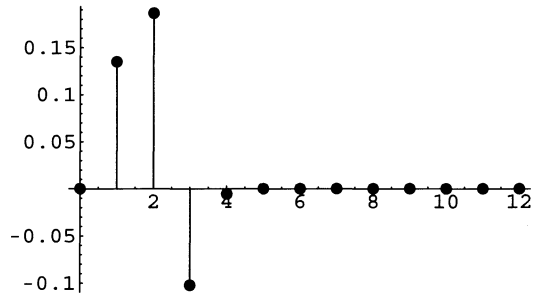
A small perturbation in the form of $\delta(k)/10$ as reference input will then produce the following movement of the steel ball:

```
InverseZTransform[% / 10, z, k]
```

$$\frac{1}{10}\,(0.000422938\,\text{KroneckerDelta}[-5+k] - 0.0559534\,\text{KroneckerDelta}[-4+k] - 1.02422\,\text{KroneckerDelta}[-3+k] + 1.86392\,\text{KroneckerDelta}[-2+k] + 1.34841\,\text{KroneckerDelta}[-1+k])$$

i.e. after five sampling intervals the ball comes to a stop.

```
Show[DiscreteSignal[%, {k, 0, 12}], PlotRange → All];
```



9.5 A Sampled-Data State-Variable Control System

Let us again take the inverted pendulum described in Section 9.1. In Section 9.1.2.3 the zero-position of the cart and the upright position of the pendulum were stabilized by an analog feedback. We shall now treat the same problem by sampling the state output for the feedback and by using a digital computer for the controller. In Section 9.1.2.2 we established the linerarized differential equations

$$(m + M)\frac{d^2x}{dt^2} - lm\frac{d^2\alpha}{dt^2} = f(t) + w_x(t),$$

$$4l\frac{d^2\alpha}{dt^2} - 3\frac{d^2x}{dt^2} - 3g\alpha(t) = w_\alpha(t),$$

where $x(t)$ is the position of the cart, $\alpha(t)$ the angle with the normal of the track of the pendulum and $f(t)$ is the force applied to the cart. The $w(t)$'s are small perturbations acting on the position of the cart and on the pendulum. The above system yields the dynamic state equations

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + f(t) \begin{pmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{22} & 0 & e_{24} \\ 0 & 0 & 0 & 0 \\ 0 & e_{42} & 0 & e_{44} \end{pmatrix} \begin{pmatrix} 0 \\ w_\alpha(t) \\ 0 \\ w_x(t) \end{pmatrix}$$

with $x_1(t) = \alpha(t)$, $x_2(t) = d\alpha/dt$, $x_3(t) = x(t)$, $x_4(t) = dx/dt$ and

$$a_{21} = -\frac{3\,g(M+m)}{l\,(4M+m)}, \quad a_{41} = -\frac{3\,g\,m}{4M+m}, \quad b_2 = \frac{3}{l\,(4M+m)}, \quad b_4 = \frac{4}{4M+m},$$

$$e_{22} = \frac{M+m}{l\,(4M+m)}, \quad e_{24} = \frac{3}{l\,(4M+m)}, \quad e_{42} = \frac{m}{4M+m}, \quad e_{44} = \frac{4}{4M+m}.$$

We assume the following physical values:

```
Clear[M]; physicalvalues = {M → 5, m → 0.5, l → 1, g → 9.81};

a21 = -3 g (M + m) / (4 M + m) / l /. physicalvalues;
a41 = -3 g m / (4 M + m) /. physicalvalues;
b2 = 3 / l / (4 M + m) /. physicalvalues;
b4 = 4 / (4 M + m) /. physicalvalues;

e22 = (M + m) / (4 M + m) / l /. physicalvalues;
e24 = 3 / (4 M + m) / l /. physicalvalues;
e42 = m / (4 M + m) /. physicalvalues;
e44 = 4 / (4 M + m) /. physicalvalues;

Am = {{0, 1, 0, 1}, {a21, 0, 0, 0}, {0, 0, 0, 1}, {a41, 0, 0, 0}};
Bm = {{0}, {b2}, {0}, {b4}};
Em = {{0, 0, 0, 0}, {0, e22, 0, e24}, {0, 0, 0, 0}, {0, e42, 0, e44}};
Cm = IdentityMatrix[4];
```

Now we refer to Sections 5.6.4. and 5.6.5. If the input and the perturbation are sampled and held with a sampling period T, i.e.,

$$\mathbf{f}(k) := \mathbf{f}(kT + t), \quad (kT \le t < (k+1)T), \quad (k = 0, 1, 2, \ldots),$$

$$\mathbf{w}(k) := \mathbf{w}(kT + t), \quad (kT \le t < (k+1)T), \quad (k = 0, 1, 2, \ldots),$$

the equivalent discrete state control system becomes (see Section 5.6.4)

$$\mathbf{x}(k+1) = \mathbf{A}_s\,\mathbf{x}(k) + \mathbf{B}_s\,\mathbf{u}(k) + \mathbf{E}_s\,\mathbf{w}(k),$$
$$\mathbf{y}(k) = \mathbf{C}_s\,\mathbf{x}(k-1) + \mathbf{D}_s\,\mathbf{u}(k-1) + \mathbf{H}_s\,\mathbf{w}(k-1).$$

We choose the sampling period equal to $T = 0.1$ and compute the discrete equivalent system.

```
T = 0.1;

ToDiscreteStateControlSystem[Am, Bm, Cm, Em, T] // Chop;

sys = %[[1]];

Es = %%[[2]]

{{0, 0.00145294, 0, 0.0016951}, {0, 0.0264458, 0, 0.0141867},
 {0, 0, 0, 0}, {0, 0.00240416, 0, 0.0194715}}

Hs = %%%[[3]]

{{0, 0.00145294, 0, 0.0016951}, {0, 0.0264458, 0, 0.0141867},
 {0, 0, 0, 0}, {0, 0.00240416, 0, 0.0194715}}
```

A full state feedback is now introduced, i.e.,

$$\mathbf{f}(k) = \mathbf{r}(k) - \mathbf{K}\,\mathbf{x}(k), \quad \mathbf{U}(z) = \mathbf{R}(z) - \mathbf{K}\,\mathbf{X}(z)$$

where $\mathbf{K}$ is a 1×4-matrix, and $\mathbf{r}(k)$ is the reference input.

```
Ks = {{k1, k2, k3, k4}};
```

The closed-loop characteristic function then becomes

```
ClosedLoopCharacteristicFunction[StateFeedbackCC[sys, Ks, z]] // Chop
```

```
1. + 0.0016951 k1 - 0.0141867 k2 + 0.00194918 k3 - 0.0194715 k4 +
 (-3.91448 - 0.0016951 k1 + 0.0439855 k2 - 0.00374375 k3 + 0.0569892 k4) z +
 (5.82896 - 0.0016951 k1 - 0.0439855 k2 + 0.0019371 k3 - 0.0569892 k4) z^2 +
 (-3.91448 + 0.0016951 k1 + 0.0141867 k2 + 0.0194715 k4) z^3 + z^4
```

```
CLL = CoefficientList[%, z] // Chop
```

```
{1. + 0.0016951 k1 - 0.0141867 k2 + 0.00194918 k3 - 0.0194715 k4,
 -3.91448 - 0.0016951 k1 + 0.0439855 k2 - 0.00374375 k3 + 0.0569892 k4,
 5.82896 - 0.0016951 k1 - 0.0439855 k2 + 0.0019371 k3 - 0.0569892 k4,
 -3.91448 + 0.0016951 k1 + 0.0141867 k2 + 0.0194715 k4, 1}
```

The four control coefficients are determined such that all zeros of the characteristic polynomial become located at zero.

```
eqns = Table[CLL[[j]] == 0, {j, Length[CLL] - 1}]
```

```
{1. + 0.0016951 k1 - 0.0141867 k2 + 0.00194918 k3 - 0.0194715 k4 == 0,
 -3.91448 - 0.0016951 k1 + 0.0439855 k2 - 0.00374375 k3 + 0.0569892 k4 == 0,
 5.82896 - 0.0016951 k1 - 0.0439855 k2 + 0.0019371 k3 - 0.0569892 k4 == 0,
 -3.91448 + 0.0016951 k1 + 0.0141867 k2 + 0.0194715 k4 == 0}
```

```
Solve[eqns, {k1, k2, k3, k4}]
```

```
{{k1 → -3174.09, k2 → -975.869, k3 → 7015.9, k4 → 1188.36}}
```

```
Kn = First[Ks /. %]
```

```
{{-3174.09, -975.869, 7015.9, 1188.36}}
```

The closed-loop z-transfer matrix between the perturbation and the output is then

```
M = ClosedLoopTransferMatrices[sys, Es, Hs, Kn, z][[2]] // Chop;
```

A slight perturbation to the cart position in form of a discrete Dirac impulse $w_x(k) = \delta(k)$ yields the following movement of the pendulum and the cart:

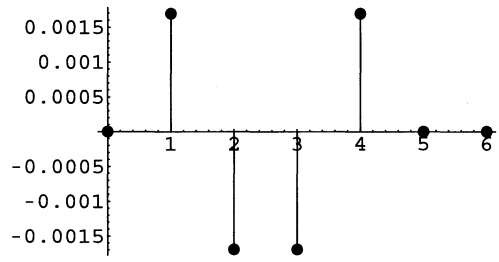
```
(M.{{0}, {0}, {0}, {1}})[[1]] // Apart
```

$$\left\{0.0016951 + \frac{0.0016951}{z^3} - \frac{0.0016951}{z^2} - \frac{0.0016951}{z}\right\}$$

```
InverseZTransform[% / z, z, k]
```

```
{(0.0016951 KroneckerDelta[-4 + k] -
    0.0016951 KroneckerDelta[-3 + k] - 0.0016951 KroneckerDelta[-2 + k] +
    0.0016951 KroneckerDelta[-1 + k]) UnitStep[-1 + k]}
```

```
Show[DiscreteSignal[%[[1]], {k, 0, 6}], PlotRange → All];
```



while the cart moves according to

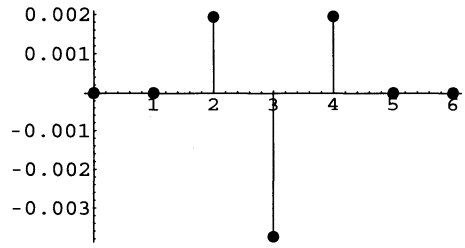
```
(M.{{0}, {0}, {0}, {1}})[[3]] // Apart
```

$$\left\{\frac{0.00194918}{z^3} - \frac{0.00374375}{z^2} + \frac{0.0019371}{z}\right\}$$

```
InverseZTransform[% / z, z, k]
```

```
{(0.00194918 KroneckerDelta[-4 + k] - 0.00374375 KroneckerDelta[-3 + k] +
    0.0019371 KroneckerDelta[-2 + k]) UnitStep[-1 + k]}
```

```
Show[DiscreteSignal[%[[1]], {k, 0, 6}], PlotRange → All];
```



Chapter 10
Heat Conduction and Vibration Problems

10.1 Flow of Heat

The main reference about heat conduction in connection with Laplace transformation is [Jaeger and Carslaw (2000)]. Further references are [Churchill (1972)], [Proudnikov and Ditkine (1979)], [Chabat and Lavrentiev (1972)]. We start with some problems concerning linear flow of heat. Later we will treat problems with cylindrical symmetry.

■ 10.1.1 Problems with Linear Flow of Heat

Let $y(x, t)$ denote the temperature as a function of the time t and of the space variable x. The partial differential equation governing linear flow of heat is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (a < x < b,\ t > 0) \tag{10.1}$$

with $\kappa^2 = K/(\rho c)$. The mass density is ρ and c is the specific heat of the solid. The constant K is defined such that $-K\,\partial y/\partial x$ represents the rate of flow of heat per unit time and per unit area in the direction of increasing x. For more details concerning the modeling of the flow of heat, see [Jaeger and Carslaw (2000)]. The initial condition (initial temperature) is specified as

$$y(x, 0) = y_0(x) \tag{10.2}$$

and the boundary conditions (given boundary temperatures) are

$$y(a, t) = v_1(t), \quad y(b, t) = v_2(t). \tag{10.3}$$

If $Y(x, s)$ denotes the Laplace transform of $y(x, t)$, the image equation of (10.1) becomes

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2} Y(x, s) = -\frac{1}{\kappa^2} y_0(x), \tag{10.4}$$

$$Y(a, s) = V_1(s), \quad Y(b, s) = V_2(s), \tag{10.5}$$

where $V_1(s)$ and $V_2(s)$ are the Laplace transforms of the boundary data $v_1(t)$ and $v_2(t)$. We first consider boundary value problems on a semi-infinite region.

10.1.1.1 Zero Initial Temperature and a Semi-infinite Region

Let us assume that $0 < x < \infty$.

a) The right boundary condition $Y(b, s) = V_2(s)$ is replaced by a boundedness condition that $y(x, t)$, and therefore $Y(x, s)$, has to remain bounded for all positive x. By assuming $y_0(x) = 0$, i.e. zero initial temperature in the solid, the general solution of (10.4) is

$$Y(x, s) = A(s)\, e^{-x\sqrt{s}/\kappa} + B(s)\, e^{x\sqrt{s}/\kappa}.$$

The boundedness condition implies that $B(s)$ must be zero. Because $a = 0$, the left boundary condition implies that $A(s) = V_1(s)$. Finally we obtain

$$Y(x, s) = V_1(s)\, e^{-x\sqrt{s}/\kappa}$$

for the image function (Laplace transform) of the solution. This expression suggests to adopt the input-output system point of view, and regard this problem as an input-output system as shown in Fig.10.1 with a nonrational transfer function

$$G(x, s) = e^{-x\sqrt{s}/\kappa}.$$

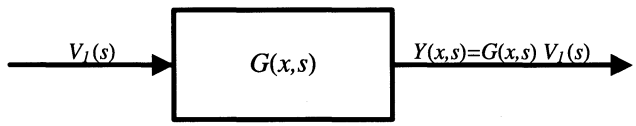


Fig.10.1: The heat conduction problem as an input-output system.

The input is the applied temperature at the left boundary and the output is the resulting temperature distribution in the solid. We now use the Package to obtain the inverse Laplace transform of the transfer function.

```
InverseLaplaceTransform[Exp[-x Sqrt[s] / κ], s, t]
```

$$\frac{e^{-\frac{x^2}{4\,t\,\kappa^2}}\,x}{2\sqrt{\pi}\;t^{3/2}\,\kappa}$$

By use of the convolution theorem the formal solution can be written in the form

$$y(x, t) = \frac{x}{2\kappa\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2}\, e^{-x^2/(4\kappa^2(t-\tau))}\, v_1(\tau)\, d\tau.$$

If at $x = a = 0$ the temperature is held at constant level v_0, then $V_1(s) = v_0/s$ and the solution takes the form

```
InverseLaplaceTransform[v0 Exp[-x Sqrt[s] / κ] / s, s, t]
```

$$\text{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa}\right] v_0$$

where Erfc denotes the complementary error function. In Notebook1011 the plots of the solution as a function of the variables x and t are shown.

b) A Radiation Boundary Condition:

Suppose now that the solid is heated at the left boundary $x = 0$ by radiation from a medium at

constant temperature V_0 (initial temperature is still zero). The boundary condition at $x = 0$ in this case is

$$\frac{\partial y(0, t)}{\partial x} = h\,[\,y(0, t) - V_0].$$

The image of this equation is

$$\frac{\partial Y(0, s)}{\partial x} = h\,[\,Y(0, s) - V_0/s].$$

Since

$$Y(x, s) = A(s)\,e^{-x\sqrt{s}/\kappa}$$

we finally obtain

$$Y(x, s) = V_0\,\frac{h\kappa}{s\left(\sqrt{s} + h\kappa\right)}\,e^{-x\sqrt{s}/\kappa}$$

for the Laplace transform of the solution. The inverse Laplace transform gives the solution

```
InverseLaplaceTransform[
  V0 h κ Exp[-x Sqrt[s] / κ] / (s (Sqrt[s] + h κ)), s, t] // Together
```

$$\left(\text{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa}\right] - e^{h x + h^2 t \kappa^2}\,\text{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa} + h\sqrt{t}\,\kappa\right]\right) V_0$$

In Notebook1011 the plots of the solution as a function of the variables x and t are shown.

bb) A Periodic Radiation Boundary Condition:

The solid is now heated at the left boundary $x = 0$ by radiation from a medium with the periodic temperature $v(t) = V_0 + V_1 \cos \omega t$ (initial temperature is still zero). The boundary condition at $x = 0$ then becomes

$$\frac{\partial y(0, t)}{\partial x} = h\,[\,y(0, t) - v(t)].$$

Proceeding as above we obtain

$$Y(x, s) = V_0\,\frac{h\kappa}{s\left(\sqrt{s} + h\kappa\right)}\,e^{-x\sqrt{s}/\kappa} + V_1\,\frac{s\,h\kappa}{(s^2 + \omega^2)\left(\sqrt{s} + h\kappa\right)}\,e^{-x\sqrt{s}/\kappa}$$

for the Laplace transform of the solution. The inverse Laplace transform of the first term of the right-hand side is

$$y_1(x, t) = V_0\,x\left\{\text{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa}\right] - e^{h x + h^2 t \kappa^2}\,\text{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa} + h\sqrt{t}\,\kappa\right]\right\}.$$

For the second term we have the contributions of the two residues at $-i\omega$, $i\omega$:

```
F = V1 h κ s Exp[-x Sqrt[s] / κ] / (s^2 + ω^2) / (Sqrt[s] + h κ);
```

```
ResidueLaplace[F, {I ω, -I ω}, s, t]
```

$$\frac{e^{-\frac{x\sqrt{-i\omega}}{\kappa}-it\omega}\, h\kappa V_1}{2\left(h\kappa+\sqrt{-i\omega}\right)}+\frac{e^{-\frac{x\sqrt{i\omega}}{\kappa}+it\omega}\, h\kappa V_1}{2\left(h\kappa+\sqrt{i\omega}\right)}$$

which gives, when simplified,

$$y_{21}(x,t)=V_1\,\frac{h\kappa e^{-x\sqrt{\omega/2}/\kappa}}{h^2\kappa^2+\sqrt{2\omega}\,H\kappa+\omega}$$
$$\times\left\{\left(h\kappa+\sqrt{\frac{\omega}{2}}\right)\cos\left(\omega t-\frac{x}{\kappa}\sqrt{\frac{\omega}{2}}\right)+\sqrt{\frac{\omega}{2}}\,\sin\left(\omega t-\frac{x}{\kappa}\sqrt{\frac{\omega}{2}}\right)\right\},$$

and the term

$$\frac{V_1\,h\kappa}{2\pi i}\int_{-\infty}^{(0+)}e^{st}\,\frac{s}{(s^2+\omega^2)\left(\sqrt{s}+h\kappa\right)}\,e^{-x\sqrt{s}/\kappa}\,ds,$$

see Sections 6.1.2.1 and 6.1.3. For this Hankel–type integral we can find an asymptotic expansion, valid for large values of t.

```
HI = HankelIntegral[F, {s, 0}][t];

AsymptoticSeries[HI, {t, Infinity, 2}]
```

$$\text{Landau}O\left[\frac{1}{t^4}\right]-\frac{15\,h\left(-\frac{1}{h^4\kappa^4}-\frac{x}{h^3\kappa^4}-\frac{x^2}{2h^2\kappa^4}-\frac{x^3}{6h\kappa^4}\right)\kappa V_1}{8\sqrt{\pi}\,t^{7/2}\,\omega^2}+\frac{3\,h\left(-\frac{1}{h^2\kappa^2}-\frac{x}{h\kappa^2}\right)\kappa V_1}{4\sqrt{\pi}\,t^{5/2}\,\omega^2}$$

Thus

$$y_{22}(x,t)=-V_1\,\frac{3\,(1+hx)}{4\sqrt{\pi}\,t^{5/2}\,\omega^2\,h\,\kappa}+\ldots\ldots\;(t\to\infty)$$

and the distribution of temperature is given by $y(x,t)=y_1(x,t)+y_{21}(x,t)+y_{22}(x,t)$.

c) Contact of the Solid with a Well-Stirred Fluid (or a Perfect Heat Conductor):

A fluid of mass M', specified heat c' and initial temperature u_0 is brought into contact with the boundary $x=0$ of the solid at $t=0$. The fluid is supposed to be so well-stirred that the temperature $u=u(t)$, $t>0$, remains uniformly distributed in the fluid. The boundary conditions now are

$$M'c'\,\frac{du}{dt}=K\,\frac{\partial y(0,t)}{\partial x},$$
$$y(0,t)=u(t),\quad t>0\ \text{ with }\ u(t)=u_0.$$

The image equation of these boundary conditions are

$$M'c'\,[sU(s)-u_0]=K\,\frac{\partial Y(0,s)}{\partial x},$$
$$Y(0,s)=U(s).$$

By

$$Y(x, s) = A(s)\, e^{-x\sqrt{s}/\kappa} ,$$

and recalling that $K = \varrho\, c\, \kappa^2$ and observing the above boundary conditions, we find

$$Y(x, s) = u_0 \frac{1}{\sqrt{s}\,(\sqrt{s} + h')}\, e^{-x\sqrt{s}/\kappa}$$

where $h' = \varrho\, c\, \kappa/(M'\, c')$. The solution of the problem is the inverse Laplace transform:

```
InverseLaplaceTransform[u0 Exp[-x Sqrt[s] / κ] / (Sqrt[s] (Sqrt[s] + h')), s, t]
```

$$e^{\frac{x\,h'}{\kappa} + t\,(h')^2}\ \text{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa} + \sqrt{t}\ h'\right] u_0$$

In Notebook1011 the plots of the solution as a function of the variables x and t are shown.

cc) We study the same problem, but replace the boundary condition $y(0, t) = u(t)$ for $t > 0$ by $-K\,(\partial y(0, t)/\partial x) = H\,[u(t) - y(0, t)]$. Thus

$$M'\, c'\, \frac{du}{dt} = K\, \frac{\partial y(0, t)}{\partial x} ,$$

$$-K\, \frac{\partial y(0, t)}{\partial x} = H\,[u(t) - y(0, t)], \quad t > 0, \quad u(t) = u_0$$

with the corresponding image equations

$$M'\, c'\, [s\, U(s) - u_0] = K\, \frac{\partial Y(0, s)}{\partial x}$$

$$-K\, \frac{\partial Y(0, s)}{\partial x} = H\,[u(t) - Y(0, s)], \quad t > 0, \quad U(s) = u_0/s .$$

The latter two equations and the expression $Y(x, s) = A(s)\, e^{-x\sqrt{s}/\kappa}$ yields the Laplace transform of the solution

$$Y(x, s) = u_0\, h \frac{1}{\sqrt{s}\,(s + h\sqrt{s} + h\, h')}\, e^{-x\sqrt{s}/\kappa} ,$$

where $h = \kappa H/K$. The inverse Laplace transform is the solution of the problem. As we obtain an If-construct, we need to make an assumption.

```
Short[InverseLaplaceTransform[
  u0 h Exp[-x Sqrt[s] / κ] / (Sqrt[s] (s + h Sqrt[s] + h h')), s, t], 3]
```

$$h\ \text{If}\left[h^2 - 4\,h\,h' > 0,\ \frac{\text{<<23>>}\left[\frac{\text{<<1>>}}{\text{<<1>>}}, s, t\right]}{\sqrt{h^2 - 4\,(h\,h')}} - \frac{\text{<<1>>}}{\text{<<1>>}},\right.$$

$$\left.\text{InverseLaplaceTransform}\left[\frac{e^{-\frac{x\ \text{<<1>>}\ \text{<<1>>}}{\kappa}}}{\sqrt{s}\ (1\,s + h\sqrt{s} + h\,h')}, s, t\right]\right] u_0$$

Inspection shows that we have to assume $h > 4\,h'$ in order to obtain the correct solution.

```
InverseLaplaceTransform[
 u0 h Exp[-x Sqrt[s] / κ] / (Sqrt[s] (s + h Sqrt[s] + h h')),
 s, t, Assumptions → h^2 - 4 h h' > 0]
```

$$h\,u_0\left(\frac{1}{\sqrt{h^2-4\,h\,h'}}\left(e^{\frac{x\left(h-\sqrt{h^2-4\,h\,h'}\right)}{2\kappa}+\frac{1}{4}t\left(h-\sqrt{h^2-4\,h\,h'}\right)^2}\right.\right.$$
$$\left.\mathrm{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa}+\frac{1}{2}\sqrt{t}\left(h-\sqrt{h^2-4\,h\,h'}\right)\right]\right)-\frac{1}{\sqrt{h^2-4\,h\,h'}}$$
$$\left.\left(e^{\frac{x\left(h+\sqrt{h^2-4\,h\,h'}\right)}{2\kappa}+\frac{1}{4}t\left(h+\sqrt{h^2-4\,h\,h'}\right)^2}\right.\mathrm{Erfc}\left[\frac{x}{2\sqrt{t}\,\kappa}+\frac{1}{2}\sqrt{t}\left(h+\sqrt{h^2-4\,h\,h'}\right)\right]\right)\Bigg)$$

Notebook1011 gives the plots of the solution as a function of the variables x and t.

d) Heat Generation within the Solid:

Heat is now produced in the solid for $t > 0$ at the rate $k\,t^{n/2}$ ($n = -1, 0, 1, \ldots$) per unit time per unit volume. At $x = 0$ we assume that there is radiation into a medium at zero temperature. The model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = -\frac{k}{K}t^{n/2} \quad (0 < x,\ t > 0) \tag{10.6}$$

with the boundary condition

$$\frac{\partial y(0,\,t)}{\partial x} = h\,y(0,\,t).$$

The image equation of (10.6) becomes

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2}Y(x,\,s) = -\frac{k}{K}\frac{\Gamma(n/2+1)}{s^{n/2+2}}$$

and has the solution (which remains bounded for all x)

$$Y(x,\,s) = A(s)\,e^{-x\sqrt{s}/\kappa} + \frac{k\kappa^2}{K}\frac{\Gamma(n/2+1)}{s^{n/2+2}}.$$

Taking into account the above boundary condition, we find

$$Y(x,\,s) = \frac{k\kappa^2}{K}\frac{\Gamma(n/2+1)}{s^{n/2+2}}\left[1 - h\kappa\frac{e^{-x\sqrt{s}/\kappa}}{\sqrt{s}+hs}\right].$$

The solution is obtained by taking the inverse Laplace transform. In Notebook1011 the lengthy expressions for the solutions for different values for n are computed and the solutions as a function of the variables x and t are plotted.

dd) We assume that heat is produced at a constant rate k in the region $0 < a < x < b$ for $t > 0$. The boundary at $x = 0$ is kept at zero for all times. The model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = -\frac{k}{K}[u(x-a) - u(x-b)], \quad (0 < x,\ t > 0)$$

with the boundary condition $y(x,\,0) = 0$ for $t > 0$. $u = u(t)$ is the unit-step or Heaviside function. Recall that we always assume zero initial conditions. The image equations are

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2} Y(x, s) = 0 \quad (0 < x < a),$$

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2} Y(x, s) = -\frac{k}{K s} \quad (a < x < b),$$

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2} Y(x, s) = 0 \quad (b < x).$$

We are looking for a solution $Y(x, s)$ that remains bounded and such that $Y(x, s)$ and $\partial Y/\partial x$ are continuous at $x = a$ and $x = b$. Therefore, we write

$$Y(x, s) = A_1(s)\, e^{-x\sqrt{s}/\kappa} + B_1(s)\, e^{x\sqrt{s}/\kappa} \quad (0 < x < a),$$

$$Y(x, s) = A_2(s)\, e^{-x\sqrt{s}/\kappa} + B_2(s)\, e^{x\sqrt{s}/\kappa} + \frac{\kappa^2 k}{K s} \quad (a < x < b),$$

$$Y(x, s) = A_3(s)\, e^{-x\sqrt{s}/\kappa} \quad (x < b).$$

The boundary condition implies $B_1(s) = -A_1(s)$. The imposed continuity conditions for $Y(x, s)$ and $\partial Y/\partial x$ at $x = a$ and $x = b$ yield four equations for the remaining four unknown coefficients. These four equations are

$$A_1(s)\, e^{-a\sqrt{s}/\kappa} - A_1(s)\, e^{a\sqrt{s}/\kappa} = A_2(s)\, e^{-a\sqrt{s}/\kappa} + B_2(s)\, e^{a\sqrt{s}/\kappa} + \frac{\kappa^2 k}{K s},$$

$$A_2(s)\, e^{-b\sqrt{s}/\kappa} + B_2(s)\, e^{b\sqrt{s}/\kappa} + \frac{\kappa^2 k}{K s} = A_3(s)\, e^{-b\sqrt{s}/\kappa},$$

$$-A_1(s)\, e^{-a\sqrt{s}/\kappa} - A_1(s)\, e^{a\sqrt{s}/\kappa} = -A_2(s)\, e^{-a\sqrt{s}/\kappa} + B_2(s)\, e^{a\sqrt{s}/\kappa},$$

$$-A_2(s)\, e^{-b\sqrt{s}/\kappa} + B_2(s)\, e^{b\sqrt{s}/\kappa} = -A_3(s)\, e^{-b\sqrt{s}/\kappa}.$$

In Notebbook1011 the above system is solved with *Mathematica*. The expressions for the coefficients are substituted into the three expressions for $Y(x, s)$ and the inverse Laplace transform is computed. Fairly large expressions are obtained, but with numerical values for the parameters, a 3-d plot of the solution in function of x and t is obtained. A plot corresponding to the physical values $\kappa \to 0.1$, $k \to 1$, $K \to 0.25$, $h \to 2$, $a \to 0.25$, $b \to 0.75$ is shown in Fig. 10.2.

d) Boundary $x = 0$ maintained at temperature $V \sin(\omega t + \phi)$:

The initial temperature of the solid is always zero. The model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2} \frac{\partial y}{\partial t} = 0, \quad (0 < x,\ t > 0)$$

with $y(x, 0) = 0$ and $y(0, t) = V(\cos\phi \sin\omega t + \sin\phi \cos\omega t)$. We obtain the image function

$$Y(x, s) = \frac{V(\omega \cos\phi + s \sin\phi)}{e^{x\sqrt{s}}(s^2 + \omega^2)}.$$

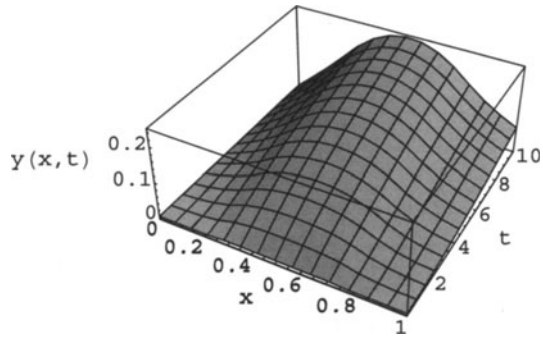


Figure 10.2: Solution of the heat conduction problem with an imposed heating of rectangular shape.

By taking the inverse Laplace transform we obtain the solution in the following form:

```
Y = V (ω Cos[ϕ] + s Sin[ϕ]) / (s^2 + ω^2) Exp[-x Sqrt[s]];

y = InverseLaplaceTransform[Y, s, t]
```

$$V\left(\frac{\int_0^\infty \frac{e^{-t\,u1}\,\mathrm{Sin}\left[\sqrt{u1}\,x\right]\,(\omega\,\mathrm{Cos}[\phi]-u1\,\mathrm{Sin}[\phi])}{u1^2+\omega^2}\,du1}{\pi} + e^{-\frac{x\sqrt{\omega}}{\sqrt{2}}}\,\mathrm{Sin}\left[\phi - \frac{x\sqrt{\omega}}{\sqrt{2}} + t\,\omega\right]\right)$$

Inspection shows that the term with the integral is the transitory part, and the second term is the steady-state part of the solution. The integral could not be evaluated but it is a Laplace integral; thus we can apply Watson's lemma to compute an asymptotic expansion for large values of t. This asymptotic expansion will inform us of how the transitory part will decay as time evolves. We compute the first two terms of the asymptotic expansion of the Laplace integral.

```
Y1 = (Sin[Sqrt[u] * x] * (ω * Cos[ϕ] - u * Sin[ϕ])) / (u^2 + ω^2);

AsymptoticSeries[LaplaceIntegral[Y1, {u, 0}][t], {t, Infinity, 2}]
```

$$\frac{\sqrt{\pi}\,x\,\mathrm{Cos}[\phi]}{2\,t^{3/2}\,\omega} + \mathrm{Landau0}\left[\frac{1}{t^3}\right] + \frac{3\sqrt{\pi}\left(-\frac{x^3\,\mathrm{Cos}[\phi]}{6\,\omega} - \frac{x\,\mathrm{Sin}[\phi]}{\omega^2}\right)}{4\,t^{5/2}}$$

10.1.1.2 Zero Initial Temperature and a Finite Region

We assume that $0 < x < \ell$.

a) We consider a configuration that has no flow of heat at the boundary $x = 0$ and where the boundary $x = \ell$ is maintained at constant temperature V for $t > 0$. The model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 < x < \ell,\ t > 0),$$

$$\frac{\partial y(0,t)}{\partial x} = 0 \text{ (no flow)}, \quad y(\ell, t) = V.$$

The image equations corresponding to the two above relations are

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2} Y(x,s) = 0, \quad (0 < x < \ell),$$

$$\frac{\partial Y(0,s)}{\partial x} = 0, \quad Y(\ell, s) = V/s.$$

The general solution of the first equation again is

$$Y(x,s) = A(s)\, e^{-x\sqrt{s}/\kappa} + B(s)\, e^{x\sqrt{s}/\kappa}.$$

By taking into account the two image boundary conditions, we obtain

$$Y(x,s) = V \frac{\cosh\left(x\sqrt{s}/\kappa\right)}{s\cosh\left(\ell\sqrt{s}/\kappa\right)}.$$

This image function is a meromorphic function with an infinite number of poles in spite of the appearance of the square root which suggests a branch point at 0. Therefore the solution will be an infinite series.

```
InverseLaplaceTransform[V Cosh[x Sqrt[s] / κ] / (s Cosh[ℓ Sqrt[s] / κ]), s, t]
```

$$V\left(1 + \frac{1}{\pi}\left(4\,\text{Summation}\left[\frac{(-1)^{n1}\, e^{-\frac{(-1+2\,n1)^2\,\pi^2\,t\,\kappa^2}{4\,\ell^2}}\,\text{Cos}\left[\frac{(-1+2\,n1)\,\pi\,x}{2\,\ell}\right]}{-1+2\,n1}, \{n1, 1, \infty\}\right]\right)\right)$$

By use of the NamedRule `TruncateSummation[n]`, the series can be truncated and after inserting numerical values for the parameters, the solution can be plotted (for details see Notebook1011). It will be observed that for small values of t the approximation by a truncated series is of poor quality, but for larger values it becomes very good. For a series that converges rapidly for small values of t, use the option `{Transient -> True}`. You obtain a series in terms of complementary error functions that has the required property. See Notebook1011 for details.

```
InverseLaplaceTransform[
 V Cosh[x Sqrt[s] / κ] / (s Cosh[ℓ Sqrt[s] / κ]), s, t, Transient → True]
```

$$V\,\text{Summation}\left[(-1)^{k1}\left(\text{Erfc}\left[\frac{-\frac{x}{\kappa}+\frac{(1+2\,k1)\,\ell}{\kappa}}{2\sqrt{t}}\right] + \text{Erfc}\left[\frac{\frac{x}{\kappa}+\frac{(1+2\,k1)\,\ell}{\kappa}}{2\sqrt{t}}\right]\right), \{k1, 0, \infty\}\right]$$

Let us now compute the average temperature $< y(t) >$ of the slab. We have

$$< y(t) > = \frac{1}{\ell}\int_0^{\ell} y(x,t)\, dx$$

or, in the image domain,

$$< Y(s) > = \frac{1}{\ell}\int_0^{\ell} Y(x,s)\, dx = \frac{V\kappa}{\ell}\,\frac{\sinh\left(\ell\sqrt{s}/\kappa\right)}{s^{3/2}\cosh\left(\ell\sqrt{s}/\kappa\right)}.$$

In Notebook1011 the rather complicated expression for the inverse Laplace transform is computed and the average temperature as a function of time is obtained.

Below are some exercises, their solutions are given in the notebook1011.

Problem 10.1. The left boundary $x = 0$ of the solid is kept at zero temperature and the right boundary $x = \ell$ is maintained at constant temperature V for $t > 0$.

Problem 10.2. At the left boundary $x = 0$ there is no flow of heat but at the right boundary $x = \ell$ of the solid there is a constant flow of heat Q into the solid.

Problem 10.3. The two boundaries at $x = 0$ and $x = \ell$ are kept at zero temperature and heat is produced at a constant rate of $a\,t^{n/2}$ for $t > 0$ per unit time and per unit volume.

10.1.1.3 Non-Zero Initial Temperature and a Finite Region

a) Constant initial temperature V:

Let us assume that there is no flow of heat at the left boundary $x = 0$ but a radiation at $x = \ell$ into a medium initially kept at zero. The model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 < x < \ell,\ t > 0),$$

$$\frac{\partial y(0, t)}{\partial x} = 0 \text{ (no flow)}, \quad \frac{\partial y(\ell, t)}{\partial x} = -h\,y(\ell, t) \text{ (radiation)},$$

$$y(x, 0) = V \text{ (initial temperature)}.$$

The image equations are

$$\frac{\partial^2 Y}{\partial x^2} - \frac{1}{\kappa^2}[s\,Y(x, s) - V] = 0, \quad (0 < x < \ell),$$

$$\frac{\partial Y(0, s)}{\partial x} = 0, \quad \frac{\partial Y(\ell, s)}{\partial x} = -h\,Y(\ell, s).$$

By use of the general solution of the above differential equation

$$Y(x, s) = A(s)\,e^{-x\sqrt{s}/\kappa} + B(s)\,e^{x\sqrt{s}/\kappa} + V/s$$

we obtain

$$Y(x, s) = V\left\{\frac{1}{s} - \frac{h\kappa\cosh\left(x\sqrt{s}/\kappa\right)}{s\left[\sqrt{s}\sinh\left(\ell\sqrt{s}/\kappa\right) + h\kappa\cos\left(\ell\sqrt{s}/\kappa\right)\right]}\right\}.$$

The inverse Laplace transform of the second term inside the curly braces is an infinite series. We have a new situation in that the terms of the infinite series depend on the roots of an algebraic transcendental equation that can be solved only numerically. Invoking the inverse Laplace transform, an output object named `LzOut` is obtained whose first argument is the infinite series and the second argument is a list. The first entry in the list contains the algebraic equation, the second its independent variable, the third a formula for the starting value depending on the summation index, and the last one the summation index itself. The formula

for the starting value is used for the procedure `FindRoot` which is contained in the procedure `ComputeZeros`.

```
F1 = h κ Cosh[x Sqrt[s] / κ] / s /
     (Sqrt[s] Sinh[l Sqrt[s] / κ] + h κ Cosh[l Sqrt[s] / κ]);

f1 = InverseLaplaceTransform[F1, s, t]
```

$$\text{LzOut}\Big[1 - 2\,h\,\kappa\,\text{Summation}\Big[\frac{e^{-\frac{t\,\kappa^2\,v1[k1]^2}{l^2}}\,\text{Cos}\big[\frac{x\,v1[k1]}{l}\big]\,\text{Sec}[v1[k1]]}{h\,(1+h\,l)\,\kappa + \frac{\kappa\,v1[k1]^2}{l}},\ \{k1,\ 1,\ \infty\}\Big],$$

$$\{v1\,\text{Tan}[v1] == h\,l,\ v1,\ 0.7 + (-1 + k1)\,\pi,\ k1\}\Big]$$

In order to evaluate and truncate the infinite series, we first have to give numerical values for the parameters, for example,

```
fln = f1 /. {κ → 0.1, l → 1, h → 2};
```

Now we truncate the series by taking for example the first 10 terms. The zeros or roots of the algebraic equation are then accurately computed by use of the starting value formula. This is done by `ComputeZeros`.

```
flntrunc = ComputeZeros[10][fln]
```

$$\begin{aligned}1 - 0.4\,(&2.94614\,e^{-0.0115966\,t}\,\text{Cos}[1.07687\,x] - 0.591802\,e^{-0.132758\,t}\,\text{Cos}[3.6436\,x] +\\ &0.212117\,e^{-0.432745\,t}\,\text{Cos}[6.57833\,x] - 0.103449\,e^{-0.927284\,t}\,\text{Cos}[9.62956\,x] +\\ &0.0603062\,e^{-1.61857\,t}\,\text{Cos}[12.7223\,x] - 0.039265\,e^{-2.50703\,t}\,\text{Cos}[15.8336\,x] +\\ &0.0275282\,e^{-3.5928\,t}\,\text{Cos}[18.9547\,x] - 0.0203426\,e^{-4.87592\,t}\,\text{Cos}[22.0815\,x] +\\ &0.015634\,e^{-6.3564\,t}\,\text{Cos}[25.2119\,x] - 0.0123851\,e^{-8.03426\,t}\,\text{Cos}[28.3448\,x])\end{aligned}$$

The approximate solution can now be plotted (assuming $V = 1$) as a function of x and t (Fig.10.3).

```
f = 1 - flntrunc;
```

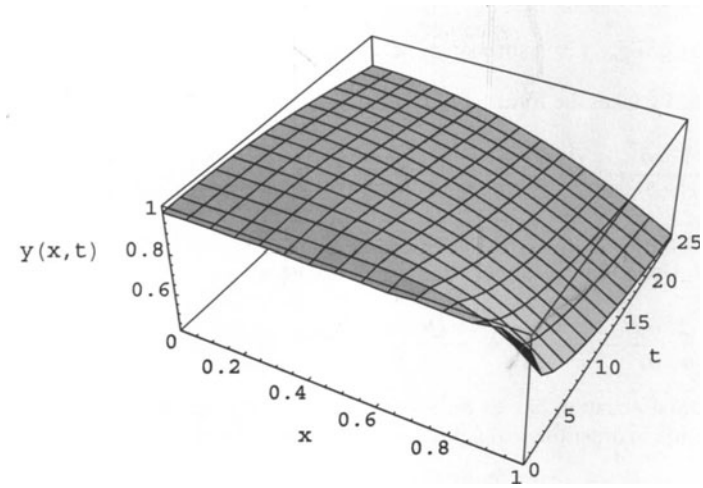


Fig.10.3: Solution of the heat conduction problem with non-zero initial temperature.

■ 10.1.2 Problems with Cylindrical or Spherical Symmetry

10.1.2.1 Modeling

The mathematical model for heat conduction in 3-dimensional space is

$$\nabla^2 y - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = -\frac{Q(x, y, z)}{K}$$

where $y(x, y, z, t)$ is the temperature and $Q(x, y, z)$ the heat supplied at point $P(x, y, z)$ per unit time and unit volume inside a solid. The constant K is the thermal conductivity of the substance of the solid. This constant is defined by

$$\vec{f} = K\overrightarrow{\operatorname{grad} y}$$

where $\vec{f}$ is the flux vector at point P. In cylindrical coordinates (r, ϕ, z) with

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z,$$

the Laplace operator ∇^2 takes the form

$$\nabla^2 y = \frac{\partial^2 y}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \phi^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{\partial^2 y}{\partial z^2}.$$

We shall consider applications where there is an axial symmetry such that y does not depend either on ϕ or on z. Under this assumption the model becomes

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = -\frac{Q(x, y, z)}{K}.$$

In spherical coordinates (r, ϑ, ϕ), with

$$x = r\sin\vartheta\cos\phi, \quad y = r\sin\vartheta\sin\phi, \quad z = r\cos\vartheta,$$

the Laplace operator ∇^2 takes the form

$$\nabla^2 y = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial y}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial y}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 y}{\partial \phi^2}.$$

Again, we shall consider only applications with a radial symmetry such that y does not depend either on ϕ or on ϑ. Under this assumption the model becomes

$$\frac{\partial^2 y}{\partial r^2} + \frac{2}{r}\frac{\partial y}{\partial r} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = -\frac{Q(x, y, z)}{K}.$$

The partial differential equation has to be accompanied by appropriate boundary conditions and an initial condition in order to yield a problem with a unique solution.

10.1.2.2 The Full Cylinder

Assume that a solid has the form of a cylinder of radius a, where its circular end surfaces are insulated such that there is no heat flow through them.

Example. Zero initial temperature and cylindrical surface maintained at constant temperature V:

The model is given by

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 \le r < a,\ t > 0),$$

with boundary condition $y(a, t) = V$ for $t > 0$ and initial condition $y(r, 0) = 0$ for $0 \le r < a$. The image equation is

$$\frac{\partial^2 Y}{\partial r^2} + \frac{1}{r}\frac{\partial Y}{\partial r} - \frac{s}{\kappa^2} Y(r, s) = 0, \quad (0 \le r < a) \tag{10.7}$$

with boundary condition $Y(a, s) = V/s$. The latter differential equation is a special case of Bessel's differential equation of the second kind and of order ν:

$$Y'' + \frac{1}{r} Y' - \left(\alpha^2 + \frac{\nu^2}{r^2}\right) Y(r) = 0$$

where $\alpha = \sqrt{s}/\kappa$ and $\nu = 0$.

In the case of integral ν, two linearly independent solutions are given by $I_\nu(\alpha r)$ and $K_\nu(\alpha r)$, the modified Bessel function of order ν and Macdonald's function of order ν, respectively. Therefore the general solution of (10.7) is

$$Y(r, s) = A(s)\, I_0\left(r\sqrt{s}/\kappa\right) + B(s)\, K_0\left(r\sqrt{s}/\kappa\right).$$

For physical reasons the solution $y(r, t)$, and as a consequence $Y(r, s)$, has to remain bounded as r tends to zero. Since Macdonald's function $K_0(x)$ tends to infinity as $x \to 0+$, we choose $B(s) = 0$. The boundary condition requires that

$$Y(a, s) = A(s)\, I_0\left(a\sqrt{s}/\kappa\right) = V/s,$$

which finally yields the image function of the solution:

$$Y(r, s) = V\,\frac{I_0\left(r\sqrt{s}/\kappa\right)}{s\, I_0\left(a\sqrt{s}/\kappa\right)},$$

```
InverseLaplaceTransform[
 V BesselI[0, r Sqrt[s] / κ] / BesselI[0, a Sqrt[s] / κ] / s, s, t]
```

```
LzOut[V - 2 V Summation[e^(-t κ^2 v2[k2]^2 / 10000) BesselJ[0, 1/100 r v2[k2]] / (BesselJ[1, v2[k2]] v2[k2]), {k2, 1, ∞}],
 {BesselJ[0, v2] == 0, v2, 2.35619 + (-1 + k2) π, k2}]
```

This is the solution of the problem. It is an infinite series where each term contains the zeros of Bessel's function of order 0. We arrive at the same situation as in Section 10.1.1.3, Example a). After substituting numerical values for the parameters, we use `ComputeZeros` to truncate

the series and accurately compute the zeros of the Bessel function $J_0(v) = 0$, that are needed in the terms of the series. See Notebook1012 for details.

Example. Zero initial temperature and a medium radiating into the solid:

Let us again assume insulated end surfaces such that there is no heat flow through them. The solid is heated by radiation from a medium kept at a constant temperature V. The model then is

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 \le r < a,\ t > 0).$$

The boundary condition is

$$\frac{\partial y(a, t)}{\partial r} = h\,[V - y(a, t)].$$

This gives the following image relations:

$$\frac{\partial^2 Y}{\partial r^2} + \frac{1}{r}\frac{\partial Y}{\partial r} - \frac{s}{\kappa^2} Y(r, s) = 0, \quad (0 \le r < a),$$

$$\frac{\partial Y(a, s)}{\partial r} = h\,[V - Y(a, s)].$$

Straightforward computation, using the image boundary condition, solves the ordinary differential equation. One obtains

$$Y(r, s) = h\,V\,\kappa\,\frac{I_0\left(r\sqrt{s}\,/\kappa\right)}{s\left[\sqrt{s}\,I_1\left(a\sqrt{s}\,/\kappa\right) + h\,\kappa\,I_0\left(a\sqrt{s}\,/\kappa\right)\right]},$$

```
h V κ InverseLaplaceTransform[BesselI[0, r Sqrt[s] / κ] / s /
   (Sqrt[s] BesselI[1, a Sqrt[s] / κ] + h κ BesselI[0, a Sqrt[s] / κ]), s, t]
```

$$\texttt{LzOut}\left[\texttt{h V}\,\kappa\left(\frac{1}{\texttt{h}\,\kappa} - \frac{1}{\kappa}\left(200\ \texttt{Summation}\left[\frac{e^{-\frac{t\,\kappa^2\,\texttt{v4[k4]}^2}{10000}}\ \texttt{BesselJ}\left[0,\ \frac{1}{100}\,\texttt{r v4[k4]}\right]}{\texttt{BesselJ[0, v4[k4]]}\ (10000\,\texttt{h}^2 + \texttt{v4[k4]}^2)},\ \{\texttt{k4},\ 1,\ \infty\}\right]\right)\right),$$

$$\{\texttt{v4 BesselJ[1, v4]} == 100\,\texttt{h BesselJ[0, v4]},\ \texttt{v4},\ (-0.5 + \texttt{k4})\,\pi,\ \texttt{k4}\}\Big]$$

This is the solution. The infinite series consists of terms containing the positive roots of the equation $v\,J_1(v) = a\,h\,J_0(v)$. After giving numerical values for the parameters, one can use `ComputeZeros` in order to truncate the series and to accurately compute the zeros of the above equation. See Notebook1012 for details and a plot of the solution. Also in the Notebook1012 you can find the solutions of the following problems.

Problem 10.4. Assume that the initial temperature of the solid is zero. The end surfaces are insulated. At the cylindrical surface $r = a$ the temperature $k\,t$ is applied.

Problem 10.5. The initial temperature is zero. End surfaces are insulated. Through the surface $r = a$ a constant flux of heat into the solid is applied. The boundary condition is of the form

$$-K\,\frac{\partial y(a, t)}{\partial r} = F.$$

Problem 10.6. The initial temperature is zero. Inside the cylinder heat is produced at the constant rate A per unit volume per unit time. The surface $r = a$ is maintained at zero.

10.1.2.3 The Hollow Cylinder

a) Constant flux of heat through the internal surface, external surface maintained at zero level: We consider a hollow cylinder and assume that there is a constant flux F of heat at $x = a$ through the inner surface into the cylinder while the outer surface at $x = b$ is maintained at zero. Again let us assume zero initial temperature. The model is

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (a < r < b,\ t > 0),$$

$$\frac{\partial y(a,t)}{\partial r} = -\frac{F}{K}, \quad y(b,t) = 0 \quad \text{(boundary conditions)},$$

$$y(r,0) = 0, \quad (a < r < b) \quad \text{(initial condition)}.$$

The image equations become then

$$\frac{\partial^2 Y}{\partial r^2} + \frac{1}{r}\frac{\partial Y}{\partial r} - \frac{s}{\kappa^2}Y(r,s) = 0, \quad (a < r < b),$$

$$\frac{\partial Y(a,s)}{\partial r} = -\frac{F}{Ks}, \quad Y(b,s) = 0\ .$$

The general solution of the above differential equation again is

$$Y(r,s) = A(s)\,I_0\left(r\sqrt{s}/\kappa\right) + B(s)\,K_0\left(r\sqrt{s}/\kappa\right)$$

with the modified Bessel function I_0 and the Macdonald's function K_0. Using the two boundary conditions, and then solving the resulting system of linear equations, we obtain the following expression for the Laplace transform of the solution:

$$Y(r,s) = -\frac{F\kappa}{K}\,\frac{K_0\left(b\sqrt{s}/\kappa\right)I_0\left(r\sqrt{s}/\kappa\right) - I_0\left(b\sqrt{s}/\kappa\right)K_0\left(r\sqrt{s}/\kappa\right)}{s\sqrt{s}\left[I_0\left(b\sqrt{s}/\kappa\right)K_1\left(a\sqrt{s}/\kappa\right) + I_1\left(a\sqrt{s}/\kappa\right)K_0\left(b\sqrt{s}/\kappa\right)\right]}.$$

Let us give an outline of what is going on next. First, the expressions in the numerator and in the denominator are in spite of the square roots entire functions. Thus we have a meromorphic image function. Furthermore the expression

$$Q(s) = \sqrt{s}\left[I_0\left(b\sqrt{s}/\kappa\right)K_1\left(a\sqrt{s}/\kappa\right) + I_1\left(a\sqrt{s}/\kappa\right)K_0\left(b\sqrt{s}/\kappa\right)\right]$$

does not vanish for $s = 0$, hence $Y(r,s)$ has a simple pole at $s = 0$. Therefore $Y(r,s)$ is a meromorphic function of the type $P(s)/(s\,Q(s))$. The general expansion theorem applies and we can write the solution in the form

$$y(r,t) = \frac{P(0)}{Q(0)} + \sum_{k=1}^{\infty}\frac{P(\nu_k)}{\nu_k\,Q'(\nu_k)}\exp(\nu_k\,t)$$

where ν_k are the simple and positive roots of $Q(s)$. Finding the zeros of $Q(s)$ is equivalent to computing all positive roots of the equation $J_1(a\,\nu/\kappa)\,Y_0(a\,\nu/\kappa) = J_0(a\,\nu/\kappa)\,Y_1(a\,\nu/\kappa)$.

```
InverseLaplaceTransform[
 -F κ / K * (BesselK[0, b Sqrt[s] / κ] BesselI[0, r Sqrt[s] / κ] -
    BesselI[0, b Sqrt[s] / κ] BesselK[0, r Sqrt[s] / κ])
  s^(-3 / 2) / (BesselI[0, b Sqrt[s] / κ] BesselK[1, a Sqrt[s] / κ] +
     BesselI[1, a Sqrt[s] / κ] BesselK[0, b Sqrt[s] / κ]), s, t]
```

$$\text{LzOut}\Big[\frac{a\,F\,\text{Log}[\frac{b}{r}]}{K}+\frac{1}{K}\Big(F\,\pi\,\kappa\,\text{Summation}\Big[\Big(e^{-t\,v2[k2]^2}\,\text{BesselJ}\Big[0,\frac{b\,v2[k2]}{\kappa}\Big]^2\Big(-\text{BesselJ}\Big[1,\frac{a\,v2[k2]}{\kappa}\Big]\text{BesselY}\Big[0,\frac{r\,v2[k2]}{\kappa}\Big]+\text{BesselJ}\Big[0,\frac{r\,v2[k2]}{\kappa}\Big]\text{BesselY}\Big[1,\frac{a\,v2[k2]}{\kappa}\Big]\Big)\Big)\Big/\Big(\Big(-\text{BesselJ}\Big[0,\frac{b\,v2[k2]}{\kappa}\Big]^2+\text{BesselJ}\Big[1,\frac{a\,v2[k2]}{\kappa}\Big]^2\Big)v2[k2]\Big),\{k2,1,\infty\}\Big]\Big),\Big\{\text{BesselJ}\Big[1,\frac{a\,v2}{\kappa}\Big]\text{BesselY}\Big[0,\frac{b\,v2}{\kappa}\Big]==\text{BesselJ}\Big[0,\frac{b\,v2}{\kappa}\Big]\text{BesselY}\Big[1,\frac{a\,v2}{\kappa}\Big],v2,\text{None},k2\Big\}\Big]$$

This shows that a general formula for an appropriate starting value for computing the exact zero ν_k (as a function of the index k) is not readily available. If you compute the inverse Laplace transform, you again obtain the `LzOut` object, but in the place where there should be the general formula for the starting value there is a `None` instead. `ComputeZeros` will try to compute the required zeros by another method. If it fails you must help by hand. Instead of `ComputeZeros` use `FindZeros`. By invoking `FindZeros` you obtain the plot of the function whose zeros are needed. In an opening window you can insert the list of the approximate zeros that you have obtained from inspecting the graphics. These values are then taken as the starting values for the `FindRoot` algorithm which accurately computes the zeros. Depending on the number of approximate zeros you give in the list, you finally obtain a truncated series ready to be plotted. For details see Notebook1012. The plot of the solution is given in Fig. 10.4.

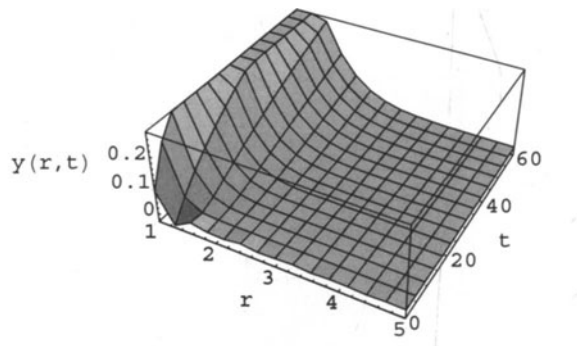


Fig.10.4: The plot of the solution of the hollow cylinder problem.

10.1.2.4 The Full Sphere

A full sphere of radius a with the initial temperature y_0 is immersed at $t = 0$ into a liquid of constant temperature zero. This is modeled by

$$\frac{\partial^2 y}{\partial r^2} + \frac{2}{r}\frac{\partial y}{\partial r} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 \le r < a,\ t > 0),$$

with the boundary condition $y(a, t) = 0$ for $t > 0$ and the initial condition $y(r, 0) = y_0$ for $0 \le r < a$. The image equation is

$$\frac{\partial^2 Y}{\partial r^2} + \frac{2}{r}\frac{\partial Y}{\partial r} - \frac{s}{\kappa^2} Y(r, s) = -\frac{y_0}{\kappa^2}, \quad (0 \le r < a) \tag{10.8}$$

with the boundary condition $Y(a, s) = 0$. Setting $Y(r, s) = Z(r, s)/r$ simplifies the differential equation:

$$\frac{\partial^2 Z}{\partial r^2} - \frac{s}{\kappa^2} Z(r, s) = -\frac{y_0}{\kappa^2} r \qquad (0 \le r < a)$$

with $Z(a, s) = 0$. The general solution is

$$Z(r, s) = A \cosh \frac{\sqrt{s}\, r}{\kappa} + B \sinh \frac{\sqrt{s}\, r}{\kappa} + \frac{y_0}{s} r.$$

On physical reasons the solution $y(r, t)$, and as a consequence $Y(r, s)$, has to remain bounded as r tends to zero. This implies that we must have $Z(0, s) = 0$, thus $A = 0$. The boundary condition $Z(a, s) = 0$ implies

$$Z(r, s) = \frac{y_0 r}{s} - \frac{y_0 a}{s} \frac{\sinh \frac{\sqrt{s}\, r}{\kappa}}{\sinh \frac{\sqrt{s}\, a}{\kappa}}.$$

```
InverseLaplaceTransform[
 y0 r / s - a y0 Sinh[r / κ Sqrt[s]] / (s Sinh[a / κ Sqrt[s]]), s, t]
```

$$r\, y_0 - a\, y_0 \left(\frac{r}{a} + \frac{1}{\pi} \left(2\, \text{Summation}\left[\frac{(-1)^{n2}\, e^{-\frac{n2^2\, \pi^2\, t\, \kappa^2}{a^2}}\, \text{Sin}\left[\frac{n2\, \pi\, r}{a}\right]}{n2}, \{n2, 1, \infty\} \right] \right) \right)$$

The solution of the problem is finally given by

$$y(r, t) = \frac{2\, a\, y_0}{r\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, e^{-\frac{n^2 \pi^2 \kappa^2}{a^2} t}}{n} \sin\left(\frac{n\pi r}{a}\right).$$

■ 10.1.3 Temperature Control

Maintaining a constant temperature in some region by controling the heat supply is a current problem in automatic control. Often the location where the temperature is measured is different from where the heat is supplied. This leads to a certain time-lag between the measurement of the temperature and the appropriate change of the heating current. A typical example is described below. For other examples see [Jaeger and Carslaw (2000)].

10.1.3.1 Example of an Automatic Temperature Control

We consider a semi-infinite solid $x > 0$ that loses heat through its left boundary $x = 0$ at a rate proportional to its surface temperature $y(0, t)$ with proportionality constant H. Let us assume that heat is supplied at this left boundary at the rate $u(t)$. The temperature in the solid is measured at $x = \ell$ and is fed back in order to control the supply of heat. We use a cascade controller configuration with a controller described by its transfer function $C(s)$. The reference temperature to be maintained at the left boundary is $r(t)$. The steady-state rate of heat supply which would be needed to compensate the loss of heat is $Q_0 = H\,r(t)$. The mathematical model then is:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 < x\,,\ t > 0)$$

$$-K\frac{\partial y(0,t)}{\partial x} = H\,r(t) - H\,y(0,t) + \frac{\partial}{\partial t}\int_0^t c(t-\tau)\,[r(\tau) - y(\ell,\tau)]\,d\tau\,.$$

With $y(x, t) \circ\!-\!\bullet\, Y(x, S)$, $c(t) \circ\!-\!\bullet\, C(s)$, $r(t) \circ\!-\!\bullet\, R(s)$ the image equations become

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2}Y(x,s) = 0\,,$$

$$-K\frac{\partial Y(0,s)}{\partial x} = H\,R(s) - H\,Y(0,s) + s\,C(s)\,[R(s) - Y(\ell,s)]\,.$$

The solution, which remains bounded, of the first differential equation is $Y(x, s) = A\exp\left[-x\sqrt{s}/\kappa\right]$. By inserting this expression into the second equation yields

$$Y(x,s) = \frac{[h + k\,s\,C(s)]\,e^{-x\sqrt{s}/\kappa}}{\sqrt{s} + h + k\,s\,C(s)\,e^{-\ell\sqrt{s}/\kappa}}\,R(s)$$

with $h = \kappa H/K$, $k = \kappa/K$. We have a control problem with a nonrational closed-loop transfer function. If $c(t) = \lambda =$ constant, i.e., $s\,C(s) = \lambda$, this is a simple P-regulator and for a constant reference temperature y_0,

$$Y(x,s) = y_0\,\frac{[h + k\,\lambda]\,e^{-x\sqrt{s}/\kappa}}{s\left[\sqrt{s} + h + k\,\lambda\,e^{-\ell\sqrt{s}/\kappa}\right]}\,.$$

Assuming a constant steady-state, i.e. that $\lim_{t\to\infty} y(x, t)$ exists, we have

$$\lim_{s\to 0} sY(x,s) = \lim_{s\to 0} y_0\,\frac{[h + k\,\lambda]\,e^{-x\sqrt{s}/\kappa}}{\sqrt{s} + h + k\,\lambda\,e^{-\ell\sqrt{s}/\kappa}} = y_0\,\frac{h + k\,\lambda}{h + k\,\lambda} = y_0$$

which is correct. But this will happen only if the closed-loop system is stable. Since we have a nonrational transfer function, things are less simple. Let us use the complex inversion formula. We have ($\gamma > 0$)

$$y(x,t) = y_0\,\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\,\frac{[h + k\,\lambda]\,e^{-x\sqrt{s}/\kappa}}{s\left[\sqrt{s} + h + k\,\lambda\,e^{-\ell\sqrt{s}/\kappa}\right]}\,ds\,.$$

The expressions $e^{-x\sqrt{s}/\kappa}$ and $e^{-\ell\sqrt{s}/\kappa}$ remain bounded for $|s| \to \infty$ and $|\arg s| < \pi$ provided we take the principal branch of the square root. We refer to Section 6.1.3 and use the integration contour of the type shown in Fig.6.2.

$$y(x,t) = \sum_{\nu} y_{\nu}(x,t) + \frac{1}{2\pi i}\int_{A(0,\psi)} e^{ts}\,\frac{[h+k\,\lambda]\,e^{-x\sqrt{s}/\kappa}}{s\left[\sqrt{s}+h+k\,\lambda\,e^{-\ell\sqrt{s}/\kappa}\right]}\,ds\,.$$

The left part of the integration contour is formed by two rays forming an angle of $\pm\psi$ with the positive real axes, where $\pi/2 < \psi < \pi$ and the branch point 0 is bypassed to the right by a small circular arc. The first term of the right-hand side contains all contributions from the poles, located between the right vertical integration path and the two left rays of the integration contour. The poles are the zeros of

$$\sqrt{s}+h+k\,\lambda\,e^{-\ell\sqrt{s}/\kappa} \qquad (s \neq 0).$$

In order to obtain some information on the poles we consider the equation

$$p+h+k\,\lambda\,e^{-\ell p/\kappa} = 0.$$

Setting $p = \xi + i\eta$, we obtain the two equations

$$\xi+h+k\lambda\,e^{-\ell\xi/\kappa}\cos\left(\frac{\ell}{\kappa}\eta\right) = 0\,, \quad \eta - k\lambda\,e^{-\ell\xi/\kappa}\sin\left(\frac{\ell}{\kappa}\eta\right) = 0\,.$$

Inspection shows that

- there are no real poles,
- the poles occur in pairs of conjugate complex poles,
- because $(\xi+h)^2+\eta^2 = [\,(k\lambda)\,e^{-\ell\xi/\kappa}\,]^2$, we conclude that if $\xi > 0$, all poles are located inside the circle of radius $k\lambda$ centered at $(-h, 0)$. This implies that if

  $$0 < \lambda < \frac{h}{k} = H,$$

  all poles are located in the left plane and will produce exponentially decaying oscillating terms.

In Notebbook10131 the first four pairs of solutions for the physical values $\kappa = 0.1$, $K = 0.25$, $H = 1.25$, $\ell = 0.2$, $\lambda = 1$ are computed. We obtain (only the ones with positive imaginary part are listed) the p-values:

```
{-0.720049 + 1.67404 I, -2.37643 + 46.32856 I,
  -3.92532 + 1026.95 I, -5.49789 + 23848.89 I};
```

The poles are given by $s_k = p_k^2$:

```
{-2.28394 - 2.41078 i, -2140.69 - 220.193 i,
 -1.05461×10^6 - 8062.21 i, -5.6877×10^8 - 262237. i}
```

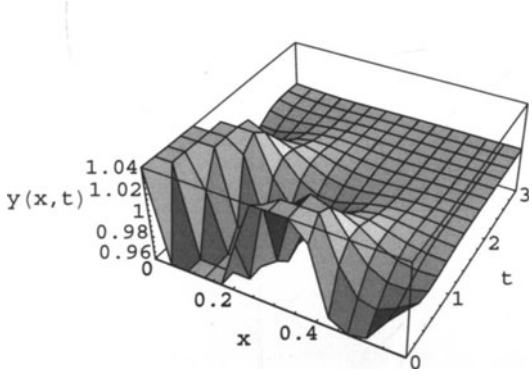
It is enough to retain the first pole since the other poles have very large negative real parts. Thus

```
polesC = {-2.28394 - 2.41078 i};
```

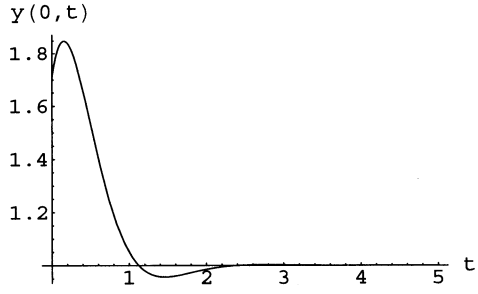
In order to obtain their contributions we can use the command `ExpansionTheorem`.

```
num = (h + k λ) Exp[-x √s / κ] /. {λ → 1, h → 0.5, k → 0.4, l → 0.2, κ → 0.1};

denom = s (√s + h + k λ Exp[-l √s / κ]) /.
   {λ → 1, h → 0.5, k → 0.4, l → 0.2, κ → 0.1};

oscillpart = ExpansionTheorem[num, denom, {}, polesC, s, t];
```

The contributions of the steady-state and the decaying oscillating part are computed via the residues at the poles and are plotted below ($y_0 = 1$).



Also shown is the controlled temperature at the surface $x = 0$ as a function of time.



Let us now compute an asymptotic expression for large values of t for

$$\frac{1}{2\pi i}\int_{A(0,\psi)} e^{ts}\,\frac{[h+k\,\lambda]\,e^{-x\sqrt{s}/\kappa}}{s\left[\sqrt{s}+h+k\,\lambda\,e^{-t\sqrt{s}/\kappa}\right]}\,ds\,.$$

We refer to the theory of Section 6.2.3.2, see also Section 6.2.5.3.

```
Int = HankelIntegral[num / denom, {s, 0}][t]
```

$$\texttt{HankelIntegral}\left[\frac{0.9\,e^{-10.\sqrt{s}\,x}}{\left(0.5+0.4\,e^{-2.\sqrt{s}}+\sqrt{s}\right)s},\ \{s,\ 0\}\right][t]$$

```
AsymptoticSeries[Int, {t, Infinity, 2}]
```

$$1. + \frac{0.507771\,(-0.246914 - 11.1111\,x)}{\sqrt{t}} - \frac{0.253885\,(1.0852 + 9.32785\,x - 12.3457\,x^2 - 185.185\,x^3)}{t^{3/2}} + \frac{1}{t^{5/2}}\,(0.380828\,(-1.33753 - 1.12449\,x + 54.26\,x^2 + 155.464\,x^3 - 102.881\,x^4 - 925.926\,x^5)) + \text{LandauO}\left[\frac{1}{t^{7/2}}\right]$$

The constant term 1 has to be discarded since it had already been accounted for in the procedure `ExpansionTheorem`.

10.2 Waves and Vibrations in Elastic Solids

A main reference for waves and vibrations in elastic solids is [Graff (1991)]. A further reference is [Churchill (1972)].

■ 10.2.1 Torsional Vibrations

The governing equation for torsional motion in a rod is given by (see [Graff (1991)])

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\varrho J}{C}\,\frac{\partial^2 \theta}{\partial t^2}$$

where $\theta(x, t)$ is the angular displacement of the rod and J is the polar moment of inertia of the cross-section, ϱ the mass density, and C the torsional rigidity of the rod. For a rod of circular cross-section we have $\varrho J/C = G/\varrho$, where G is the modulus of elasticity in shear. The torque is related to the angular displacement by

$$T = C\,\frac{\partial \theta}{\partial x}$$

where $C = J\,G$ for a rod of circular cross-section.

10.2.1.1 Resonance Phenomena

Let us consider the transmission of the torque from the driving engine of a vessel to the propeller along a long shaft. We assume that the engine has a uniform angular speed Ω. The cylindrical shaft has length L, is made of steel with modulus of elasticity in shear G and with polar moment of inertial J. The propeller has n blades and a moment of inertia J_p. Due to the presence of the bottom of the ship, a periodic perturbation of the form $B \sin(n\,\Omega\,t)$ acts on the propeller. If $\theta(x, t)$ denotes the angular displacement of the shaft, we have

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 \theta}{\partial t^2}, \qquad c = \sqrt{\frac{G}{\varrho}}\,.$$

The boundary condition near the engine is

$$\theta(0, t) = \Omega\,t.$$

The boundary condition near the propeller is

$$J_p \frac{\partial^2 \theta(L, t)}{\partial t^2} = GJ \left[\frac{\theta_L}{L} - \frac{\partial \theta(L, t)}{\partial x} \right] + B \sin(n \Omega t).$$

The initial conditions are

$$\theta(x, 0) = \frac{x}{L} \theta_L, \quad \frac{\partial \theta(x, 0)}{\partial t} = \Omega.$$

θ_L is the total steady-state twist between the engine and the propeller under steady-state working condition. We introduce the twist with respect to a steady-state revolution without perturbation. It is defined as $y(x, t) := \theta(x, t) - \Omega t - x\, \theta_L / L$. We have

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2},$$

$$y(0, t) = 0, \quad J_p \frac{\partial^2 y(L, t)}{\partial t^2} = - GJ \frac{\partial y(L, t)}{\partial x} + f(t),$$

$$y(x, 0) = 0, \quad \frac{\partial \theta(x, 0)}{\partial t} = 0,$$

where $f(t) = B \sin(n \Omega t)$.

If $Y(x, s)$ is the Laplace transform of $y(x, t)$ with respect to t, the following image system results:

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s^2}{c^2} Y(x, s) = 0,$$

$$Y(0, s) = 0, \quad s^2 Y(L, s) = - \frac{GJ}{J_p} \frac{\partial Y(L, s)}{\partial x} + \frac{1}{J_p} F(s).$$

The general solution of the differential equation is

$$Y(x, s) = A(s) \cosh(s\, x / c) + B(s) \sinh(s\, x / c).$$

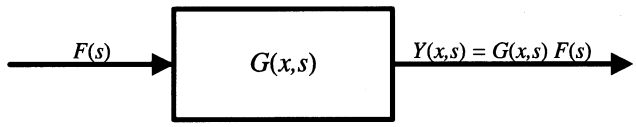
The left-boundary condition implies $A(s) = 0$. The right-boundary condition impies that

$$B = \frac{c\, F(s)\, L s}{GJ L s^2 \operatorname{Cosh}[\frac{Ls}{c}] + c L s^3 \operatorname{Sinh}[\frac{Ls}{c}] J_p}.$$

Finally, we obtain

$$Y(x, s) = \frac{1}{J_p} \frac{\sinh(\frac{s x}{c})}{s \left[s \sinh(\frac{Ls}{c}) + \frac{GJ}{c J_p} \cosh(\frac{Ls}{c}) \right]} F(s).$$

This is represented by the following input-output system



with the transfer function

$$G(x, s) = \frac{1}{J_p} \frac{\sinh(\frac{s\,x}{c})}{s\left[\, s \sinh(\frac{L\,s}{c}) + \frac{G\,J}{c\,J_p} \cosh(\frac{L\,s}{c}) \right]}.$$

This transfer function is a meromorphic function of s. The singularity $s = 0$ is removable. The zeros of the expression between the brackets are the simple poles of $G(x, s)$. These poles occur in pairs and are of the form $s_k = \pm\, i\,\alpha_k$. It is found that the α_k's are the solutions of the equation

$$\tan\left(\frac{L\,\alpha_k}{c}\right) = \frac{G\,J}{c\,J_p}\,\frac{1}{\alpha_k} = \frac{G\,J\,L}{c^2\,J_p}\,\frac{c}{L\,\alpha_k} = \frac{\varrho\,J\,L}{J_p}\,\frac{c}{L\,\alpha_k}.$$

If the positive solutions u_k of the equation

$$\tan u_k = \frac{\varrho\,J\,L}{J_p}\,\frac{1}{u_k}$$

are computed, we have

$$\alpha_k = u_k\,c/L = u_k\,\frac{1}{L}\sqrt{\frac{G}{\varrho}}.$$

The α_k are the resonance frequencies of the propeller shaft. This is seen as follows. If $F(s)$ is $\omega/(s^2 + \omega^2) = \omega/((s - i\,\omega)\,(s + i\,\omega))$ and $\omega = \alpha_k$ for some k, then the otherwise simple poles of $G(x, s)\,F(s)$ become poles of order 2, and the solution will contain terms proportional to t. This means that the amplitude of the vibration will go to infinity because we neglected damping in the model. In reality, damping is always present and the amplitude will only take large, but limited values at resonance. Let us study a real situation.

Example. $L = 50\,m$, $G = 8.9\,10^{10}\,N/m^2$, $\varrho = 7.84\,10^3\,\mathrm{kg}/m^3$, $J = 4.09\,10^{-4}\,m^4$. The propeller has n = 4 blades and a moment of inertia of $J_p = 250\,\mathrm{kg}\,m^2$. This gives $u_1 = 0.025$, $\alpha_1 = 53.91$, which corresponds to 514.8 revolutions per minute, i.e. with four blades we would have the fundamental resonance at 128.7 revolutions per minute, a value to be concerned about. Let us now compute an approximation of the solution. We will take the first 20 terms of the series of the original function. Thus we need the first 20 poles of $G(x, s)\,\Omega/(s^2 + \Omega^2)$.

```
physicalvalues =
 {L → 50, G → 8.9 10^10, ϱ → 7.84 10^3, J → 4.09 10^-4, Jp → 250};
c = c /. (c → Sqrt[G/ϱ] /. physicalvalues);

polesC =
 Table[c I / 50 (u /. FindRoot[Tan[u] == ((ϱ J L / Jp) /. physicalvalues) / u,
       {u, Pi (k - 1/2) - 0.5/k}]), {k, 20}] // Flatten

{48.8235 i, 224.512 i, 430.143 i, 639.64 i, 850.214 i, 1061.23 i, 1272.48 i,
 1483.85 i, 1695.3 i, 1906.81 i, 2118.35 i, 2329.93 i, 2541.52 i, 2753.13 i,
 2964.75 i, 3176.39 i, 3388.03 i, 3599.68 i, 3811.33 i, 4022.99 i}
```

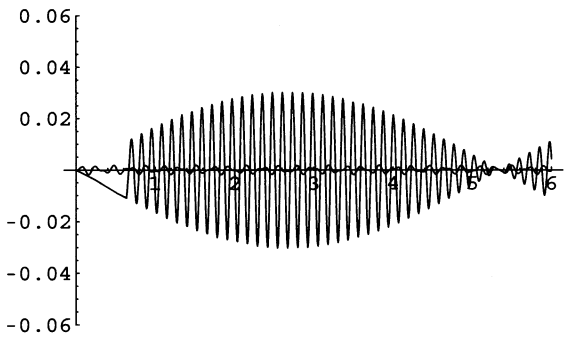
Now we use the expansion theorem for meromorphic functions. We take the excitation $1000\sin(\Omega\,t)$, first with $\Omega = 30$ away from a resonance frequency, then with $\Omega = 50$ near the first resonance frequency.

```
Ω = 30;
y1 = ExpansionTheorem[ (1000 Ω Sinh[x s / c] / s / Jp) /. physicalvalues,
     (s Sinh[L s / c] + G J / (c Jp) Cosh[L s / c]) (s^2 + Ω^2) /. physicalvalues,
     {}, Prepend[polesC, Ω I] /. physicalvalues, s, t] // Chop;
```

```
Ω = 50;
y2 = ExpansionTheorem[ (1000 Ω Sinh[x s / c] / s / Jp) /. physicalvalues,
    (s Sinh[L s / c] + G J / (c Jp) Cosh[L s / c]) (s^2 + Ω^2) /. physicalvalues,
    {}, Prepend[polesC, Ω I] /. physicalvalues, s, t] // Chop;
```

The following are the two vibrations in the middle of the propeller shaft. Near the resonance frequency we observe vibrations with large periodically varying amplitudes.

```
Plot[Evaluate[{y1, y2} /. x → 25], {t, 0, 6}, PlotRange → {-0.06, 0.06}];
```



■ 10.2.2 Transverse Motions of Thin Rods or Beams

10.2.2.1 The Governing Equation

We assume the basic hypothesis of the Bernoulli–Euler theory of beams [Graff (1991)]:

- Plane cross-sections initially perpendicular to the axis of the beam remain plane and perpendicular to the neutral axis during bending (shearing deformations are neglected).
- Rotational-inertia effects are neglected (assumption of low frequencies).
- Curvature of the beam is approximated by the second derivative of the elastic line (assumption of small deflections).

Under these assumptions the governing equation for the transverse motion of a thin rod or beam is given by

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \varrho A\frac{\partial^2 y}{\partial t^2} = q(x,t),$$

where $y(x, t)$ is the transverse deflection of the neutral axis of the beam, ϱ is the mass density per unit volume, A is the cross-sectional area, I is the moment of inertia of the cross-section with respect to the neutral axis, and $q(x, t)$ is the loading of the beam. In the following we assume that Young's modulus of elasticity E and the cross-section of the beam are constant, i.e. do not depend on x. In this case the above partial differential equation reduces to

$$\frac{\partial^4 y}{\partial x^4} + 4a^4\frac{\partial^2 y}{\partial t^2} = \frac{1}{EI}q(x,t), \quad a = \sqrt[4]{\frac{\varrho A}{4EI}}. \tag{10.9}$$

The equation of motion has to be completed by boundary conditions. At the boundary $x = a$ ($a = 0$ or $a = \ell$) the beam can be

- pinned, i.e.,

$$y(a, t) = 0, \quad \frac{\partial^2 y(a, t)}{\partial x^2} = 0,$$

- fixed or clamped, i.e.,

$$y(a, t) = 0, \quad \frac{\partial y(a, t)}{\partial x} = 0,$$

- free, i.e.,

$$\frac{\partial^2 y(a, t)}{\partial x^2} = 0, \quad \frac{\partial^3 y(a, t)}{\partial x^3} = 0.$$

Futhermore, initial conditions have to be specified.

$$y(x, 0) = f(x), \quad \frac{\partial y(x, 0)}{\partial t} = v(x) \qquad (0 < x < \ell).$$

10.2.2.2 Motion of a Tuning Fork

We consider a tuning fork (a beam) which is clamped at $x = 0$. At the instant $t = 0$ we apply a slap to the free end at $x = \ell$. The strike can be modeled by the following boundary condition at $x = \ell$:

$$\frac{\partial^2 y(\ell, t)}{\partial x^2} = 0, \quad -EI \frac{\partial^3 y(\ell, t)}{\partial x^3} = h\,\delta(t).$$

The boundary conditions at $x = 0$ (fixed end) is

$$y(0, t) = 0, \quad \frac{\partial y(0, t)}{\partial x} = 0.$$

We suppose zero initial conditions, i.e.,

$$y(x, 0) = 0, \quad \frac{\partial y(x, 0)}{\partial t} = 0 \qquad (0 < x < \ell).$$

If $Y(x, s)$ is the Laplace transform of $y(x, t)$ with respect to t, the image equation of (10.9) with the above initial conditions becomes

$$\frac{\partial^4 Y}{\partial x^4} + 4a^4 s^2 Y(x, s) = 0.$$

The transformed boundary conditions are

$$Y(0, s) = 0, \quad \frac{\partial Y(0, s)}{\partial x} = 0, \quad \frac{\partial^2 Y(\ell, s)}{\partial x^2} = 0, \quad \frac{\partial^3 Y(\ell, s)}{\partial x^3} = -k.$$

In Notebook1021 the above differential equation with respect to the boundary conditions is solved. The solution can be written in the following form:

$$Y(x, s) = \frac{k}{2a^3}\left\{\frac{\sinh(a\,x\sqrt{s})\cosh(a\,\ell\sqrt{s})\cos[a\,(\ell-x)\sqrt{s}]}{s^{3/2}\,[2+\cosh(2\,a\,\ell\sqrt{s})+\cos(2\,a\,\ell\sqrt{s})]}\right.$$
$$\left.-\frac{\sin(a\,x\sqrt{s})\cos(a\,\ell\sqrt{s})\cosh[a\,(\ell-x)\sqrt{s}]}{s^{3/2}\,[2+\cosh(2\,a\,\ell\sqrt{s})+\cos(2\,a\,\ell\sqrt{s})]}\right\}.$$

In spite of the square root, $Y(x, s)$ is a meromorphic function and we have no branch point at $s = 0$ (look at the Taylor series expansions of the various sub-expressions). We have a simple pole at $s = 0$, while the other poles are also simple and given by

$$\nu_k = \frac{i}{2a^2\ell^2}u_k^2, \quad \nu_{-k} = -\frac{i}{2a^2\ell^2}u_k^2, \qquad k = 1, 2, 3, \ldots .$$

The u_k are the positive solution of the equation $\cos u_k = -1/\cosh u_k$. We have the same situation as in the Subsection 10.1.2.3: $Y(x, s) = P(s)/(s\,Q(s))$ with

$$P(s) = \frac{k}{2a^3}\left\{\frac{\sinh(a\,x\sqrt{s})\cosh(a\,\ell\sqrt{s})\cos[a\,(\ell-x)\sqrt{s}]}{\sqrt{s}}\right.$$
$$\left.-\frac{\sin(a\,x\sqrt{s})\cos(a\,\ell\sqrt{s})\cosh[a\,(\ell-x)\sqrt{s}]}{\sqrt{s}}\right\}.$$

$$Q(s) = 2 + \cosh(2\,a\,\ell\sqrt{s}) + \cos(2\,a\,\ell\sqrt{s}).$$

The inverse Laplace transform will be of the form

$$y(x, t) = \frac{P(0)}{Q(0)} + \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} \frac{P(\nu_k)}{\nu_k\,Q'(\nu_k)}\exp(\nu_k\,t).$$

This represents an infinite superposition of harmonic oscillations with the angular frequencies

$$\omega_k = \frac{\nu_k}{i} = \frac{1}{2a^2\ell^2}u_k^2$$
$$= \frac{1}{\ell^2}\sqrt{\frac{EI}{\rho A}}\,u_k^2 \quad (k = 1, 2, 3, \ldots, \quad \cos u_k = -1/\cosh u_k).$$

The fundamental angular frequency is

$$\omega_0 = \frac{4}{\ell^2}\sqrt{\frac{EI}{\rho A}}\,u_0^2 = \frac{3.51602}{\ell^2}\sqrt{\frac{EI}{\rho A}}.$$

Observe that $P(0) = 0$, and that the terms of the infinite series corresponding to the index $-k$ are the complex conjugates of those for $+k$. We obtain

$$y(x, t) = 2\sum_{k=1}^{\infty}\frac{1}{\omega_k}\left\{\mathrm{Im}\left[\frac{P(i\,\omega_k)}{Q'(i\,\omega_k)}\right]\cos\omega_k\,t + \mathrm{Re}\left[\frac{P(i\,\omega_k)}{Q'(i\,\omega_k)}\right]\sin\omega_k\,t\right\}.$$

In Notebook1021 this expression is evaluated by use of the command `ExpansionTheorem` and one can watch an animation of the motion of the tuning fork.

10.2.2.3 Loading of the form $q(x, t) = q(x)\exp(-i\omega t)$

If the expression $y(x, t) = w(x)\exp(-i\omega t)$ is inserted into Equation (10.9), we obtain the ordinary differential equation

$$\frac{d^4 w}{dx^4} - b^4 w(x) = q(x)/EI, \quad b^4 = 4a^4\omega^2 .$$

We now apply the Laplace transformation with respect to x. Let $W(s)$ and $Q(s)$ denote the Laplace transforms of $w(x)$ and $q(x)$, respectively. The Laplace transform $W(s)$ can then be written as

$$W(s) = w(0)\frac{s^3}{s^4 - b^4} + w'(0)\frac{s^2}{s^4 - b^4} + w''(0)\frac{s}{s^4 - b^4} + w^{(3)}(0)\frac{1}{s^4 - b^4} + \frac{1}{EI}\frac{Q(s)}{s^4 - b^4} .$$

Two of the four constants $w(0)$, $w'(0)$, $w''(0)$ and $w^{(3)}(0)$ are zero because of the boundary conditions at $x = 0$. The other two must be found by applying the boundary conditions at $x = \ell$. This must be done in the original x-domain. Let us introduce the following four functions.

```
w0 =
 InverseLaplaceTransform[s^3 / (s^4 - b^4), s, x] /. ToHyperbolicFunctions
```

$$\frac{1}{2}\,\text{Cos}[b\,x] + \frac{1}{2}\,\text{Cosh}[b\,x]$$

```
w1 = InverseLaplaceTransform[s^2 / (s^4 - b^4), s, x] /.
   ToHyperbolicFunctions // Simplify
```

$$\frac{\text{Sin}[b\,x] + \text{Sinh}[b\,x]}{2\,b}$$

```
w2 =
 InverseLaplaceTransform[s / (s^4 - b^4), s, x] /. ToHyperbolicFunctions
```

$$-\frac{\text{Cos}[b\,x]}{2\,b^2} + \frac{\text{Cosh}[b\,x]}{2\,b^2}$$

```
w3 = InverseLaplaceTransform[1 / (s^4 - b^4), s, x] /.
   ToHyperbolicFunctions // Simplify
```

$$\frac{-\text{Sin}[b\,x] + \text{Sinh}[b\,x]}{2\,b^3}$$

By the convolution theorem we can write

$$w(x) = \frac{w(0)}{2}[\cos(b\,x) + \cosh(b\,x)] + \frac{w'(0)}{2b}[\sin(b\,x) + \sin(b\,x)] + \frac{w''(0)}{2b^2}[-\cos(b\,x) + \cos(b\,x)] + \frac{w^{(3)}(0)}{2b^3}[-\sin(b\,x) + \sinh(b\,x)] + \frac{1}{2EIb^3}\int_0^x [-\sin(b(x-\xi)) + \sinh(b(x-\xi))]\, q(\xi)\, d\xi .$$

After having determined $w(0)$, $w'(0)$, $w''(0)$ and $w^{(3)}(0)$ from the boundary conditions, the solution is

$$y(x, t) = w(x) \exp(-i\omega t).$$

Let us apply this procedure to the following example: A beam, pinned at $x = 0$ and $x = \ell$, is loaded by a harmonic excitation at the center $x = \ell/2$. Thus

$$q(x, t) = k\, \delta(x - \ell/2) \exp(-i\omega t).$$

The boundary conditions are

$$y(0, t) = 0, \quad \frac{\partial^2 y(0, t)}{\partial x^2} = 0, \quad y(\ell, t) = 0, \quad \frac{\partial^2 y(\ell, t)}{\partial x^2} = 0.$$

The function $w(x)$ now becomes

$$w(x) = \frac{A}{2b}[\sin(b x) + \sin(b x)] + \frac{B}{2b^3}[-\sin(b x) + \sinh(b x)] + \frac{k}{2EIb^3}\int_0^x [-\sin(b(x - \xi)) + \sinh(b(x - \xi))]\, \delta(\xi - \ell/2)\, d\xi$$

where A and B must be determined by the boundary conditions at $x = \ell$. Using the unit-step function $u = u(x)$, we can write

$$w(x) = \frac{A}{2b}[\sin(b x) + \sin(b x)] + \frac{B}{2b^3}[-\sin(b x) + \sinh(b x)] + u(x - \ell/2)\, \frac{k}{2EIb^3}[-\sin(b(x - \ell/2)) + \sinh(b(x - \ell/2))].$$

In Notebook1021 the constants A and B are computed and the solution finally becomes

$$y(x, t) = \frac{c}{2b^3}\left(\frac{\sin(b x)}{\cos(b\ell/2)} - \frac{\sinh(b x)}{\cosh(b\ell/2)} - 2\, u(x - \ell/2)\,(\sin[b\,(x - \ell/2)] + \sinh[b\,(x - \ell/2)]\,)\right) \exp(-i\omega t)$$

with $c = k/(2EI)$ and $b = a\sqrt{\omega}$.

10.2.2.4 Normal Modes and the Motion of a Xylophone Bar

Let us solve

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0, \quad c = \sqrt{\frac{EI}{\varrho A}} \tag{10.10}$$

by separating variables. We set $y(x, t) = f(x)\, g(t)$ and obtain two ordinary differential equations

$$\frac{d^4 f}{dx^4} - \frac{\omega^2}{c^2} f(x) = 0, \quad \frac{d^2 g}{dt^2} + \omega^2 g(t) = 0$$

where ω is a constant independent of x and t. A boundary condition at $x = \ell$ for $y(x, t)$ of the form $(\partial/\partial x)^k\, y(\ell, t) = 0$ becomes a boundary condition of the form $f^{(k)}(\ell) = 0$ for the first of the two ordinary differential equations. The solution of this differential equation is of the form

$$f(x) = A_1 (\cos \beta x + \cosh \beta x) + A_2 (\cos \beta x - \cosh \beta x) + B_1 (\sin \beta x + \sinh \beta x) + B_2 (\sin \beta x - \sinh \beta x)$$

with $\beta = \sqrt{\omega / c}$. The boundary conditions imply that only a discrete sequence ω_k ($k = 1, 2, 3 \ldots$) of values are possible. These values are the natural frequencies of the boundary value problem of the beam. The corresponding solution $f_k(x)$ of

$$\frac{d^4 f_k}{dx^4} - \frac{\omega_k{}^2}{c^2} f_k(x) = 0 \tag{10.11}$$

is called the normal mode corresponding to the frequency ω_k. We now consider two normal modes f_n and f_m belonging to two different frequencies ω_n and ω_m.

$$\frac{d^4 f_n}{dx^4} - \frac{\omega_n{}^2}{c^2} f_n(x) = 0, \quad \frac{d^4 f_m}{dx^4} - \frac{\omega_m{}^2}{c^2} f_m(x) = 0.$$

By multiplying the first equation by f_m and the second by f_m, then taking the difference between the two equations, and finally integrating over the interval $[0, \ell]$ of the beam, we obtain

$$\int_0^\ell (f_n{}^{(4)}(x) f_m(x) - f_n(x) f_m{}^{(4)}(x)) \, dx = \frac{\omega_n{}^2 - \omega_m{}^2}{c^2} \int_0^\ell f_n(x) f_m(x) \, dx .$$

By repeating four times an integration by parts of the first term in the integral of the left-hand side, we obtain

$$[f_m(x) f_n^{(3)}(x) - f_m{}'(x) f_n{}''(x) + f_m{}''(x) f_n{}'(x) + f_m^{(3)}(x) f_n(x)]_0^\ell = \frac{\omega_n{}^2 - \omega_m{}^2}{c^2} \int_0^\ell f_n(x) f_m(x) \, dx .$$

We note that any combination of the various simple boundary conditions such as pinned-pinned, pinned-clamped, clamped-clamped , and so on, causes the left-hand side to vanish. The result is that

$$\langle f_n, f_n \rangle := \int_0^\ell f_n(x) f_m(x) \, dx = 0 \quad \text{for } n \neq m,$$

i.e. the normal modes form an orthogonal function system on the interval of the beam. If $h(x)$ is a given function on $[0, \ell]$, we can expand it in the normal modes and write

$$h(x) = \sum_{k=1}^{\infty} \frac{\langle h, f_k \rangle}{\langle f_k, f_k \rangle} f_k(x) .$$

For a pinned-pinned beam, it is found in Notebook1031.nb, that the natural frequencies are $\omega_k = c [k \pi / \ell]^2$ and that the corresponding normal modes are given by $f_k(x) = \sin(\frac{k\pi}{\ell} x)$ with $\langle f_k, f_k \rangle = \ell / 2$. In this case we have

$$h(x) = \sum_{n=1}^{\infty} b_k \sin\left(\frac{k\pi}{\ell} x\right), \quad b_k = \frac{2}{\ell} \int_0^\ell h(x) \sin\left(\frac{k\pi}{\ell} x\right) dx.$$

Now consider the inhomogeneous equation of the beam.

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{q(x,t)}{EI}, \quad c = \sqrt{\frac{EI}{\varrho A}}.$$

We take the Laplace transform with respect to t by assuming zero initial conditions and obtain

$$\frac{\partial^4 Y}{\partial x^4} + \frac{s^2}{c^2} Y(x,s) = \frac{1}{EI} Q(x,s).$$

We expand $Y(x,s)$ and $Q(x,s)$ in a series of the normal modes of the boundary problem.

$$Y(x,s) = \sum_{k=1}^{\infty} Y_k(s) f_k(x), \quad Y_k(s) = \frac{\langle Y, f_k\rangle}{\langle f_k, f_k\rangle},$$

$$Q(x,s) = \sum_{k=1}^{\infty} Q_k(s) f_k(x), \quad Q_k(s) = \frac{\langle Q, f_k\rangle}{\langle f_k, f_k\rangle}.$$

We substitute these expressions into the above differential equation and use (7.11) to obtain

$$\sum_{k=1}^{\infty}\left[\frac{{\omega_k}^2 + s^2}{c^2} Y_k(s) - \frac{Q_k(s)}{EI}\right] f_k(x) = 0,$$

from which

$$Y_k(s) = \frac{c^2}{EI}\frac{1}{{\omega_k}^2 + s^2} Q_k(s)$$

results and

$$Y(x,s) = \frac{c^2}{EI}\sum_{k=1}^{\infty} f_k(x)\frac{1}{{\omega_k}^2 + s^2} Q_k(s).$$

By the convolution theorem we finally obtain the result

$$y(x,t) = \frac{c^2}{EI}\sum_{k=1}^{\infty}\frac{f_k(x)}{\omega_k}\int_0^t \sin[\omega_k(t-\tau)]\, q_k(\tau)\, d\tau$$

with

$$q_k(\tau) = \frac{\langle q(x,\tau), f_k(x)\rangle}{\langle f_k(x), f_k(x)\rangle}.$$

Let us now consider the case of a simple supported beam of length ℓ (the wooden bar of a xylophone) subjected to a strike at $x = \xi$. It is modeled by

$$q(x,t) = P\,\delta(x-\xi)\,\delta(t).$$

We have

$$q_k(\tau) = \frac{2P}{\ell}\,\delta(\tau)\sin\left(\frac{k\pi}{\ell}\xi\right),$$

$$\int_0^t \sin[\omega_k(t-\tau)]\; q_k(\tau)\, d\tau \;=\; \frac{2P}{\ell}\sin\!\left(\frac{k\pi}{\ell}\,\xi\right)\int_0^t \sin[\omega_k(t-\tau)]\;\delta(\tau)\, d\tau$$

$$= \frac{2P}{\ell}\sin\!\left(\frac{k\pi}{\ell}\,\xi\right)\sin(\omega_k\, t) \;=\; \frac{2P}{\ell}\sin\!\left(\frac{k\pi}{\ell}\,\xi\right)\sin(c\;[k\pi/\ell]^2\, t)\,,$$

$$y(x,\,t) \;=\; \frac{2P\ell\, c}{\pi^2\, E\, I}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\!\left(\frac{k\pi}{\ell}\,x\right)\sin\!\left(\frac{k\pi}{\ell}\,\xi\right)\sin(c\;[k\pi/\ell]^2\, t)\;.$$

If the bar is hit in the middle $\xi = \ell/2$, we have

$$y(x,\,t) \;=\; \frac{2P\ell\, c}{\pi^2\, E\, I}\sum_{k=1,3,5,\ldots}^{\infty}\frac{(-1)^{k-1}}{k^2}\sin\!\left(\frac{k\pi}{\ell}\,x\right)\sin\!\left(c\,\frac{k^2\,\pi^2}{\ell^2}\,t\right).$$

Chapter 11
Further Techniques

11.1 Duhamel's Formulas

Duhamel's formulas allow us to write the solution of a complicated problem as a convolution integral containing the solution of a simpler associated problem.

■ 11.1.1 Duhamel's Formulas for Heat Conduction Problems

Let us consider the problem of heat conduction

$$\begin{aligned} &\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = 0, \quad (0 < x < \ell,\ t > 0), \\ &y(0, t) = 0, \quad y(\ell, t) = f(t), \\ &y(x, 0) = 0, \quad 0 < x < \ell, \end{aligned} \tag{11.1}$$

where

- the partial differential equation is homogeneous,
- one of the two boundary conditions is zero,
- the other boundary condition is a given function of time $f(t)$,
- the initial condition is zero.

Assuming $y(x, t) \circ\!-\!\bullet\ Y(x, s)$, the image system of (11.1) is

$$\begin{aligned} &\frac{\partial^2 Y}{\partial x^2} - \frac{s}{\kappa^2} Y(x, s) = 0, \quad (0 < x < \ell), \\ &Y(0, s) = 0, \quad Y(\ell, s) = F(s). \end{aligned}$$

With problem (11.1) we associate one of the two simpler problems

$$\begin{aligned} &\frac{\partial^2 h}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial h}{\partial t} = 0, \quad (0 < x < \ell,\ t > 0), \\ &h(0, t) = 0, \quad h(\ell, t) = u(t), \\ &h(x, 0) = 0, \quad 0 < x < \ell. \end{aligned} \tag{11.2}$$

where the given function $f(t)$ is replaced by the unit-step, or

$$\begin{aligned} &\frac{\partial^2 g}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial g}{\partial t} = 0, \quad (0 < x < \ell,\ t > 0), \\ &g(0, t) = 0, \quad g(\ell, t) = \delta(t), \\ &g(x, 0) = 0, \quad 0 < x < \ell. \end{aligned} \tag{11.3}$$

where the given function $f(t)$ is replaced by the Dirac impulse.

With $h(x, t) \circ\!-\!\bullet H(x, s)$ the image system of (11.2) becomes

$$\frac{\partial^2 H}{\partial x^2} - \frac{s}{\kappa^2} H(x, s) = 0, \quad (0 < x < \ell),$$
$$H(0, s) = 0, \quad H(\ell, s) = 1/s.$$

The important point now is that if $H(x, s)$ solves the above problem, the image function

$$Y(x, s) := s\, H(x, s)\, F(s)$$

solves (11.1). This is seen as follows:

- Because $H(x, s)$ is the solution of a homogeneous ordinary linear differential equation, the scalar multiple $s\, F(s)\, H(x, s)$ is a solution, too.
- $Y(0, s) := s\, H(0, s)\, F(s) = s\, 0\, F(s) = 0.$
- $Y(\ell, s) := s\, H(\ell, s)\, F(s) = s\ \ 1/s\ F(s) = F(s).$

The convolution theorem implies that the solution of problem (11.1) can be written in the form

$$y(x, t) = \frac{\partial}{\partial t} \int_0^t h(x, t-\tau)\, f(\tau)\, d\tau = \int_0^t \frac{\partial}{\partial t} h(x, t-\tau)\, f(\tau)\, d\tau \tag{11.4}$$

where $h(x, t)$ is the solution of the simpler problem (11.2). This is the classical *Duhamel formula*. If instead of (11.2) we use the solution of (11.3), we obtain

$$y(x, t) = \int_0^t g(x, t-\tau)\, f(\tau)\, d\tau. \tag{11.5}$$

The method just explained for the above simple boundary value problem can be applied word by word to more involved problems, where the differential operator $\partial^2 / \partial x^2$ is replaced by the complete Laplace operator ∇^2 for all three space coordinates, or where the boundary conditions are in a modified form.

Example. Suppose that a semi-infinite solid is heated at the left boundary $x = 0$ by radiation from a medium with a variable temperature $f(t)$. The initial temperature of the solid is zero. The boundary condition at $x = 0$ in this case is

$$\frac{\partial y(0, t)}{\partial x} = k\,[\, y(0, t) - f(t)].$$

The image equation is

$$\frac{\partial Y(0, s)}{\partial x} = k\,[\, Y(0, s) - F(s)].$$

With $Y(x, s) = A(s) \exp\left[-x\sqrt{s}\,/k\right]$

we obtain

$$Y(x, s) = \frac{k\kappa}{\sqrt{s} + k\kappa}\, e^{-x\sqrt{s}/\kappa}\ F(s).$$

In Section 10.1.1.1, 2nd Example, we solved this problem for the constant medium temperature $f(t) = 1$ and obtained

$$H(x, s) = \frac{k\kappa}{s\left(\sqrt{s} + k\kappa\right)} e^{-x\sqrt{s}/\kappa}$$

where $h(x, t)$ satisfied the same partial differential equation but with the boundary condition

$$\frac{\partial h(0, t)}{\partial x} = k\,[\,h(0, t) - u(t)]\,,$$

i.e. in the image domain with

$$\frac{\partial H(0, s)}{\partial x} = k\left[H(0, s) - \frac{1}{s}\right].$$

By setting $Y(x, s) = s\,F(s)\,H(x, s)$, we verify

$$\frac{\partial Y(0, s)}{\partial x} = s\,F(s)\,\frac{\partial H(0, s)}{\partial x} = s\,F(s)\,k\left[H(0, s) - \frac{1}{s}\right]$$
$$= k\,[\,s\,F(s)\,H(0, s) - F(s)] = k\,[\,Y(0, s) - F(s)]$$

and therefore obtain the Duhamel formula

$$y(x.t) = \frac{\partial}{\partial t}\int_0^t h(x, t-\tau)\,f(\tau)\,d\tau = \int_0^t \frac{\partial}{\partial t}\,h(x, t-\tau)\,f(\tau)\,d\tau$$

with

$$h(x, t) = \mathcal{L}^{-1}\left[\frac{k\kappa}{s\left(\sqrt{s} + k\kappa\right)} e^{-x\sqrt{s}/\kappa}\right](t)\,.$$

```
h = InverseLaplaceTransform[k κ Exp[-x Sqrt[s] / κ] / (s (Sqrt[s] + k κ)), s, t]
```

$$k\kappa\left(\frac{\text{Erfc}\left[\frac{x}{2\sqrt{t\,\kappa}}\right]}{k\kappa} - \frac{e^{k\,x+k^2\,t\,\kappa^2}\,\text{Erfc}\left[\frac{x}{2\sqrt{t\,\kappa}} + k\sqrt{t\,\kappa}\right]}{k\kappa}\right)$$

```
dh = D[h, t] // Simplify
```

$$k\kappa\left(\frac{e^{-\frac{x^2}{4\,t\,\kappa^2}}}{\sqrt{\pi}\,\sqrt{t}} - e^{k\,(x+k\,t\,\kappa^2)}\,k\kappa\,\text{Erfc}\left[\frac{x + 2\,k\,t\,\kappa^2}{2\sqrt{t\,\kappa}}\right]\right)$$

By virtue of Duhamel's formula, the solution is given in the form

$$y(x, t) = k\kappa\int_0^t \left\{\frac{1}{\sqrt{\pi}\,\sqrt{t-\tau}}\,\text{Exp}\left[-\frac{x^2}{4\,(t-\tau)\,\kappa^2}\right]\right.$$
$$\left. - k\kappa\,\text{Exp}[k\,(x + k(t-\tau)\,\kappa^2)]\,\text{Erfc}\left[\frac{x + 2\,k\,(t-\tau)\,\kappa^2}{2\sqrt{t-\tau}\,\kappa}\right]\right\} f(\tau)\,d\tau\,.$$

■ 11.1.2 Duhamel's Formulas for the Wave Equation

Let for example

$$
\begin{aligned}
&\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - b\frac{\partial y}{\partial t} - k\,y(x,t) = 0\,,\\
&a_1\,y(0,t) + a_2\frac{\partial y(0,t)}{\partial x} = 0\,,\\
&b_1\,y(\ell,t) + b_2\frac{\partial y(\ell,t)}{\partial x} = f(t)\,,\\
&y(x,0) = \frac{\partial y(x,0)}{\partial t} = 0 \qquad (0 < x < \ell)
\end{aligned}
\tag{11.6}
$$

be a given boundary value problem, where the important point again is that

- the partial differential equation is homogeneous,
- one of the two boundary conditions is zero,
- the other boundary condition contains a given function of time $f(t)$,
- the initial conditions are zero.

Again, we replace the above problem by a simpler one, where $f(t)$ is replaced by a unit-step (or the Dirac impulse):

$$
\begin{aligned}
&\frac{\partial^2 h}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} - b\frac{\partial h}{\partial t} - k\,h(x,t) = 0\,,\\
&a_1\,h(0,t) + a_2\frac{\partial h(0,t)}{\partial x} = 0\,,\\
&b_1\,h(\ell,t) + b_2\frac{\partial h(\ell,t)}{\partial x} = u(t) \qquad (\text{or } \ldots = \delta(t))\,,\\
&h(x,0) = \frac{\partial h(x,0)}{\partial t} = 0 \qquad (0 < x < \ell).
\end{aligned}
\tag{11.7}
$$

By taking the Laplace transform, we obtain the image system

$$
\begin{aligned}
&\frac{\partial^2 Y}{\partial x^2} - \left[\frac{s^2}{c^2} + b\,s + k\right] Y(x,s) = 0\,,\\
&a_1\,Y(0,s) + a_2\frac{\partial Y(0,s)}{\partial x} = 0\,,\\
&b_1\,Y(\ell,s) + b_2\frac{\partial Y(\ell,s)}{\partial x} = F(s)\,,
\end{aligned}
\tag{11.8}
$$

respectively,

$$
\begin{aligned}
&\frac{\partial^2 H}{\partial x^2} - \left[\frac{s^2}{c^2} + b\,s + k\right] H(x,s) = 0\,,\\
&a_1\,H(0,s) + a_2\frac{\partial H(0,s)}{\partial x} = 0\,,\\
&b_1\,H(\ell,s) + b_2\frac{\partial H(\ell,s)}{\partial x} = \frac{1}{s} \qquad (\text{or } \ldots = 1).
\end{aligned}
$$

If $H(x,s)$ is a solution of the above boundary value problem, then $Y(x,s) = s\,F(s)\,H(x,s)$ (or $F(s)\,H(x,s)$) is the solution of (11.8), because

- the ordinary differential equation is homogeneous

- the two boundary conditions are satisfied, since

$$a_1\, Y(0, s) + a_2 \frac{\partial Y(0, s)}{\partial x} = s\, F(s)\left[a_1\, H(0, s) + a_2 \frac{\partial H(0, s)}{\partial x}\right] = 0$$

$$b_1\, Y(\ell, s) + b_2 \frac{\partial Y(\ell, s)}{\partial x} = s\, F(s)\left[b_1\, H(\ell, s) + b_2 \frac{\partial H(\ell, s)}{\partial x}\right]$$

$$= s\, F(s)\, \frac{1}{s} = F(s).$$

A similar result holds in the case of the Dirac impulse in place of $u(t)$. We obtain, by the convolution theorem, Duhamel's formula

$$y(x, t) = \frac{\partial}{\partial t} \int_0^t h(x, t-\tau)\, f(\tau)\, d\tau = \int_0^t \frac{\partial}{\partial t} h(x, t-\tau)\, f(\tau)\, d\tau \tag{11.9}$$

for the case where $f(t)$ is replaced by $u(t)$, respectively,

$$y(x, t) = \int_0^t h(x, t-\tau)\, f(\tau)\, d\tau \tag{11.10}$$

in the case where $f(t)$ is replaced by $\delta(t)$.

As an example, let us consider the longitudinal vibrations of a bar with a mass M attached to the right end $x = \ell$. The left end $x = 0$ of the bar is fixed. Let us assume further that a longitudinal force $f(t)$ acts on the mass M. The elastic displacement within the mass is assumed negligible, but not within the bar. There the longitudinal displacement is described by $y(x, t)$. The force exerted by the bar onto the attached mass M is $-a\, E\, \partial y(\ell, t)/\partial x$, where a is the area of the cross section of the bar and E is Young's modulus of elasticity. We assume further that the bar initially is at rest and unstrained. If ρ denotes the mass density of the bar, the model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = 0 \qquad \left(c^2 = \frac{E}{\rho}\right), \quad (0 < x < \ell),$$

$$y(0, t) = 0, \quad M \frac{\partial^2 y(\ell, t)}{\partial t^2} + a\, E \frac{\partial y(\ell, t)}{\partial x} = f(t),$$

$$y(x, 0) = \frac{\partial y(x, 0)}{\partial t} = 0.$$

Let us consider the simpler associated problem

$$\frac{\partial^2 h}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 h}{\partial t^2} = 0,$$

$$h(0, t) = 0, \quad M \frac{\partial^2 h(\ell, t)}{\partial t^2} + a\, E \frac{\partial h(\ell, t)}{\partial x} = \delta(t),$$

$$h(x, 0) = \frac{\partial h(x, 0)}{\partial t} = 0.$$

where the given force is replaced by a Dirac impulse. Via the Laplace transform with respect to t we obtain for $H(x, s)$ ■ – ▫ $h(x, t)$

$$\frac{\partial^2 H}{\partial x^2} - \frac{s^2}{c^2} H(x, s) = 0,$$

$$H(0, s) = 0, \quad M s^2 H(\ell, s) + a E \frac{\partial H(\ell, s)}{\partial x} = 1.$$

A straightforward verification shows

$$H(x, s) = \frac{c}{a E} \frac{\sinh(s x / c)}{s [\cosh(s \ell / c) + k s \sinh(s \ell / c)]}$$

solves the above problem with $k = c M / (a E)$. Hence $Y(x, s) := F(s) H(x, s)$ is the solution of the originally given problem because

- $Y(x, s)$ satisfies the same homogeneous ordinary differential equation as $H(x, s)$,
- $Y(0, s) = F(s) H(0, s) = 0$,
- $M s^2 Y(\ell, s) + a E \partial Y(\ell, s) / \partial x = F(s) [M s^2 H(\ell, s) + a E \partial H(\ell, s) / \partial x] = F(s) \cdot 0 = 0$.

The convolution theorem implies that $y(x, t)$ is given by Duhamel's formula

$$y(x, t) = \int_0^t h(x, t - \tau) f(\tau) d\tau$$

where $h(x, t)$ is the inverse Laplace transform of $H(x, s)$.

Before we compute $h(x, t)$, observe that $\omega_\mu = s_\mu / i$, where the s_μ's are the simple roots of the denominator

$$P(s) := \cosh(s \ell / c) + k s \sinh(s \ell / c) = 0$$

are the eigen-angular frequencies for which resonance in the bar will taken place. Let the τ_μ's be the positive solutions of the equation

$$\frac{1}{\tau_\mu} = \frac{M}{m} \tan(\tau_\mu),$$

and $\omega_\mu = c \tau_\mu / \ell$. $m = a \ell \rho$ is the mass of the bar. Let us now write $H(x, s)$ in the form

$$H(x, s) = \frac{\ell}{a E} \frac{\sinh(s \ell / c \cdot x / \ell)}{(s \ell / c) [\cosh(s \ell / c) + k s \sinh(s \ell / c)]}$$

and let us take the inverse Laplace transform of

$$H^\dagger(\xi, s) = \frac{\ell}{a E} \frac{\sinh(s \xi) / s}{\cosh s + k' s \sinh s},$$

$$\left(k' = k c / \ell = \frac{M}{m}\right), \quad \left(0 < \xi = \frac{x}{\ell} < 1\right).$$

$H^\dagger(\xi, s)$ is a meromorphic function in s ($s = 0$ is not a pole). We can use Expansion Theorem I (see Sections 6.1.1.1 and 6.1.2.2) and obtain

$$h^\dagger(\xi, t) = \frac{2 \ell}{a E} \sum_{\mu=1}^{\infty} \frac{\sin(\tau_\mu \xi) / \tau_\mu}{\sin \tau_\mu + k' (\sin \tau_\mu + \tau_\mu \cos \tau_\mu)} \sin(\tau_\mu t).$$

By the similarity rule

$$h(x,t) = \frac{2c}{\ell a E} \sum_{\mu=1}^{\infty} \frac{\sin(\tau_\mu x/\ell)/\tau_\mu}{\sin\tau_\mu + k'(\sin\tau_\mu + \tau_\mu \cos\tau_\mu)} \sin\left(\tau_\mu \frac{c}{\ell} t\right),$$

and Duhamel's formula implies

$$y(x,t) = \frac{2c}{\ell a E} \sum_{\mu=1}^{\infty} \frac{\sin(\tau_\mu x/\ell)/\tau_\mu}{\sin\tau_\mu + M/m(\sin\tau_\mu + \tau_\mu \cos\tau_\mu)} \int_0^t \sin\left[\tau_\mu \frac{c}{\ell}(t-\nu)\right] f(\nu)\, d\nu.$$

The resonance phenomenon is set in evidence if $f(t) = \sin(\tau_\mu \frac{c}{\ell} t)$ for some μ. For the corresponding convolution term we have

```
Convolution[Sin[ω u], Sin[ω u], u][t] /. ComputeConvolution
```

$$-\frac{1}{2}\,\texttt{t Cos[t }\omega\texttt{]} + \frac{\texttt{Sin[t }\omega\texttt{]}}{2\,\omega}$$

i.e. a term with an increasing amplitude. If, for example, $k' = M/m = 5$, then the first 20 roots that give the τ_μ 's, are

```
τ = Table[x /. FindRoot[1 / x == 5 Tan[x], {x, k Pi + 1}], {k, 0, 19}]
```

```
{0.432841, 3.20394, 6.31485, 9.44595, 12.5823, 15.7207,
 18.8602, 22.0002, 25.1407, 28.2814, 31.4223, 34.5633, 37.7044,
 40.8456, 43.9868, 47.1281, 50.2695, 53.4108, 56.5522, 59.6936}
```

11.2 Green's Functions

Green's function is an important technique for solving initial and boundary value problems with partial differential equations. Similar to Duhamel's formula, an associated and simpler problem must first be solved, one that consists of constructing a Green's function. The solution of the original (given) problem can then be expressed as an integral containing Green's function. Integral relations based on Green's second theorem of vector analysis are fundamental when constructing a Green's function. For parabolic and hyperbolic partial differential equations, the method of Laplace transformation can often be helpful. Point sources, with the concept of fundamental solutions, are often mentioned in the context of Green's function, too. For many problems one obtains the same solution, but the approach is different. For a wider discussion of the Green's function technique see for example [Zauderer (1983)].

■ 11.2.1 Parabolic Partial Differential Equations

11.2.1.1 Integral Relations

For Green's function technique we first have to construct the basic integral relation. Let

$$L := -\frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right) + q(x), \qquad (0 < x < \ell,\ t > 0)$$

be the given differential operator. We consider the parabolic partial differential equation

$$\rho(x)\,\frac{\partial y}{\partial t} + L\,y(x,\,t) = \rho(x)\,f(x,\,t) \tag{11.11}$$

with the boundary conditions

$$\alpha_1\,y(0,\,t) + \beta_1\,\frac{\partial y(0,\,t)}{\partial x} = g_1(t)\,,\quad \alpha_2\,y(\ell,\,t) + \beta_2\,\frac{\partial y(\ell,\,t)}{\partial x} = g_2(t) \tag{11.12}$$

and the initial condition

$$y(x,\,0) = \varphi(x) \qquad (0 < x < \ell)\,. \tag{11.13}$$

This is a more general initial and boundary value problem. It also covers the conduction of heat or diffusion problems where

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\,\frac{\partial y}{\partial t} = h(x,\,t),\quad (0 < x < \ell,\ t > 0)\,,$$

i.e. $p(x) = 1,\ q(x) = 0,\ \rho(x) = 1/\kappa^2,\ f(x,\,t) = -\,\kappa^2\,h(x,\,t)$.

Let $D = \{(x,\,t) \mid 0 < x < \ell,\ 0 < t < T\}$ denote the rectangular two-dimensional region with boundary ∂D, where $T > 0$ is an arbitrary fixed number. Let us consider the following integral over D:

$$\int\int_D \left[w\left(\rho\,\frac{\partial y}{\partial t} + L\,y\right) - y\left(-\rho\,\frac{\partial w}{\partial t} + L\,w\right)\right] dx\,dt$$

where $w(x,\,t)$ is a function specified later. We have

$$w\,L\,y - y\,L\,w = y\,\frac{\partial}{\partial x}\left(p\,\frac{\partial w}{\partial x}\right) - w\,\frac{\partial}{\partial x}\left(p\,\frac{\partial y}{\partial x}\right),$$

thus

$$\begin{aligned} &w\left(\rho\,\frac{\partial y}{\partial t} + L\,y\right) - y\left(-\rho\,\frac{\partial w}{\partial t} + L\,w\right) \\ &= y\,\frac{\partial}{\partial x}\left(p\,\frac{\partial w}{\partial x}\right) - w\,\frac{\partial}{\partial x}\left(p\,\frac{\partial y}{\partial x}\right) + \frac{\partial}{\partial t}\,(\rho\,y\,w) \\ &= \frac{\partial}{\partial x}\left(y\,p\,\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial x}\left(w\,p\,\frac{\partial y}{\partial x}\right) + \frac{\partial}{\partial t}\,(\rho\,y\,w)\,. \end{aligned}$$

The integral over the region D can then be written as

$$\int\int_D \frac{\partial}{\partial x}\left[y\,p\,\frac{\partial w}{\partial x} - w\,p\,\frac{\partial y}{\partial x}\right] - \frac{\partial}{\partial t}\,(-\,\rho\,y\,w)\,dx\,dt\,.$$

By applying Green's theorem in the plane of vector analysis on the above expression, we obtain

$$\int_{\partial D} -\,\rho\,y\,w\,dx + \left[y\,p\,\frac{\partial w}{\partial x} - w\,p\,\frac{\partial y}{\partial x}\right] dt.$$

This line integral over the boundary ∂D then becomes

$$
\begin{aligned}
&- \int_0^{\ell} \rho(x)\, y(x, 0)\, w(x, 0)\, \mathbf{d}x \\
&+ \int_0^{T} \left[y(\ell, t)\, p(\ell) \frac{\partial w(\ell, t)}{\partial x} - w(\ell, t)\, p(\ell) \frac{\partial y(\ell, t)}{\partial x} \right] \mathbf{d}t \\
&+ \int_0^{\ell} \rho(x)\, y(x, T)\, w(x, T)\, \mathbf{d}x \\
&- \int_0^{T} \left[y(0, t)\, p(0) \frac{\partial w(0, t)}{\partial x} - \left(w(0, t)\, p(0) \frac{\partial y(0, t)}{\partial x} \right) \right] \mathbf{d}t .
\end{aligned}
$$

Let us now specify the function $w(x, t)$. We request that

$$
-\rho(x) \frac{\partial w}{\partial t} + L\, w(x, t) = \delta(x - \xi)\, \delta(t - \tau) , \tag{11.14}
$$

where (ξ,τ) is a fixed point in the interior of the region D. This implies

$$
\begin{aligned}
&\int\int_D \left[w\left(\rho \frac{\partial y}{\partial t} + L y \right) - y\left(-\rho \frac{\partial w}{\partial t} + L w \right) \right] \mathbf{d}x\, \mathbf{d}t \\
&= \int\int_D \left[w\left(\rho \frac{\partial y}{\partial t} + L y \right) - y(x, t)\, \delta(x - \xi)\, \delta(t - \tau) \right] \mathbf{d}x\, \mathbf{d}t \\
&= \int\int_D w(x, t) \left(\rho \frac{\partial y}{\partial t} + L y \right) \mathbf{d}x\, \mathbf{d}t - y(\xi, \tau) \\
&= \int\int_D w(x, t)\, \rho(x)\, f(x, t)\, \mathbf{d}x\, \mathbf{d}t - y(\xi, \tau) .
\end{aligned}
$$

Furthermore, we request that $w(x, t)$ satisfies the homogeneous boundary conditions

$$
\alpha_1\, w(0, t) + \beta_1 \frac{\partial w(0, t)}{\partial x} = 0 , \quad \alpha_2\, w(\ell, t) + \beta_2 \frac{\partial w(\ell, t)}{\partial x} = 0 \tag{11.15}
$$

and the initial (terminal) condition $w(x, T) = 0$, $(0 < x < \ell)$. In using the inital condition the integral over the boundary ∂D then becomes

$$
\begin{aligned}
&\int_{\partial D} - \rho\, y\, w\, \mathbf{d}x + \left[y\, p \frac{\partial w}{\partial x} - w\, p \frac{\partial y}{\partial x} \right] \mathbf{d}t \\
&= - \int \rho(x)\, \varphi(x)\, w(x, 0)\, \mathbf{d}x \\
&+ \int_0^{T} \left[y(\ell, t)\, p(\ell) \frac{\partial w(\ell, t)}{\partial x} - w(\ell, t)\, p(\ell) \frac{\partial y(\ell, t)}{\partial x} \right] \mathbf{d}t \\
&- \int_0^{T} \left[y(0, t)\, p(0) \frac{\partial w(0, t)}{\partial x} - w(0, t)\, p(0) \frac{\partial y(0, t)}{\partial x} \right] \mathbf{d}t
\end{aligned}
$$

which is equal to

$$
\int\int_D w(x, t)\, \rho(x)\, f(x, t)\, \mathbf{d}x\, \mathbf{d}t - y(\xi, \tau) .
$$

Observe that, by choosing $w(x, T) = 0$, the values of $y(x, T)$ become irrelevant. Therefore, we have *the basic integral relation*

$$y(\xi, \tau) = \int\int_D w(x, t)\,\rho(x)\,f(x, t)\,dx\,dt + \int_0^{\ell} \rho(x)\,\varphi(x)\,w(x, 0)\,dx$$
$$+ \int_0^T \Big[- y(\ell, t)\,p(\ell)\,\frac{\partial w(\ell, t)}{\partial x} + w(\ell, t)\,p(\ell)\,\frac{\partial y(\ell, t)}{\partial x}\Big]\,dt$$
$$+ \int_0^T \Big[y(0, t)\,p(0)\,\frac{\partial w(0, t)}{\partial x} - w(0, t)\,p(0)\,\frac{\partial y(0, t)}{\partial x}\Big]\,dt.$$

If $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = 1$ *(Dirichlet boundary conditions), then*

$$y(0, t) = g_1(t), \quad y(\ell, t) = g_2(t).$$

By (11.12) and (11.15),

$$y(\xi, \tau) = \int\int_D w(x, t)\,\rho(x)\,f(x, t)\,dx\,dt$$
$$+ \int_0^{\ell} \rho(x)\,\varphi(x)\,w(x, 0)\,dx - \int_0^T p(\ell)\,g_2(t)\,\frac{\partial w(\ell, t)}{\partial x}\,dt \qquad (11.16)$$
$$+ \int_0^T p(0)\,g_1(t)\,\frac{\partial w(0, t)}{\partial x}\,dt.$$

If $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$ *(Neumann boundary conditions),* then

$$\frac{\partial y(0, t)}{\partial x} = g_1(t), \quad \frac{\partial y(\ell, t)}{\partial x} = g_2(t),$$

and we obtain

$$y(\xi, \tau) = \int\int_D w(x, t)\,\rho(x)\,f(x, t)\,dx\,dt$$
$$+ \int_0^{\ell} \rho(x)\,\varphi(x)\,w(x, 0)\,dx + \int_0^T w(\ell, t)\,p(\ell)\,g_2(t)\,dt \qquad (11.17)$$
$$- \int_0^T w(0, t)\,p(0)\,g_1(t)\,dt.$$

The function $G(x, t; \xi, \tau) := w(x, t)$, satisfying (11.14), (11.15) and the terminal condition $G(x, T; \xi, \tau) = 0$ is called *Green's function* for the boundary value problem (11.11), (11.12) and (11.13). Let us summarize the above steps for the heat conduction problem.

Proposition. *The solution for the initial and boundary value problem for the finite interval* $0 < x < \ell$,

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\,\frac{\partial y}{\partial t} = h(x, t), \quad (0 < x < \ell,\ t > 0),$$
$$y(0, t) = g_1(t), \quad y(\ell, t) = g_2(t), \quad y(x, 0) = \varphi(x),$$

is given by

$$
\begin{aligned}
y(\xi, \tau) &= -\int_0^T \int_0^{\ell} G(x, t, \xi, \tau)\, h(x, t)\, dx\, dt \\
&+ \frac{1}{\kappa^2} \int_0^{\ell} \varphi(x)\, G(x, 0; \xi, \tau)\, dx - \int_0^T g_2(t) \frac{\partial G(\ell, t; \xi, \tau)}{\partial x}\, dt \\
&+ \int_0^T g_1(t) \frac{\partial G(0, t; \xi, \tau)}{\partial x}\, dt ,
\end{aligned}
\tag{11.18}
$$

where Green's function $G(x, t; \xi, \tau)$ satisfies

$$
\begin{aligned}
&\frac{1}{\kappa^2} \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = -\delta(x - \xi)\, \delta(t - \tau) , \\
&w(0, t) = 0 , \quad w(\ell, t) = 0, \ w(x, T) = 0 .
\end{aligned}
\tag{11.19}
$$

The modification for a semi-infinite solid is the following:

Boundary value problem for a semi-infinite interval:

$$
\begin{aligned}
y(\xi, \tau) &= -\int_0^T \int_0^{\infty} G(x, t, \xi, \tau)\, h(x, t)\, dx\, dt \\
&+ \frac{1}{\kappa^2} \int_0^{\infty} \varphi(x)\, G(x, 0; \xi, \tau)\, dx + \int_0^T g_1(t) \frac{\partial G(0, t; \xi, \tau)}{\partial x}\, dt
\end{aligned}
\tag{11.20}
$$

where Green's function $G(x, t; \xi, \tau)$ satisfies

$$
\begin{aligned}
&\frac{1}{\kappa^2} \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = -\delta(x - \xi)\, \delta(t - \tau) \qquad (t, \tau < T, \ x, \xi > 0), \\
&w(0, t) = 0 , \quad w(x, T) = 0
\end{aligned}
\tag{11.21}
$$

The constant T is arbitrarily large and $G(x, t; \xi, \tau)$ satisfies a boundedness condition for $x \to \infty$ that replaces the second boundary condition.

11.2.1.2 Green's Function for a Finite Interval

Let us assume that $0 < x < \ell$. We want to solve the initial and boundary value problem (11.19). As to the initial value problem we have a so-called backward initial value problem, i.e. we are looking for the solution $w(x, t)$ for $t < T$ such that $w(x, T) = 0$, see also Section 6.3.4. Let us first transform it into an ordinary initial value problem. By setting

$$
w(x, t) = : v(x, T - t), \quad v(x, \vartheta) = w(x, T - \vartheta)
$$

with $\vartheta = T - t$, $t = T - \vartheta$, we obtain

$$
\begin{aligned}
&-\frac{\partial^2 v}{\partial x^2} + \frac{1}{\kappa^2} \frac{\partial v}{\partial \vartheta} = \delta(x - \xi)\, \delta(\vartheta - (T - \tau)), \\
&v(0, \vartheta) = 0, \quad v(\ell, \vartheta) = 0, \ v(x, 0) = 0.
\end{aligned}
$$

The Laplace transformation $v(x, \vartheta) \circ\!-\!\bullet\ V(x, s)$ takes it to the image system

$$-\frac{\partial^2 V}{\partial x^2} + \frac{s}{\kappa^2} V(x, s) = \delta(x-\xi)\exp[-(T-\tau)s],$$

$$V(0, s) = 0, \quad V(\ell, s) = 0.$$

The solution of this boundary value problem for an ordinary differential equation

- must satisfy the homogeneous differential equation for $x \neq \xi$,
- must be continuous at $x = \xi$,
- must have a first derivative with a jump of magnitude $-\exp[-(T-\tau)s]$ at $x = \xi$.

It can be shown that the solution of this problem can be put into the following form, see for example [Friedman (1966)] and the theorem in Section 11.3.1.2:

$$V(x, s) = \frac{V_-(x)\,V_+(\xi)\,u(\xi - x) + V_+(x)\,V_-(\xi)\,u(x-\xi)}{J(V_+, V_-)} \exp[-(T-\tau)s]$$

where $V_-(x)$ is the solution of the homogeneous equation satisfying the left boundary condition and $V_+(x)$ is the one satisfying the right-boundary condition. The conjunct

$$J(V_+, V_-) := V_+(x)\,V_-'(x) - V_+'(x)\,V_-(x)$$

is constant and can be evaluated at any point of the interval $[0, \ell]$.

In our case we obtain

$$V_-(x) = \sinh\left(\frac{\sqrt{s}\,x}{\kappa}\right), \quad V_+(x) = \sinh\left(\frac{\sqrt{s}\,(\ell - x)}{\kappa}\right),$$

$$J(V_+, V_-) = \frac{\sqrt{s}}{\kappa}\sinh\left(\frac{\sqrt{s}\,\ell}{\kappa}\right).$$

If we introduce

$$h(x, \xi, \vartheta) = \mathcal{L}^{-1}\left[\frac{\sinh(\sqrt{s}\,x/\kappa)\sinh(\sqrt{s}\,(\ell-\xi)/\kappa)}{(\sqrt{s}/\kappa)\sinh(\sqrt{s}\,\ell/\kappa)}\right](x, \vartheta)$$

the solution can be written as

$$v(x, \vartheta) = u(\vartheta - T + \tau)\{u(\xi - x)\,h(x, \xi, \vartheta - T + \tau) + u(x-\xi)\,h(\xi, x, \vartheta - T + \tau)\},$$

and, going back to $w(x, t)$, as

$$w(x, t) = u(\tau - t)\{u(\xi - x)\,h(x, \xi, \tau - t) + u(x-\xi)\,h(\xi, x, \tau - t)\}.$$

Let us now compute $h(x, \xi, \vartheta)$. Because

$$\sinh(\sqrt{s}\,x/\kappa)\sinh(\sqrt{s}\,(\ell-\xi)/\kappa)$$
$$= \frac{1}{2}\{\cosh(\sqrt{s}\,(x+\ell-\xi)/\kappa) - \cosh(\sqrt{s}\,(x-\ell+\xi)/\kappa)\}$$

we first obtain

```
InverseLaplaceTransform[(Cosh[Sqrt[s] a] / Sqrt[s] / Sinh[Sqrt[s] ℓ]) //
  ScaleChange[1 / κ^2, s], s, ϑ]
```

$$\kappa^2 \left(\frac{1}{\ell} + \frac{1}{\ell}\left(2\,\text{Summation}\left[(-1)^{n2}\, e^{-\frac{n2^2\,\pi^2\,\vartheta\,\kappa^2}{\ell^2}}\,\text{Cos}\left[\frac{a\,n2\,\pi}{\ell}\right],\ \{n2,\ 1,\ \infty\}\right]\right)\right)$$

and, after some simplification, the symmetric expression in x, ξ,

$$h(x, \xi, \vartheta) = \frac{2\kappa^2}{\ell} \sum_{n=1}^{\infty} \exp\left[-\frac{n^2 \pi^2 \kappa^2}{\ell^2}\,\vartheta\right] \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi \xi}{\ell}\right).$$

Green's function for this problem finally becomes

$$G(x, t; \xi, \tau) =$$

$$u(\tau - t)\,\frac{2\kappa^2}{\ell} \sum_{n=1}^{\infty} \exp\left[-\frac{n^2 \pi^2 \kappa^2}{\ell^2}\,(\tau - t)\right] \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi \xi}{\ell}\right). \tag{11.22}$$

Example. Let us consider the following two-point boundary value problem with a non-vanishing initial condition (heat conduction in a wall):

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = h(x, t), \quad (0 < x < \ell,\ t > 0),$$

$$y(0, t) = y_0\,, \quad y(\ell, t) = y_\ell, \quad y(x, 0) = f(x)\,.$$

We can proceed further with the given non-homogeneous boundary conditions, but the substitution

$$y(x, t) = v(x, t) + \frac{x}{\ell}\,(y_\ell - y_0) + y_0$$

together with $\partial^2 y / \partial x^2 = \partial^2 v / \partial x^2$, $\partial y / \partial t = \partial v / \partial t$ and

$$y(x, 0) = v(x, 0) + \frac{x}{\ell}\,(y_\ell - y_0) + y_0$$

yields the simpler homogeneous boundary value problem

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial v}{\partial t} = h(x, t), \quad (0 < x < \ell,\ t > 0)\,,$$

$$v(0, t) = 0\,, \quad v(\ell, t) = 0, \quad v(x, 0) = f(x) - \frac{x}{\ell}\,(y_\ell - y_0) - y_0 = :\varphi(x).$$

From the above general expressions we immediately obtain

$$v(\xi, \tau) = -\int_0^\tau \int_0^\ell G(x, t; \xi, \tau)\, h(x, t)\, \boldsymbol{d}x\, \boldsymbol{d}t + \frac{1}{\kappa^2} \int_0^\ell \varphi(x)\, G(x, 0; \xi, \tau)\, \boldsymbol{d}x\,,$$

and, from (11.22),

$$v(\xi, \tau) = \sum_{n=1}^{\infty} \left(A_n + B_n\, e^{-\lambda^2 n^2 \tau}\right) \sin\left(\frac{n\pi \xi}{\ell}\right) \quad \text{with}$$

$$A_n = -\frac{2\kappa^2}{\ell}\int_0^\tau\int_0^\ell e^{-\lambda^2 n^2 (\tau - t)} \sin\left(\frac{n\pi x}{\ell}\right) h(x, t)\, dx\, dt\,, \quad \lambda = \frac{\pi\kappa}{\ell},$$

$$B_n = \frac{2}{\ell}\int_0^\ell \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell}\int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx + \frac{2[(-1)^n y_\ell - y_0]}{n\pi},$$

$$\varphi(x) = f(x) - \frac{x}{\ell}(y_\ell - y_0) - y_0,$$

$$y(\xi, \tau) = v(\xi, \tau) + \frac{\xi}{\ell}(y_\ell - y_0) + y_0\,.$$

11.2.1.3 Free Space Green's Function

Let us consider the boundary value problem on the infinite interval $-\infty < x < \infty$. The corresponding Green's function is called the *free space Green's function.* It satisfies

$$\frac{1}{\kappa^2}\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = -\delta(x - \xi)\,\delta(t - \tau) \qquad (t, \tau < T,\ -\infty < x, \xi < \infty)$$
$$w(x, T) = 0\,.$$

First method: We will use the two-sided Laplace transform with respect to x (see Section 1.2). Thus let $w(x, t) \circ\!\!-\!\!\bullet\, W(s, t)$. The image equation then is

$$\frac{1}{\kappa^2}\frac{\partial W(s, t)}{\partial t} + s^2 W(s, t) = -e^{-\xi s}\,\delta(t - \tau)$$

or

$$\frac{\partial}{\partial t}\left[e^{\kappa^2 s^2 t} W(s, t)\right] = -\kappa^2 e^{-\xi s}\, e^{\kappa^2 s^2 t}\,\delta(t - \tau)\,,$$

observe that $\delta(t - \tau) = \delta(\tau - t)$. By integrating in the sense of generalized functions, we obtain

$$e^{\kappa^2 s^2 t} W(s, t) = \kappa^2 e^{-\xi s}\, e^{\kappa^2 s^2 \tau}\, u(\tau - t) + c,$$

hence

$$W(s, t) = \kappa^2 e^{-\xi s}\, e^{\kappa^2 s^2 (\tau - t)}\, u(\tau - t) + c\, e^{-\kappa^2 s^2 t}.$$

The terminal condition $w(x, T) \circ\!\!-\!\!\bullet\, W(s, T) = 0$ for an arbitrary large T implies $c = 0$. Therefore

$$W(s, t) = \kappa^2 e^{-\xi s}\, e^{\kappa^2 s^2 (\tau - t)}\, u(\tau - t).$$

By taking the inverse two-sided Laplace transform (see 1.2.2.6) we obtain

```
InverseTwoSidedLaplaceTransform[
 {κ^2 Exp[-ξ s] Exp[κ^2 (τ - t) s^2], Interval[{-Infinity, Infinity}]}, s, x]
```

$$\frac{e^{-\frac{x^2}{4\kappa^2(-t+\tau)} + \frac{x\xi}{2\kappa^2(-t+\tau)} - \frac{\xi^2}{4\kappa^2(-t+\tau)}}\,\kappa^2}{2\sqrt{\pi}\,\sqrt{\kappa^2(-t+\tau)}}$$

We obtain the *free space Green's function:*

$$w(x, t) = u(\tau - t)\,\frac{\kappa}{2\sqrt{\pi(\tau - t)}}\exp\left[-\frac{(x-\xi)^2}{4\kappa^2(\tau - t)}\right] =: G(x, t; \xi, \tau)\,. \tag{11.23}$$

Second method: The backward initial problem

$$\frac{1}{\kappa^2}\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = -\delta(x-\xi)\,\delta(t-\tau) \qquad (t, \tau < T,\ x, \xi > 0)$$
$$w(x, T) = 0$$

can also be solved by use of the ordinary Laplace transform. Setting $\vartheta = T - t$ and $v(x, T - t) := w(x, t)$, i.e. $v(x, \vartheta) = w(x, T - \vartheta)$, we obtain the usual initial value problem

$$-\frac{1}{\kappa^2}\frac{\partial v}{\partial \vartheta} + \frac{\partial^2 v}{\partial x^2} = -\delta(x-\xi)\,\delta(T - \vartheta - \tau)$$
$$= -\delta(x-\xi)\,\delta(\vartheta - (T - \tau))$$
$$(\vartheta > 0,\ x, \xi > 0), \qquad v(x, 0) = 0.$$

The ordinary Laplace transform with respect to ϑ yields the image equation

$$-\frac{s}{\kappa^2}V(x, s) + \frac{\partial^2 V}{\partial x^2} = -\delta(x-\xi)\,e^{-(T-\tau)s}\,.$$

The solution must be continuous at $x = \xi$, but its derivative has a jump of magnitude $-e^{-(T-\tau)s}$ at $x = \xi$ (also see the theorem of Section 11.3.1.2):

$$V(\xi +, s) - V(\xi -, s) = 0, \quad \frac{\partial V(\xi +, s)}{\partial x} - \frac{\partial V(\xi -, s)}{\partial x} = -e^{-(T-\tau)s}\,.$$

For $x < \xi$, the solution of the homogeneous equation is

$$V(x, s) = A\,e^{x\sqrt{s}/\kappa}, \quad x < \xi,$$

and for $x > \xi$,

$$V(x, s) = B\,e^{-x\sqrt{s}/\kappa}, \quad x > \xi.$$

These two solutions remain bounded for $x \to \pm\infty$. The above two equations determine A and B. One obtains

$$A = e^{-(T-\tau)s}\,\frac{\kappa}{2\sqrt{s}}\,e^{-\xi\sqrt{s}/\kappa}, \quad B = e^{-(T-\tau)s}\,\frac{\kappa}{2\sqrt{s}}\,e^{\xi\sqrt{s}/\kappa}$$

which gives

$$V(x, s) = e^{-(T-\tau)s}\,\frac{\kappa}{2\sqrt{s}}\,e^{-(\xi-x)\sqrt{s}/\kappa}, \quad x < \xi,$$

$$V(x, s) = e^{-(T-\tau)s}\,\frac{\kappa}{2\sqrt{s}}\,e^{(\xi-x)\sqrt{s}/\kappa}, \quad x > \xi,$$

i.e.,

$$V(x,\, s) = e^{-(T-\tau)s}\, \frac{\kappa}{2\sqrt{s}}\; e^{-|\xi - x|\sqrt{s}/\kappa}\,, \quad -\infty < x < \infty\,.$$

By taking the inverse Laplace transform we obtain

```
κ / 2 InverseLaplaceTransform[Exp[- Abs[ξ - x] √s /κ] / √s, s, ϑ]
```

$$\frac{e^{-\frac{\mathrm{Abs}[-x+\xi]^2}{4\,\vartheta\,\kappa^2}}\,\kappa}{2\sqrt{\pi}\,\sqrt{\vartheta}}$$

The original shift rule yields

$$v(x,\, \vartheta) = u(\vartheta - (T-\tau))\, \frac{e^{-\frac{(-x+\xi)^2}{4(\vartheta-(T-\tau))\kappa^2}}\,\kappa}{2\sqrt{\pi(\vartheta - (T-\tau))}}$$

and, by going back to $w(x,\, t)$,

$$w(x,\, t) = v(x,\, T-t) = u(\tau - t)\, \frac{\kappa}{2\sqrt{\pi\,(\tau - t)}} \exp\left[-\frac{(-x+\xi)^2}{4\,(\tau - t)\,\kappa^2}\right]$$

as before.

The free space solution of

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\,\frac{\partial y}{\partial t} = h(x,\, t), \quad (-\infty < x < \infty,\; t > 0)\,,$$
$$y(x,\, 0) = \varphi(x)$$

is, therefore, given by

$$y(\xi,\, \tau) = \frac{\kappa}{2}\int_0^{\tau}\int_{-\infty}^{\infty} \frac{h(x,\, t)}{\sqrt{\pi\,(\tau - t)}} \exp\left[-\frac{(x-\xi)^2}{4\,\kappa^2(\tau - t)}\right] dx\, dt$$
$$+ \frac{1}{2\,\kappa\sqrt{\pi\,\tau}}\int_{-\infty}^{\infty} \varphi(x) \exp\left[-\frac{(x-\xi)^2}{4\,\kappa^2\,\tau}\right] dx\,.$$

11.2.1.4 Half-Space Green's Function

Let us revisit the half-space problem and use the method of images in order to construct the appropriate Green's function. By the same process as above the problem

$$\frac{1}{\kappa^2}\,\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = -\,\delta(x-\xi)\,\delta(t-\tau) \;+\; \delta(x+\xi)\,\delta(t-\tau) \qquad (t,\, \tau < T)$$
$$w(x,\, T) = 0$$

has the solution

$$w(x,t) = u(\tau - t)\frac{\kappa}{2\sqrt{\pi(\tau - t)}}\exp\left[-\frac{(x-\xi)^2}{4\kappa^2(\tau - t)}\right] - u(\tau - t)\frac{\kappa}{2\sqrt{\pi(\tau - t)}}\exp\left[-\frac{(x+\xi)^2}{4\kappa^2(\tau - t)}\right]. \tag{11.24}$$

We observe, however, that for this $w(x, t)$ the boundary condition $w(0, t) = 0$ is now satisfied, i.e. the boundary condition at $x = 0$ for the half-space problem. Furthermore, the added term $\delta(x + \xi)\,\delta(t - \tau)$ is zero, if $0 < x, \xi$. We conclude that (11.24) is the requested Green's function for the half-space problem. The solution for the half-space problem

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = h(x,t), \quad (0 < x,\ t > 0),$$
$$y(0,t) = g_1(t), \quad y(x,0) = \varphi(x)$$

is therefore given by

$$\begin{aligned} y(\xi,\tau) = {} & \kappa\int_0^\tau\int_0^\infty \frac{h(x,t)}{2\sqrt{\pi(\tau - t)}} \left\{\exp\left[-\frac{(x-\xi)^2}{4\kappa^2(\tau - t)}\right] - \exp\left[-\frac{(x+\xi)^2}{4\kappa^2(\tau - t)}\right]\right\} dx\,dt \\ & + \frac{1}{2\kappa\sqrt{\pi\tau}}\int_0^\infty \varphi(x)\left\{\exp\left[-\frac{(x-\xi)^2}{4\kappa^2\tau}\right] - \exp\left[-\frac{(x+\xi)^2}{4\kappa^2\tau}\right]\right\} dx \\ & + \frac{\xi}{2\kappa\sqrt{\pi}}\int_0^\tau \frac{g_1(t)}{(\tau - t)^{3/2}}\exp\left[-\frac{\xi^2}{4\kappa^2(\tau - t)}\right] dt. \end{aligned} \tag{11.25}$$

Observe that Green's function constructed above is the sum of the free space Green's function plus an added term.

■ 11.2.2 Hyperbolic Partial Differential Equations

As the construction of Green's function for the hyperbolic partial differential equations parallels that for the parabolic case, we mainly proceed in a series of exercises for the reader. See also [Zauderer (1983)] for example.

11.2.2.1 Integral Relations

Let

$$L := -\frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right) + q(x), \qquad (0 < x < \ell,\ t > 0)$$

be the given differential operator. We consider the hyperbolic partial differential equation

$$\rho(x)\frac{\partial^2 y}{\partial t^2} + L\,y(x,t) = \rho(x)\,f(x,t) \tag{11.26}$$

with the boundary conditions

$$\alpha_1\, y(0,\, t) + \beta_1 \frac{\partial y(0,\, t)}{\partial x} = g_1(t)\,, \quad \alpha_2\, y(\ell,\, t) + \beta_2 \frac{\partial y(\ell,\, t)}{\partial x} = g_2(t) \tag{11.27}$$

and the initial conditions

$$y(x,\, 0) = \varphi(x)\,, \quad \frac{\partial y(x,\, 0)}{\partial t} = \psi(x) \quad (0 < x < \ell)\,. \tag{11.28}$$

This also covers the wave equation, where

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = h(x,\, t), \quad (0 < x < \ell,\ t > 0),$$

i.e. $p(x) = 1,\ q(x) = 0,\ \rho(x) = 1/c^2,\ f(x,\, t) = -c^2\, h(x,\, t)$.

Let $D = \{(x,\, t) \mid 0 < x < \ell,\ 0 < t < T\}$ denote the rectangular region with boundary ∂D, where $T > 0$ is an arbitrary fixed number. Let us consider the integral

$$\int\int_D \left[w\left(\rho \frac{\partial^2 y}{\partial t^2} + L\, y\right) - y\left(\rho \frac{\partial^2 w}{\partial t^2} + L\, w\right)\right] \boldsymbol{d}x\, \boldsymbol{d}t$$

over D, where $w(x,\, t)$ is a function to be specified later.

Problem 11.1. Show that

$$\begin{aligned} y(\xi,\, \tau) \;=\; & \int_0^T \int_0^\ell w(x,\, t)\, \rho(x)\, f(x,\, t)\, \boldsymbol{d}x\, \boldsymbol{d}t \\ & + \int_0^\ell \rho(x)\; \psi(x)\, w(x,\, 0) - \rho(x)\; \varphi(x)\, \frac{\partial w(x,\, 0)}{\partial t}\, \boldsymbol{d}x \\ & - \int_0^T p(\ell)\, g_2(t)\, \frac{\partial w(\ell,\, t)}{\partial x}\, \boldsymbol{d}t + \int_0^T p(0)\, g_1(t)\, \frac{\partial w(0,\, t)}{\partial x}\, \boldsymbol{d}t \end{aligned}$$

if $\beta_1 = \beta_2 = 0$, $\alpha_1 = \alpha_2 = 1$, and

$$\begin{aligned} y(\xi,\, \tau) \;=\; & \int_0^T \int_0^\ell w(x,\, t)\, \rho(x)\, f(x,\, t)\, \boldsymbol{d}x\, \boldsymbol{d}t \\ & + \int_0^\ell \rho(x)\; \psi(x)\, w(x,\, 0) - \rho(x)\, \varphi(x)\, \frac{\partial w(x,\, 0)}{\partial t}\, \boldsymbol{d}x \\ & - \int_0^T w(\ell,\, t)\, p(\ell)\, g_2(t)\, \boldsymbol{d}t - \int_0^T w(0,\, t)\, p(0)\, g_1(t)\, \boldsymbol{d}t \end{aligned}$$

if $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$, provided the function $w(x,\, t)$ satisfies

$$\rho(x) \frac{\partial^2 w}{\partial t^2} + L\, w(x,\, t) = \delta(x - \xi)\, \delta(t - \tau), \qquad (t < T)\,,$$

$$\alpha_1\, w(0,\, t) + \beta_1 \frac{\partial w(0,\, t)}{\partial x} = 0\,, \quad \alpha_2\, w(\ell,\, t) + \beta_2 \frac{\partial w(\ell,\, t)}{\partial x} = 0\,,$$

$$w(x, T) = 0, \quad \frac{\partial w(x, T)}{\partial t} = 0 .$$

The function $G(x, t; \xi, \tau) := w(x, t)$ satisfying the above conditions is called *Green's function* for the hyperbolic boundary value problem (11.26), (11.27), (11.28).

For a semi-infinite solid, we have ($\beta_1 = 0, \alpha_1 = 1$)

$$y(\xi, \tau) = \int_0^T \int_0^\infty w(x, t)\, \rho(x)\, f(x, t)\, \boldsymbol{dx}\, \boldsymbol{dt} + \int_0^\infty \rho(x)\, \psi(x)\, w(x, 0)$$
$$-\rho(x)\, \varphi(x) \frac{\partial w(x, 0)}{\partial t}\, \boldsymbol{dx} - \int_0^T p(0)\, g_1(t) \frac{\partial w(0, t)}{\partial x}\, \boldsymbol{dt}$$

or ($\beta_1 = 1, \alpha_1 = 0$)

$$y(\xi, \tau) = \int_0^T \int_0^\infty w(x, t)\, \rho(x)\, f(x, t)\, \boldsymbol{dx}\, \boldsymbol{dt} + \int_0^\ell \rho(x)\, \psi(x)\, w(x, 0)$$
$$-\rho(x)\, \varphi(x) \frac{\partial w(x, 0)}{\partial t}\, \boldsymbol{dx} - \int_0^T w(0, t)\, p(0)\, g_1(t)\, \boldsymbol{dt}$$

where Green's function $w(x, t) = G(x, t; \xi, \tau)$ must satisfy

$$\rho(x) \frac{\partial^2 w}{\partial t^2} + L\, w(x, t) = \delta(x - \xi)\, \delta(t - \tau), \qquad (t < T),$$

$$\alpha_1\, w(0, t) + \beta_1 \frac{\partial w(0, t)}{\partial x} = 0 ,$$

$$w(x, T) = 0, \quad \frac{\partial w(x, T)}{\partial t} = 0 .$$

The constant T is arbitrarily large and $w(x, t)$ has to satisfy a boundedness condition for $x \to \infty$.

11.2.2.2 Green's Function for the Wave Equation in a Finite Interval

We consider the boundary value problem for the finite space interval $0 < x < \ell$.

$$-\frac{\partial^2 y}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = -\, h(x, t),$$
$$y(0, t) = g_1(t), \quad y(\ell, t) = g_2(t),$$
$$y(x, 0) = \varphi(x), \quad \frac{\partial y(x, 0)}{\partial t} = \psi(x) \quad (0 < x < \ell).$$

For Green's function we have to solve the homogeneous boundary value problem with the two terminal conditions. Let us first transform it into an ordinary initial value problem. By setting

$$w(x, t) = : v(x, T - t), \quad v(x, \vartheta) = w(x, T - \vartheta)$$

with $\vartheta = T - t$, $t = T - \vartheta$, we obtain

$$-\frac{\partial^2 v}{\partial x^2}+\frac{1}{c^2}\frac{\partial^2 v}{\partial \vartheta^2}=\delta(x-\xi)\,\delta(\vartheta-(T-\tau)),$$

$$v(x,0)=0,\quad \frac{\partial v(x,0)}{\partial \vartheta}=0.$$

With $v(x,\vartheta)\circ\!-\!\bullet\, V(x,s)$ we obtain the image system

$$-\frac{\partial^2 V}{\partial x^2}+\frac{s^2}{c^2}V(x,s)=\delta(x-\xi)\exp[-(T-\tau)s],$$

$$v(0,\vartheta)=0,\quad v(\ell,\vartheta)=0.$$

Problem 11.2. Show that with

$$h(x,\xi,\vartheta)=\mathcal{L}^{-1}\Big[\frac{\sinh(s\,x/c)\sinh(s\,(\ell-\xi)/c)}{(s/c)\sinh(s\,\ell/c)}\Big](x,\vartheta),$$

the solution can be written as

$$v(x,\vartheta)=u(\vartheta-T+\tau)\,\{u(\xi-x)\,h(x,\xi,\vartheta-T+\tau)+u(x-\xi)\,h(\xi,x,\vartheta-T+\tau)\},$$

and, by going back to $w(x,t)$, as

$$w(x,t)=u(\tau-t)\,\{u(\xi-x)\,h(x,\xi,\tau-t)+u(x-\xi)\,h(\xi,x,\tau-t)\}.$$

Let us now compute $h(x,\xi,\vartheta)$. Because

$$\sinh(s\,x/c)\sinh(s\,(\ell-\xi)/c)$$
$$=\frac{1}{2}\{\cosh(s\,(x+\ell-\xi)/c)-\cosh(s\,(x-\ell+\xi)/c)\}$$

we obtain

```
InverseLaplaceTransform[
 (Cosh[s a] / s / Sinh[s ℓ]) // ScaleChange[1 / c, s], s, ϑ]
```

$$c\left(\frac{c\,\vartheta}{\ell}+\texttt{Summation}\Big[\frac{2\,(-1)^{k1}\,\texttt{Cos}[\frac{a\,k1\,\pi}{\ell}]\,\texttt{Sin}[\frac{c\,k1\,\pi\,\vartheta}{\ell}]}{k1\,\pi},\ \{k1,\ 1,\ \infty\}\Big]\right)$$

which then yields the symmetric expression in x,ξ,

$$h(x,\xi,\vartheta)=\frac{2c}{\pi}\sum_{n=1}^{\infty}\frac{\sin(c\,\pi\,n\,\vartheta/\ell)}{n}\sin\Big(\frac{n\pi x}{\ell}\Big)\sin\Big(\frac{n\pi\xi}{\ell}\Big).$$

Green's function for this problem finally becomes

$$G(x,t;\xi,\tau)=u(\tau-t)\,\frac{2c}{\pi}\sum_{n=1}^{\infty}\frac{\sin(c\,\pi\,n\,(\tau-t)/\ell)}{n}\sin\Big(\frac{n\pi x}{\ell}\Big)\sin\Big(\frac{n\pi\xi}{\ell}\Big)\qquad(11.29)$$

and the solution of the boundary value problem is then given in the form

$$y(\xi, \tau) = -\int_0^\tau \int_0^\ell G(x, t; \xi, \tau)\, h(x, t)\, dx\, dt$$
$$+\frac{1}{c^2}\int_0^\ell \psi(x)\, G(x, 0; \xi, \tau) - \varphi(x)\, \frac{\partial G(x, 0; \xi, \tau)}{\partial t}\, dx$$
$$-\int_0^\tau g_2(t)\, \frac{\partial G(\ell, t; \xi, \tau)}{\partial x}\, dt + \int_0^\tau g_1(t)\, \frac{\partial G(0, t; \xi, \tau)}{\partial x}\, dt .$$

11.2.2.3 The Free Space Green's Function

Let us consider the boundary value problem on the infinite interval $-\infty < x < \infty$. The corresponding Green's function is called the *free space Green's function.* It satisfies the backward initial and boundary value problem

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = \delta(x-\xi)\,\delta(t-\tau) \qquad (t, \tau < T,\ -\infty < x, \xi < \infty)$$
$$w(x, T) = 0,\ \frac{\partial w(x, T)}{\partial t} = 0 .$$

Again setting $\vartheta = T - t$ and $v(x, T-t) := w(x, t)$, i.e. $v(x, \vartheta) = w(x, T-\vartheta)$, we obtain the usual initial value problem

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial \vartheta^2} - \frac{\partial^2 v}{\partial x^2} = \delta(x-\xi)\,\delta(T-\vartheta-\tau)$$
$$= \delta(x-\xi)\,\delta(\vartheta-(T-\tau)) \qquad (\vartheta > 0),$$
$$v(x, 0) = 0,\ \frac{\partial v(x, T)}{\partial \vartheta} = 0 .$$

The ordinary Laplace transform with respect to ϑ takes it to the image equation

$$+\frac{s^2}{c^2} V(x, s) - \frac{\partial^2 V}{\partial x^2} = \delta(x-\xi)\, e^{-(T-\tau)s} .$$

The solution must be continuous at $x = \xi$, but its derivative has a jump of magnitude $-e^{-(T-\tau)s}$ at $x = \xi$ (also see the theorem of Section 11.3.1.2):

$$V(\xi+, s) - V(\xi-, s) = 0, \quad \frac{\partial V(\xi+, s)}{\partial x} - \frac{\partial V(\xi-, s)}{\partial x} = -e^{-(T-\tau)s}.$$

For $x < \xi$, the solution of the homogeneous equation is

$$V(x, s) = A\, e^{xs/c}, \quad x < \xi ,$$

and for $x > \xi$, it is

$$V(x, s) = B\, e^{-xs/c}, \quad x > \xi .$$

These two solutions remain bounded for $x \to \pm\infty$ and $\Re s > 0$. The two equations above determine A and B. One obtains

$$V(x, s) = e^{-(T-\tau)s}\, \frac{c}{2s}\, \{ e^{-(\xi-x)s/c}\, u(\xi - x) + e^{-(x-\xi)s/c}\, u(x-\xi)\}$$

which can be written as

$$V(x, s) = e^{-(T-\tau)s} \frac{c}{2s} e^{-|\xi - x|s/c}, \quad -\infty < x < \infty .$$

By taking the inverse Laplace transform we obtain

$$v(x, \vartheta) = \frac{c}{2} u\left[\vartheta - \left(T - \tau + \frac{|\xi - x|}{c}\right)\right],$$

and, getting back to $w(x, t)$,

$$w(x, t) = v(x, T - \vartheta) = \frac{c}{2} u\left(\tau - t - \frac{|\xi - x|}{c}\right) =: G(x, t; \xi, \tau) .$$

An equivalent form is

$$G(x, t; \xi, \tau) = w(x, t) = \frac{c}{2} u[x - \xi + c(\tau - t)]\ u[\xi - x + c(\tau - t)] . \tag{11.30}$$

We also need the derivative (in the sense of generalized functions), i.e.,

$$\frac{\partial u(t-a)}{\partial t} = \delta(t - a),$$

therefore

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} &= -\frac{c^2}{2} \{\delta[x - \xi + c(\tau - t)]\ u[\xi - x + c(\tau - t)] \\ &\quad + u[x - \xi + c(\tau - t)]\ \delta[\xi - x + c(\tau - t)]\} \\ &= -\frac{c^2}{2} \{\delta[x - \xi + c(\tau - t)] + \delta[\xi - x + c(\tau - t)]\} \end{aligned}$$

and

$$\frac{\partial w(x, 0)}{\partial t} = -\frac{c^2}{2} \{\delta(x + c\tau - \xi) + \delta(x - c\tau - \xi)\} .$$

The free space solution of

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2} \frac{\partial^2 y}{\partial t^2} = h(x, t), \quad (-\infty < x < \infty,\ t > 0),$$

$$y(x, 0) = \varphi(x), \quad \frac{\partial y(x, 0)}{\partial t} = \psi(x)$$

becomes

$$\begin{aligned} y(\xi, \tau) = &-\frac{c}{2} \int_0^{\tau} \int_{\xi - c(\tau - t)}^{\xi + c(\tau - t)} h(x, t)\, dx\, dt \\ &+ \frac{1}{2c} \int_{\xi - c\tau}^{\xi + c\tau} \psi(x)\, dx + \frac{1}{2} \{\varphi(\xi - c\tau) + \varphi(\xi + c\tau)\} . \end{aligned}$$

11.2.2.4 Half-Space Green's Function

We return to the half-space problem with zero initial conditions

$$-\frac{\partial^2 y}{\partial x^2}+\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}=-h(x,t) \qquad (0<x<\infty,\ 0<t),$$

$$y(0,t)=g(t),$$

$$y(x,0)=\varphi(x)=0,\quad \frac{\partial y(x,0)}{\partial t}=\psi(x)=0 \qquad (0<x<\infty).$$

Green's function $w(x,t)=G(x,t;\xi,\tau)$ must satisfy

$$-\frac{\partial^2 w}{\partial x^2}+\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}=\delta(x-\xi)\,\delta(t-\tau) \qquad (t,\tau<T,\ 0<x,\xi<\infty),$$

$$w(0,t)=0,$$

$$w(x,T)=0,\quad \frac{\partial w(x,T)}{\partial t}=0.$$

Problem 11.3. By proceeding as above show that

$$V(x,s)=u(\xi-x)\,\frac{-\exp[-((\xi+x)/c+(T-\tau))s]+\exp[-(((\xi-x)/c+(T-\tau))s]}{2s/c}$$
$$+\,u(x-\xi)\,\frac{-\exp[-((x+\xi)/c+(T-\tau))s]+\exp[-((x-\xi)/c+(T-\tau))s]}{2s/c}.$$

Therefore

$$v(x,\vartheta)=u(\xi-x)\,h(x,\xi)+u(x-\xi)\,h(\xi,x),$$

$$h(x,\xi)=\frac{c}{2}\{-u[\vartheta-((\xi+x)/c+(T-\tau))]+u[\vartheta-((\xi-x)/c+(T-\tau))]\}$$

and, going back to $w(x,t)$, Green's function becomes

$$w(x,t)=G(x,t;\xi,t)$$
$$=\frac{c}{2}u(\xi-x)\{-u[-(\xi+x)/c+(\tau-t)]+u[-(\xi-x)/c+(\tau-t)]\}$$
$$+\frac{c}{2}u(x-\xi)\{-u[-(x+\xi)/c+(\tau-t)]+u[-(x-\xi)/c+(\tau-t)]\}$$
$$=-\frac{c}{2}u[-(x+\xi)/c+(\tau-t)]+\frac{c}{2}u[-|\xi-x|/c+(\tau-t)].$$

This can also be represented as

$$G(x,t;\xi,t)=$$
$$\frac{c}{2}\{-u[\tau-t-(\xi+x)/c]+u[c(\tau-t)-\xi+x]\,u[c(\tau-t)+\xi-x]\}. \tag{11.31}$$

In the last section we determined the free space Green's function as

$$G_F(x,t;\xi,\tau)=\frac{c}{2}u[x-\xi+c(\tau-t)]\,u[\xi-x+c(\tau-t)]$$

hence we can write

$$G(x,t;\xi,t)=G_F(x,t;\xi,\tau)-\frac{c}{2}u[\tau-t-(\xi+x)/c].$$

But we also have

$$-G_F(x, t; -\xi, \tau) =$$
$$-\frac{c}{2}\, u[x+\xi+c(\tau-t)]\; u[-\xi-x+c(\tau-t)] = -\frac{c}{2}\, u[-\xi-x+c(\tau-t)]$$

and therefore

$$G(x, t; \xi, t) = G_F(x, t; \xi, \tau) - G_F(x, t; -\xi, \tau)$$

in agreement with the method of images.

The derivative in the generalized sense is

$$\frac{\partial w(x, t)}{\partial x} = \frac{1}{2}\,\delta[\tau - t - (\xi + x)/c] + \frac{c}{2}\,\delta[c(\tau-t) - \xi + x]\, u[c(\tau-t) + \xi - x]$$
$$-\frac{c}{2}\, u[c(\tau-t) - \xi + x]\,\delta[c(\tau-t) + \xi - x],$$
$$\frac{\partial w(0, t)}{\partial x} = \frac{1}{2}\,\delta[\tau - t - \xi/c]$$
$$+\frac{c}{2}\,\delta[c(\tau-t) - \xi]\, u[c(\tau-t) + \xi] - \frac{c}{2}\, u[c(\tau-t) - \xi]\,\delta[c(\tau-t) + \xi]\ .$$

The last term is zero and the second one becomes (regarded as a function of t)

$$\frac{c}{2}\,\delta[c(\tau-t) - \xi]\, u[c(\tau-t) + \xi] =$$
$$\frac{c}{2}\,\delta[c(\tau-t) - \xi] = \frac{c}{2}\,\delta[c\,t - c\tau + \xi] = \frac{1}{2}\,\delta[t - \tau + \xi/c],$$

thus

$$\frac{\partial w(0, t)}{\partial x} = \delta[\tau - t - \xi/c].$$

This finally gives

$$y(\xi, \tau) = -\frac{c}{2}\, u(\tau - \xi/c)$$
$$\times \left[\int_0^{\tau-\xi/c} \int_{c(\tau-t)-\xi}^{c(\tau-t)+\xi} h(x, t)\, dx\, dt + \int_{\tau-\xi/c}^{\tau} \int_{\xi-c(\tau-t)}^{\xi+c(\tau-t)} h(x, t)\, dx\, dt\right]$$
$$-\frac{c}{2}\, u(\xi/c - \tau) \int_0^{\tau} \int_{\xi-c(\tau-t)}^{\xi+c(\tau-t)} h(x, t)\, dx\, dt + g(\tau - \xi/c).$$

11.2.2.5 Green's Function for the Klein–Gordon and Telegrapher's Equation

A generalization of the wave equation is the Klein–Gordon partial differential equation. We consider the free space boundary value problem

$$-\gamma^2 \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} + c^2\, y(x, t) = f(x, t) \qquad (-\infty < x < \infty,\ 0 < t)$$

with the initial conditions

$$y(x, 0) = \varphi(x), \quad \frac{\partial y(x, 0)}{\partial t} = \psi(x) \qquad (-\infty < x < \infty).$$

The integral relation of Section 11.2.2.1 covers this case with $\rho(x) = 1$, $p(x) = \gamma^2$, $q(x) = c^2$. Green's function therefore satisfies

$$-\gamma^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} + c^2 w(x, t) = \delta(x - \xi)\,\delta(t - \tau) \qquad (-\infty < x, \xi < \infty,\ 0 < t)$$

$$w(x, T) = 0, \quad \frac{\partial w(x, T)}{\partial t} = 0$$

with $0 < t, \tau < T$. With respect to t, we have a backward initial value problem. By setting $w(x, t) = v(x, T - t) = v(x, \vartheta)$, i.e. $\vartheta = T - t$, we obtain an ordinary initial value problem with respect to ϑ,

$$-\gamma^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \vartheta^2} + c^2 v(x, \vartheta) = \delta(x - \xi)\,\delta(T - \vartheta - \tau),$$

$$v(x, 0) = 0, \quad \frac{\partial v(x, 0)}{\partial \vartheta} = 0.$$

We solve it via Laplace transformation. With $v(x, \vartheta) \circ\!-\!\bullet\ V(x, s)$ we obtain the image system

$$-\gamma^2 \frac{\partial^2 V}{\partial x^2} + (c^2 + s^2)\, V(x, s) = \delta(x - \xi)\ e^{-(T-\tau)s}.$$

The solution $V(x, s)$ must

- satisfy the homogeneous differential equation for $x \neq \xi$,
- be continuous at $x = \xi$,
- have a discontinuity with a jump $-e^{-(T-\tau)s}/\gamma^2$ at $x = \xi$,
- remain bounded for $x \to \pm\infty$.

The first and last requirements are satisfied with

$$V(x, s) = A \exp\left(x \sqrt{c^2 + s^2}\,/\gamma\right), \quad x < \xi,$$

$$V(x, s) = B \exp\left(-x \sqrt{c^2 + s^2}\,/\gamma\right), \quad x > \xi.$$

The second and third requirements yield

$$A \exp\left(\xi \sqrt{c^2 + s^2}\,/\gamma\right) - B \exp\left(-\xi \sqrt{c^2 + s^2}\,/\gamma\right) = 0,$$

$$\frac{\sqrt{c^2 + s^2}}{\gamma} \left\{-B \exp\left(-\xi \sqrt{c^2 + s^2}\,/\gamma\right) - A \exp\left(\xi \sqrt{c^2 + s^2}\,/\gamma\right)\right\}$$
$$= -e^{-(T-\tau)s}/\gamma^2$$

with the solution

$$A = \frac{1}{2\gamma \sqrt{c^2 + s^2}} \exp\left(-\xi \sqrt{c^2 + s^2}\,/\gamma\right) e^{-(T-\tau)s},$$

$$B = \frac{1}{2\gamma\sqrt{c^2 + s^2}} \exp\left(\xi\sqrt{c^2 + s^2}\,/\gamma\right) e^{-(T-\tau)s}.$$

Therefore

$$V(x, s) = \frac{e^{-(T-\tau)s}}{2\gamma\sqrt{c^2 + s^2}} \left\{u(\xi - x)\exp\left(-(\xi - x)\sqrt{c^2 + s^2}\,/\gamma\right) + u(x - \xi)\exp\left(-(x - \xi)\sqrt{c^2 + s^2}\,/\gamma\right)\right\}.$$

This can be written as

$$V(x, s) = \frac{e^{-(T-\tau)s}}{2\gamma\sqrt{c^2 + s^2}} \exp\left(-\left|\xi - x\right|\sqrt{c^2 + s^2}\,/\gamma\right).$$

The inverse Laplace transform yields

```
InverseLaplaceTransform[ Exp[- a Sqrt[c^2 + s^2]] / Sqrt[c^2 + s^2], s, ϑ]
```

```
BesselJ[0, c √(-a² + ϑ²)] UnitStep[-a + ϑ]
```

By use of the original shift rule,

$$v(x, \vartheta) = \frac{u[\vartheta - (T - \tau)]}{2\gamma}\, u\left[\vartheta - (T - \tau) - \frac{|\xi - x|}{\gamma}\right] J_0\left[c\sqrt{(\vartheta - (T - \tau))^2 - (|\xi - x|/\gamma)^2}\right]$$

and, by going back to $w(x, t)$

$$w(x, t) = \frac{1}{2\gamma}\, u\left[\tau - t - \frac{|\xi - x|}{\gamma}\right] J_0\left[c\sqrt{(\tau - t)^2 - (|\xi - x|/\gamma)^2}\right]$$

or, equivalently

$$G(x, t; \xi, \tau) = w(x, t) = \frac{1}{2\gamma}\, u[\gamma(\tau - t) - |\xi - x|]\, J_0\left[\frac{c}{\gamma}\sqrt{\gamma^2(\tau - t)^2 - (\xi - x)^2}\right]. \tag{11.32}$$

In Section 8.1 we studied in detail the *telegrapher equation* written as

$$\frac{\partial^2 y}{\partial t^2} + \left(\frac{RC + GL}{LC}\right)\frac{\partial y}{\partial t} + \frac{GR}{LC}\, y(x, t) = \frac{1}{LC}\,\frac{\partial^2 y}{\partial x^2}$$

if $LC \neq 0$. By substituting

$$y(x, t) = \exp\left[-\frac{(RC + GL)}{2LC}\, t\right] z(x, t)$$

we can get rid of the first derivative in time and obtain the *modified telegrapher equation*

$$-\gamma^2\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} - c^2\, y(x, t) = 0$$

with $\gamma = 1/\sqrt{LC}$, $c = (RC - GL)/2LC$. By replacing c by ic in (11.32) and using the property $J_0(i\lambda) = I_0(\lambda)$, where I_0 is the modified Bessel function, we immediately obtain the free space Green's function for the modified telegrapher equations

$$-\gamma^2 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} - c^2 y(x, t) = f(x, t) \qquad (-\infty < x < \infty,\ 0 < t),$$

$$z(x, 0) = \varphi(x), \quad \frac{\partial z(x, 0)}{\partial x} = \psi(x) \qquad (-\infty < x < \infty),$$

i.e.,

$$G(x, t; \xi, \tau) = w(x, t)$$
$$= \frac{1}{2\gamma} u[\gamma(\tau - t) - |\xi - x|]\, I_0\left[\frac{c}{\gamma}\sqrt{\gamma^2(\tau - t)^2 - (\xi - x)^2}\right]. \tag{11.33}$$

The solutions for the inhomogeneous Klein–Gordon and the modified telegrapher equations are then given (see [Zauderer (1983) p. 413–415]by

$$y(\xi, \tau) = \frac{1}{2}\{\varphi(\xi - \gamma\tau) + \varphi(\xi + \gamma\tau)\}$$
$$+ \frac{1}{2\gamma}\int_{\xi-\gamma\tau}^{\xi+\gamma\tau} \psi(x)\, J_0\left[\frac{c}{\gamma}\sqrt{\gamma^2\tau^2 - (\xi - x)^2}\right] dx$$
$$- \frac{c\tau}{2}\int_{\xi-\gamma\tau}^{\xi+\gamma\tau} \varphi(x)\, \frac{J_1\left[\frac{c}{\gamma}\sqrt{\gamma^2\tau^2 - (\xi - x)^2}\right]}{\sqrt{\gamma^2\tau^2 - (\xi - x)^2}}\, dx$$
$$+ \frac{1}{2\gamma}\int_0^\tau \int_{\xi-\gamma(\tau-t)}^{\xi+\gamma(\tau-t)} f(x, t)\, J_0\left[\frac{c}{\gamma}\sqrt{\gamma^2(t - \tau)^2 - (\xi - x)^2}\right] dx\, dt$$

or, respectively,

$$z(\xi, \tau) = \frac{1}{2}\{\varphi(\xi - \gamma\tau) + \varphi(\xi + \gamma\tau)\}$$
$$+ \frac{1}{2\gamma}\int_{\xi-\gamma\tau}^{\xi+\gamma\tau} \psi(x)\, I_0\left[\frac{c}{\gamma}\sqrt{\gamma^2\tau^2 - (\xi - x)^2}\right] dx$$
$$+ \frac{c\tau}{2}\int_{\xi-\gamma\tau}^{\xi+\gamma\tau} \varphi(x)\, \frac{I_1\left[\frac{c}{\gamma}\sqrt{\gamma^2\tau^2 - (\xi - x)^2}\right]}{\sqrt{\gamma^2\tau^2 - (\xi - x)^2}}\, dx$$
$$+ \frac{1}{2\gamma}\int_0^\tau \int_{\xi-\gamma(\tau-t)}^{\xi+\gamma(\tau-t)} f(x, t)\, I_0\left[\frac{c}{\gamma}\sqrt{\gamma^2(t - \tau)^2 - (\xi - x)^2}\right] dx\, dt.$$

Green's function for the half-space boundary value problem

$$-\gamma^2 \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} \pm c^2 y(x, t) = f(x, t) \qquad (0 < x < \infty,\ 0 < t),$$

$$y(0, t) = g(t)$$

with the initial conditions

$$y(x, 0) = \varphi(x), \quad \frac{\partial y(x, 0)}{\partial t} = \psi(x) \qquad (-\infty < x < \infty)$$

is readily obtained by use of the above free space Green's function. If $G_F(x, t; \xi, \tau)$ denotes the free space Green's function, then by the method of images,

$$G(x, t; \xi, \tau) = G_F(x, t; \xi, \tau) - G_F(x, t; -\xi, \tau).$$

11.3 Fundamental Solutions

■ 11.3.1 Parabolic PDE

The above Green's function was obtained as the solution of the adjoint differential equation

$$\frac{1}{\kappa^2}\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = -\delta(x-\xi)\,\delta(t-\tau)$$

subjected to a terminal condition $w(x, T) = 0$. We had to solve a backward initial value problem. We now introduce another notion similar to that of Green's function. The *fundamental solution* of the given initial value problem

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = h(x, t), \quad (-\infty < x < \infty,\ t > 0).$$
$$y(x, 0) = 0$$

is the solution of the associated problem

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial w}{\partial t} = \delta(x-\xi)\,\delta(t-\tau),$$
$$y(x, 0) = 0.$$

Suppose $H(x, t; \xi, \tau)$ is a fundamental solution. Let us consider

$$y(x, t) := \int_{-\infty}^{\infty}\int_{0}^{T} H(x, t; \xi, \tau)\, h(\xi, \tau)\, d\tau\, d\xi, \qquad (T > t)$$

and show that this expression solves the inhomogeneous problem. We assume that differentiation and integration can be interchanged. We then have

$$\frac{\partial^2 y}{\partial x^2} = \int_{-\infty}^{\infty}\int_{0}^{T} \frac{\partial^2 H(x, t; \xi, \tau)}{\partial x^2}\, h(\xi, \tau)\, d\tau\, d\xi,$$
$$\frac{\partial y}{\partial t} = \int_{-\infty}^{\infty}\int_{0}^{T} \frac{\partial H(x, t; \xi, \tau)}{\partial t}\, h(\xi, \tau)\, d\tau.$$

Therefore

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = \int_{-\infty}^{\infty}\int_{0}^{T}\left\{\frac{\partial^2 H(x, t; \xi, \tau)}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial H(x, t; \xi, \tau)}{\partial t}\right\} h(\xi, \tau)\, d\tau\, d\xi$$

$$= \int_{-\infty}^{\infty}\int_{0}^{T} \delta(x-\xi)\,\delta(t-\tau)\,h(\xi,\tau)\,d\tau\,d\xi = h(x,t),$$

as we claimed. Now we solve the homogeneous problem, but with a non-zero initial condition. The solution of

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial w}{\partial t} = 0 \\ &w(x,0) = \varphi(x) \end{aligned}$$

can formally be found by the use of the two-sided Laplace transformation with respect to x.

$$w(x,t) \circ\!\!-\!\!\bullet\; W(s,t), \quad \varphi(x) \circ\!\!-\!\!\bullet\; \Phi(s),$$

$$s^2 W - \frac{1}{\kappa^2}\frac{\partial W(s,t)}{\partial t} = 0, \quad W(s,0) = \Phi(s)$$

$$\Longrightarrow W(s,t) = \Phi(s)\exp[\kappa^2 t\, s^2]\ .$$

By using the correspondence

$$\exp[\kappa^2 s^2 t] \bullet\!\!-\!\!\circ \frac{1}{2\kappa\sqrt{\pi t}}\, e^{-x^2/(4\kappa^2 t)}$$

we obtain, by the convolution theorem,

$$w(x,t) = \frac{1}{2\kappa\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/(4\kappa^2 t)}\,\varphi(\xi)\,d\xi\,.$$

The solution of the inhomogeneous initial value problem

$$\begin{aligned} &\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = h(x,t), \quad (-\infty < x < \infty,\ t > 0), \\ &y(x,0) = \varphi(x) \end{aligned} \tag{11.34}$$

now is the sum of the two above solutions, i.e.,

$$y(x,t) := \int_{-\infty}^{\infty}\int_{0}^{T} H(x,t;\xi,\tau)\,h(\xi,\tau)\,d\tau\,d\xi + \frac{1}{2\kappa\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/(4\kappa^2 t)}\,\varphi(\xi)\,d\xi\,.$$

It remains to find the fundamental solution $w(x,t) = H(x,t;\xi,\tau)$. It is obtained by applying the ordinary Laplace transformation. We take the one-sided Laplace transform with respect to t. The image system of

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial w}{\partial t} = \delta(x-\xi)\,\delta(t-\tau)\ , \\ &y(x,0) = 0 \end{aligned}$$

then is

$$\frac{\partial^2 W(x,s)}{\partial x^2} - \frac{s}{\kappa^2}\,W(x,s) = \delta(x-\xi)\ e^{-\tau s}\ .$$

The solution is continuous at $x = \xi$ but its derivative has a jump of magnitude $e^{-\tau s}$ at $x = \xi$:

$$W(\xi+, s) - W(\xi-, s) = 0, \quad \frac{\partial W(\xi+, s)}{\partial x} - \frac{\partial W(\xi-, s)}{\partial x} = e^{-\tau s}.$$

For $x < \xi$ we take as the solution of the homogeneous equation

$$W(x, s) = A\, e^{x\sqrt{s}/\kappa}, \quad x < \xi,$$

for $x > \xi$ we take

$$W(x, s) = B\, e^{-x\sqrt{s}/\kappa}, \quad x > \xi.$$

These two solutions remain bounded for $x \to \pm\infty$. The above two equations determine A and B. One obtains

$$A = e^{-\tau s}\frac{\kappa}{2\sqrt{s}} e^{-\xi\sqrt{s}/\kappa}, \quad B = e^{-\tau s}\frac{\kappa}{2\sqrt{s}} e^{\xi\sqrt{s}/\kappa}$$

which gives

$$W(x, s) = e^{-\tau s}\frac{\kappa}{2\sqrt{s}} e^{-(\xi-x)\sqrt{s}/\kappa}, \quad x < \xi,$$

$$W(x, s) = e^{-\tau s}\frac{\kappa}{2\sqrt{s}} e^{(\xi-x)\sqrt{s}/\kappa}, \quad x > \xi$$

i.e.,

$$W(x, s) = e^{-\tau s}\frac{\kappa}{2\sqrt{s}} e^{-|\xi-x|\sqrt{s}/\kappa}, \quad -\infty < x < \infty.$$

Taking the inverse Laplace transform we obtain

```
κ / 2 InverseLaplaceTransform[
  Exp[- Abs[ξ - x] √s /κ] / √s // TimesExp[-τ, s], s, t ]
```

$$\frac{e^{-\frac{\text{Abs}[-x+\xi]^2}{4\kappa^2(t-\tau)}}\,\kappa\,\text{UnitStep}[t-\tau]}{2\sqrt{\pi}\sqrt{t-\tau}}$$

$$w(x, t) = u[t-\tau]\frac{\kappa}{2\sqrt{\pi(t-\tau)}} e^{-\frac{(\xi-x)^2}{4\kappa^2(t-\tau)}} = H(x, t; \xi, \tau).$$

The solution of (11.34) is now given by

$$y(x, t) := \frac{\kappa}{2}\int_{-\infty}^{\infty}\int_0^t \frac{h(\xi, \tau)}{\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(\xi-x)^2}{4\kappa^2(t-\tau)}\right] d\tau\, d\xi$$
$$+ \frac{1}{2\kappa\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/(4\kappa^2 t)}\,\varphi(\xi)\, d\xi,$$

in agreement with the solution found by the Green's function technique at the end of Section 11.2.1.3. Observe that the solution can be written as

$$y(x,\,t) := \int_{-\infty}^{\infty}\int_{0}^{t} H(x,\,t;\,\xi,\,\tau)\;h(\xi,\,\tau)\,d\tau\,d\xi + \frac{1}{\kappa^2}\int_{-\infty}^{\infty} H(x,\,t;\,\xi,\,0)\;h(\xi,\,\tau)\;d\xi\,.$$

11.3.1.1 Half-Space with Radiation Boundary Condition

Another example is the *half-space problem with a radiation boundary condition.* The mathematical model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = h(x,\,t),\quad (0 < x,\;t > 0)\,,$$
$$\frac{\partial y(0,\,t)}{\partial x} = h\,y(0,\,t),\quad y(x,\,0) = \varphi(x)\,.$$

For the fundamental solution, $w(x,\,t) = H(x,\,t;\,\xi,\,\tau)$ it must hold that

$$-\frac{1}{\kappa^2}\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = \delta(x-\xi)\,\delta(t-\tau)\qquad (0 < t,\,\tau\,;\;0 < x,\,\xi < \infty)\,,$$
$$\frac{\partial w(0,\,t)}{\partial x} = h\,w(0,\,t)\,.$$

We determine this function in the form

$$w(x,\,t) = H(x,\,t;\,\xi,\,\tau) := H_F(x,\,t;\,\xi,\,\tau) + v(x,\,t)$$

where

- $H_F(x,\,t;\,\xi,\,\tau)$ is the fundamental solution,
- $v(x,\,t)$ satisfies the homogeneous heat conduction equation with zero initial condition,
- $H(x,\,t;\,\xi,\,\tau)$ satisfies the radiation boundary condition.

We use the Laplace transformation technique. The Laplace transform of the free space fundamental solution is (see the last section)

$$\mathcal{L}H_F(x,\,s;\,\xi,\,\tau) = e^{-\tau s}\,\frac{\kappa}{2\sqrt{s}}\;e^{-|\xi-x|\sqrt{s}/\kappa}\,,\quad -\infty < x,\,\xi < \infty\,,$$

thus

$$W(x,\,s) = e^{-\tau s}\,\frac{\kappa}{2\sqrt{s}}\;e^{-|\xi-x|\sqrt{s}/\kappa} + V(x,\,s).$$

$V(x,\,s)$ satisfies the image system

$$\frac{\partial^2 V}{\partial x^2} - \frac{s}{\kappa^2}\,V(x,\,s) = 0$$

with the solution $V(x,\,s) = A\exp\left[-x\sqrt{s}\,/\,\kappa\right]$ remaining bounded for $x \to \infty$.

$$W(x,\,s) = e^{-\tau s}\,\frac{\kappa}{2\sqrt{s}}\;e^{-|\xi-x|\sqrt{s}/\kappa} + A\exp\left[-x\sqrt{s}\,/\,\kappa\right]$$

must satisfy

$$\frac{\partial W(0,\,s)}{\partial x} = h\,W(0,\,s)$$

which determines the constant A. We obtain

$$A = -\frac{\kappa\left(h\kappa - \sqrt{s}\right)}{2\left(h\kappa + \sqrt{s}\right)\sqrt{s}}\, e^{-\tau s}\, e^{-\xi\sqrt{s}/\kappa},$$

$$W(x,\,s) = e^{-\tau s}\,\frac{\kappa}{2\sqrt{s}}\left\{ e^{-|\xi - x|\sqrt{s}/\kappa} + \frac{\sqrt{s} - h\kappa}{\sqrt{s} + h\kappa}\, e^{-(\xi + x)\sqrt{s}/\kappa} \right\}.$$

By taking the inverse Laplace transform

```
κ / 2 * InverseLaplaceTransform[Exp[-a Sqrt[s] / κ] / Sqrt[s], s, t]
+ κ / 2 *
  InverseLaplaceTransform[Exp[-b Sqrt[s] / κ] / (Sqrt[s] + h κ), s, t] +
 - κ^2 h / 2 InverseLaplaceTransform[
    Exp[-b Sqrt[s] / κ] / (Sqrt[s] + h κ) / Sqrt[s], s, t]
```

$$\frac{e^{-\frac{a^2}{4\,t\,\kappa^2}}\,\kappa}{2\sqrt{\pi}\,\sqrt{t}}$$

$$-\frac{1}{2}\, e^{b\,h + h^2\,t\,\kappa^2}\, h\,\kappa^2\, \text{Erfc}\left[\frac{b}{2\sqrt{t}\,\kappa} + h\sqrt{t}\,\kappa\right] +$$

$$\frac{1}{2}\,\kappa\left(\frac{e^{-\frac{b^2}{4\,t\,\kappa^2}}}{\sqrt{\pi}\,\sqrt{t}} - e^{b\,h + h^2\,t\,\kappa^2}\, h\,\kappa\, \text{Erfc}\left[\frac{b}{2\sqrt{t}\,\kappa} + h\sqrt{t}\,\kappa\right]\right)$$

we obtain the fundamental solution for the half-space problem with radiation boundary condition

$$w(x,\,t) = H(x,\,t;\,\xi,\,\tau) = u(t-\tau)\,\frac{\kappa}{2\sqrt{\pi\,(t-\tau)}}\left\{e^{-\frac{(\xi - x)^2}{4(t-\tau)\kappa^2}} + e^{-\frac{(\xi + x)^2}{4(t-\tau)\kappa^2}}\right\}$$

$$- h\,\kappa^2\, e^{(\xi + x)h + h^2\,\kappa^2(t-\tau)}\,\text{Erfc}\left[\frac{\xi + x}{2\sqrt{(t-\tau)}\,\kappa} + h\sqrt{(t-\tau)}\,\kappa\right].$$

The solution of the original boundary value problem then is

$$y(x,\,t) = \int_0^t \int_0^\infty H(x,\,t,\,\xi,\,\tau)\,h(x,\,t)\,d\xi\,d\tau + \frac{1}{\kappa^2}\int_0^\infty \varphi(\xi)\,H(x,\,t;\,\xi,\,0)\,d\xi\,.$$

Green's function technique and the fundamental solution concept are not always distinguished in the literature. Both approaches often are called Green's function techniques. For the self-adjoint elliptic and hyperbolic equations Green's function turns out to be the same as the fundamental solution. For the not self-adjoint parabolic equations the two approaches are different as we have seen above, but finally lead to the same result. In this book, as in [Zauderer (1983)], Green's function techniques are based on Green's theorem in the plane and lead to a terminal value problem. The problems below and the next section show that the fundamental solution concept, together with the means of Laplace transformation, also give general answers.

The above problem can be solved in another way by using the steps of problems 11.4 to 11.6.

Problem 11.4. The Laplace transform with respect to t of

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2}\frac{\partial y}{\partial t} = h(x, t), \quad (0 < x,\ t > 0),$$
$$\frac{\partial y(0, t)}{\partial x} = h\, y(0, t), \quad y(x, 0) = \varphi(x)$$

yields

$$-\frac{\partial^2 Y(x, s)}{\partial x^2} + \frac{s}{\kappa^2}\partial Y(x, s) = -\left\{H(x, s) + \frac{\varphi(x)}{\kappa^2}\right\}, \quad (0 < x),$$
$$\frac{\partial Y(0, s)}{\partial x} = h\, Y(0, s).$$

Problem 11.5. Now, consider the problem

$$\frac{\partial^2 W(x, \xi)}{\partial x^2} - \frac{s}{\kappa^2}\partial W(x, \xi) = \delta(x - \xi),$$
$$\frac{\partial W(0, \xi)}{\partial x} = h\, W(0, \xi)$$

and show that the solution is

$$W(x, \xi) = -\frac{\kappa}{2}\left\{\frac{\sqrt{s} - \kappa h}{\sqrt{s}\left(\sqrt{s} + \kappa h\right)} e^{-\sqrt{s}\,(x+\xi)/\kappa} + \frac{1}{\sqrt{s}} e^{-\sqrt{s}\,|x-\xi|/\kappa}\right\}.$$

With $G(x, \xi, s) := W(x, \xi)$ for the fundamental solution we then have

$$Y(x, s) = -\int_0^\infty G(x, \xi, s)\left[H(\xi, s) + \frac{\varphi(\xi)}{\kappa^2}\right] d\xi.$$

Problem 11.6. Take the inverse Laplace transform and get

$$y(x, t) = -\int_0^\infty \mathcal{L}^{-1}[G(x, \xi, s)\ H(\xi, s)\,](t)\ d\xi - \frac{1}{\kappa^2}\int_0^\infty \mathcal{L}^{-1}[G(x, \xi, s)](t)\,\varphi(\xi)\, d\xi.$$

By the convolution theorem one obtains

$$y(x, t) = -\int_0^\infty \int_0^t g(x, \xi, t - \tau)\ h(\xi, \tau)\, d\tau\ d\xi - \frac{1}{\kappa^2}\int_0^\infty g(x, \xi, t)\ \varphi(\xi)\, d\xi$$

with

$$g(x, \xi, t) = -\frac{\kappa}{2}\mathcal{L}^{-1}\left[\frac{\sqrt{s} - \kappa h}{\sqrt{s}\left(\sqrt{s} + \kappa h\right)} e^{-\sqrt{s}\,(x+\xi)/\kappa} + \frac{1}{\sqrt{s}} e^{-\sqrt{s}\,|x-\xi|/\kappa}\right](t)$$
$$= -\frac{\kappa}{2\sqrt{\pi t}}\left\{e^{-\frac{(\xi-x)^2}{4t\kappa^2}} + e^{-\frac{(\xi+x)^2}{4t\kappa^2}}\right\} + h\kappa^2 e^{(\xi+x)h + h^2\kappa^2 t}\,\mathrm{Erfc}\left[\frac{\xi + x}{2\sqrt{t}\,\kappa} + h\sqrt{t}\,\kappa\right]$$

as above.

11.3.1.2 A General Approach

In this section we establish general formulas by using Laplace transformation and a result from ordinary differential equations. Let us use the following theorem:

Theorem. *The solution of the inhomogeneous boundary value problem*

$$-\frac{d}{dx}\left(p(x)\,\frac{dy}{dx}\right)+q(x)\,y(x)=f(x)\,,$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = a\,,\quad y(\ell)\cos\beta + y'(\ell)\sin\beta = b$$

is given by

$$y(x)=\int_0^{\ell} h(x,\xi)\,f(\xi)\,d\xi - a\,\frac{p(0)}{J}[w_1(0)\sin\alpha - w_1'(0)\cos\alpha]\,w_2(x)$$

$$+\,b\,\frac{p(\ell)}{J}[w_2(\ell)\sin\beta - w_2'(\ell)\cos\beta]\,w_1(x)\,,$$

where the functions $w_1(x)$ and $w_2(x)$ are independent solutions of the homogeneous problems

$$-\frac{d}{dx}\left(p(x)\,\frac{dy}{dx}\right)+q(x)\,y(x)=0\,,\ y(0)\cos\alpha + y'(0)\sin\alpha = 0$$

and

$$-\frac{d}{dx}\left(p(x)\,\frac{dy}{dx}\right)+q(x)\,y(x)=0,\ \ y(\ell)\cos\beta + y'(\ell)\sin\beta = 0$$

respectively, and

$$h(x,\xi) = \frac{w_1(x)\,w_2(\xi)\,u(\xi-x)+w_2(x)\,w_1(\xi)\,u(x-\xi)}{J}$$

with the constant conjunct $J := J(w_2, w_1) = -p\,(w_2'\,w_1 - w_2\,w_1')$ that can be computed at any point of $(0, \ell)$.

A proof of this theorem can be found in [Friedman B. (1966)] p. 167.

Let us now consider the boundary value problem

$$-\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial y(x,t)}{\partial x}\right)+\frac{1}{\kappa^2}\,\frac{\partial y(x,t)}{\partial t}=f(x,t)\,,$$

$$y(0,t)\cos\alpha + \frac{\partial y(0,t)}{\partial x}\sin\alpha = g_1(t)\ ,\ y(\ell,t)\cos\beta + \frac{\partial y(\ell,t)}{\partial x}\sin\beta = g_2(t)\,,$$

$$y(x,0)=\varphi(x)\,,\quad (0<x<\ell)\,.$$

With the Laplace transforms

$$y(x,t)\ \circ\!-\!\bullet\ Y(x,s),\ f(x,t)\ \circ\!-\!\bullet\ F(x,s),\ g_1(t)\ \circ\!-\!\bullet\ G_1(s),\ g_2(t)\ \circ\!-\!\bullet\ G_2(s)$$

the image system becomes

$$-\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial Y}{\partial x}\right)+\frac{s}{\kappa^2}\,Y(x,s)=F(x,s)+\frac{1}{\kappa^2}\,\varphi(x)\,,$$

$$Y(0,s)\cos\alpha+\frac{\partial Y(0,s)}{\partial x}\sin\alpha=G_1(s)\,,$$

$$Y(\ell,s)\cos\beta+\frac{\partial Y(\ell,s)}{\partial x}\sin\beta=G_1(s)\,.$$

This is in fact a boundary value problem for an ordinary differential equation. Now, let us use the above theorem. With $W_1(x,s)$ and $W_2(x,s)$ satisfying

$$-\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial W_1}{\partial x}\right)+\frac{s}{\kappa^2}\,W_1=0\,,\quad W_1(0,s)\cos\alpha+\frac{\partial W_1(0,s)}{\partial x}\sin\alpha=0\,,$$

$$-\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial W_2}{\partial x}\right)+\frac{s}{\kappa^2}\,W_2=0\,,\quad W_2(\ell,s)\cos\beta+\frac{\partial W_2(\ell,s)}{\partial x}\sin\beta=0$$

respectively, we form

$$H_s(x,\xi)=\frac{W_1(x,s)\,W_2(\xi,s)\,u(\xi-x)+W_2(x,s)\,W_1(\xi,s)\,u(x-\xi)}{J(s)}$$

with

$$J(s)=-p(x)\left[\frac{\partial W_2(x,s)}{\partial x}\,W_1(x,s)-W_2(x,s)\,\frac{\partial W_1(x,s)}{\partial x}\right].$$

The conjunct $J(s)$ does not depend on x, i.e. the right-hand side can be computed for any value $x\in[0,\ell]$. The above theorem now implies

$$\begin{aligned}Y(x,s)=&\int_0^\ell H_s(x,\xi)\left\{F(\xi,s)+\frac{1}{\kappa^2}\,\varphi(\xi)\right\}d\xi\\&-p(0)\,\frac{G_1(s)}{J(s)}\left[W_1(0,s)\sin\alpha-\frac{\partial W_1(0,s)}{\partial x}\cos\alpha\right]W_2(x,s)\\&+p(\ell)\,\frac{G_2(s)}{J(s)}\left[W_2(\ell,s)\sin\beta-\frac{\partial W_2(\ell,s)}{\partial x}\cos\beta\right]W_1(x,s)\,.\end{aligned}\tag{11.35}$$

Since

$$H_s(x,0)=\frac{W_2(x,s)\,W_1(0,s)}{J(s)}\,,\quad\frac{\partial H_s(x,0)}{\partial\xi}=\frac{W_2(x,s)}{J(s)}\,\frac{\partial W_1(0,s)}{\partial x}\,,$$

$$H_s(x,\ell)=\frac{W_1(x,s)\,W_2(\ell,s)}{J(s)}\,,\quad\frac{\partial H_s(x,\ell)}{\partial\xi}=\frac{W_1(x,s)}{J(s)}\,\frac{\partial W_2(\ell,s)}{\partial x}\,,$$

we can also write

$$\begin{aligned}Y(x,s)=&\int_0^\ell H_s(x,\xi)\left\{F(\xi,s)+\frac{1}{\kappa^2}\,\varphi(\xi)\right\}d\xi\\&-p(0)\,G_1(s)\left[H_s(x,0)\sin\alpha-\frac{\partial H_s(x,0)}{\partial\xi}\cos\alpha\right]\\&+p(\ell)\,G_2(s)\left[H_s(x,\ell)\sin\beta-\frac{\partial H_s(x,\ell)}{\partial\xi}\cos\beta\right].\end{aligned}$$

By taking the inverse Laplace transform and by using the convolution theorem, we obtain

$$y(x,t) = \int_0^\ell \int_0^t h(x,\xi,t-\eta)\, f(\xi,\eta)\, d\eta\, d\xi + \frac{1}{\kappa^2} \int_0^\ell h(x,\xi,t)\, \varphi(\xi)\, d\xi$$
$$- p(0) \int_0^t g_1(\eta) \left[h(x,0,t-\eta) \sin\alpha - \frac{\partial\, h(x,0,t-\eta)}{\partial \xi} \cos\alpha \right] d\eta \qquad (11.36)$$
$$+ p(\ell) \int_0^t g_2(\eta) \left[h(x,\ell,t-\eta) \sin\beta - \frac{\partial\, h(x,\ell,t-\eta)}{\partial \xi} \cos\beta \right] d\eta$$

where

$$h(x,\xi,t) = \mathcal{L}^{-1}[H_s(x,\xi)]\,(t)$$
$$= u(\xi - x)\, \mathcal{L}^{-1}\left[\frac{W_1(x,s)\, W_2(\xi,s)}{J(s)} \right](t) + u(x-\xi)\, \mathcal{L}^{-1}\left[\frac{W_2(x,s)\, W_1(\xi,s)}{J(s)} \right](t).$$

Problem 11.7. Use (11.35) to solve

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\kappa^2} \frac{\partial y}{\partial t} = f(x,t), \quad (0 < x < \ell,\ t > 0),$$
$$y(0,t) = y_0\,, \quad y(\ell,t) = y_\ell, \quad y(x,0) = \varphi(x).$$

Verify that the result is in agreement with the solution found at the end of Section 11.2.1.2.

Problem 11.8. Use this approach to develop a similar set of formulas for the wave equation

$$-\frac{\partial}{\partial x}\left(p(x) \frac{\partial y(x,t)}{\partial x} \right) + \frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2} = f(x,t),$$
$$y(0,t)\cos\alpha + \frac{\partial y(0,t)}{\partial x} \sin\alpha = g_1(t)\,, \; y(\ell,t)\cos\beta + \frac{\partial y(\ell,t)}{\partial x} \sin\beta = g_2(t),$$
$$y(x,0) = \varphi(x), \quad \frac{\partial y(x,0)}{\partial t} = \psi(x), \quad (0 < x < \ell).$$

Problem 11.9. Modify the results if the boundary conditions are given in the general form

$$a_{11}\, y(0,t) + a_{12} \frac{\partial y(0,t)}{\partial x} = g_1(t)\,, \; a_{21}\, y(\ell,t) + a_{22} \frac{\partial y(\ell,t)}{\partial x} = g_2(t).$$

11.4 Finite Fourier Transforms

Duhamel's formulas are helpful, if the partial differential equation and one of the two boundary conditions are homogeneous whereas the other boundary condition is non-homogeneous (see Section 11.1). Green's function technique has a still wider scope of application comprising non-homogeneous partial differential equations and non-homogeneous boundary and initial conditions (see Section 11.2). Another technique for solving certain types of boundary value problems over a finite space interval uses Finite Fourier transforms. It can work as well for homogeneous and non-homogeneous problems. The Finite Fourier transform is intimately related to the theory of Fourier Series and can be regarded as a reformulation of this theory. There is a Finite Fourier Sine transform and a Finite Fourier Cosine transform. In regard to boundary value problems, the former is more important. Often, the Finite Fourier transform is applied in connection with the Laplace transform.

■ 11.4.1 Finite Fourier Sine and Cosine Transforms

Definition. Let $f(x)$ be defined on the interval $0 < x < \ell$. The *Finite Fourier Sine transform* of $f(x)$ is the sequence $F_s(n)$, $n = 1, 2, 3, \ldots$ defined as

$$F_s(n) = \int_0^\ell f(x) \sin\left(\frac{\pi}{\ell} x n\right) dx . \tag{11.37}$$

Similarly, the *Finite Fourier Cosine transform* of $f(x)$ is the sequence $F_c(n)$, $n = 0, 1, 2, 3, \ldots$ defined as

$$F_c(n) = \int_0^\ell f(x) \cos\left(\frac{\pi}{\ell} x n\right) dx . \tag{11.38}$$

Let us use the notation:

$$F_s(n) = S[f(x)](n), \quad f(x) \overset{S}{\to} F_s(n),$$

$$F_c(n) = C[f(x)](n), \quad f(x) \overset{C}{\to} F_c(n).$$

The basic operational properties are the *Differentiation Rules*. They are given below, but their derivation is left as an exercise in partial integration. For the Sine transform,

$$f''(x) \overset{S}{\to} -\left(\frac{\pi}{\ell} n\right)^2 F_s(n) + \left(\frac{\pi}{\ell} n\right)[f(0) - (-1)^n f(\ell)], \tag{11.39}$$

$$f^{(4)}(x) \overset{S}{\to} \left(\frac{\pi}{\ell} n\right)^4 F_s(n) - \left(\frac{\pi}{\ell} n\right)^3 [f(0) - (-1)^n f(\ell)] + \left(\frac{\pi}{\ell} n\right)[f''(0) - (-1)^n f''(\ell)] \tag{11.40}$$

and for the Cosine transform,

$$f''(x) \overset{C}{\to} -\left(\frac{\pi}{\ell} n\right)^2 F_s(n) - f'(0) + (-1)^n f'(\ell), \tag{11.41}$$

$$f^{(4)}(x) \overset{C}{\to} \left(\frac{\pi}{\ell} n\right)^4 F_s(n) + \left(\frac{\pi}{\ell} n\right)^2 [f'(0) - (-1)^n f'(\ell)] - f^{(3)}(0) + (-1)^n f^{(3)}(\ell) \tag{11.42}$$

provided that $f(x) \overset{S}{\to} F_s(n)$, $f(x) \overset{C}{\to} F_c(n)$ and $f(x)$ is sufficiently differentiable. It is seen that the Finite Fourier Sine transform is useful when the boundary conditions only comprise even order derivatives, whereas the Finite Cosine transform is applicable when only odd order derivatives in the boundary condition are present. The inverse Finite Fourier transforms are given by the appropriate form of the Fourier series.

The *inverse Finite Fourier Sine transform* is given by

$$\frac{f_0(x-) + f_0(x+)}{2} = \frac{2}{\ell} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{\pi}{\ell} x n\right)$$

where $f_0(x) = f_0(x + 2\ell)$ is the odd-periodic extension of period 2ℓ of $f(x)$ on the entire real axis. Therefore, if $f(x)$ is continuous on $[0, \ell]$ with $f(0) = f(\ell) = 0$, we can write

$$f(x) = \frac{2}{\ell} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{\pi}{\ell} x n\right). \tag{11.43}$$

The *inverse Finite Fourier Cosine transform* is given by

$$\frac{f_0(x-) + f_0(x+)}{2} = \frac{1}{\ell} F_c(0) + \frac{2}{\ell} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{\pi}{\ell} x n\right)$$

where $f_0(x) = f_0(x + 2\ell)$ is the even-periodic extension of period 2ℓ of $f(x)$ on the entire real axis. Therefore, if $f(x)$ is continuous on $[0, \ell]$, we can write

$$f(x) = \frac{1}{\ell} F_c(0) + \frac{2}{\ell} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{\pi}{\ell} x n\right). \tag{11.44}$$

There are many other interesting operational properties concerning the Finite Fourier transforms, see, for example, [Churchill (1972)]. A further reference for the use of the Finite Fourier transform applied to vibration problems is [Graff (1991)].

■ 11.4.2 Use of the Package

Use the `FiniteFourierSineTransform[f[x],x,n,ℓ]` command to compute the Finite Fourier Sine transform of $f(x)$ defined on $[0, \ell]$. The right interval endpoint ℓ is optional. If omitted, then $\ell = \pi$ is assumed. For example:

```
FiniteFourierSineTransform[x^2, x, n, L]
```

$$-\frac{L^3 \, (2 - 2 \, (-1)^n + (-1)^n \, n^2 \, \pi^2)}{n^3 \, \pi^3}$$

```
F = FiniteFourierSineTransform[x (Pi - x), x, n]
```

$$-\frac{2\,(-1+(-1)^n)}{n^3}$$

The `InverseFiniteFourierSineTransform[F[n],n,x,L]` command gives the inverse Finite Fourier Sine Transform. For example:

```
InverseFiniteFourierSineTransform[F, n, x]
```

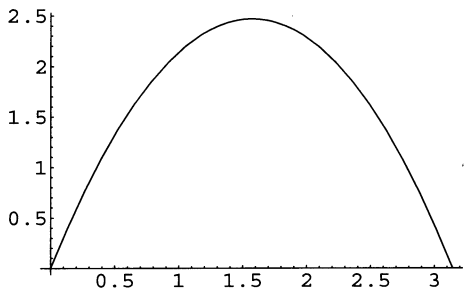
$$\frac{2\,\text{Summation}\left[-\frac{2\,(-1+(-1)^n)\,\text{Sin}[n\,x]}{n^3},\ \{n,\ 1,\ \infty\}\right]}{\pi}$$

If in the above *Mathematica* output `Summation` is replaced by `Sum`, then the infinite series can perhaps be evaluated.

```
% /. Summation → Sum
```

$$-\frac{1}{\pi}\,(2\,i\,(\text{PolyLog}[3,\,-e^{-i\,x}]-\text{PolyLog}[3,\,e^{-i\,x}]-\text{PolyLog}[3,\,-e^{i\,x}]+\text{PolyLog}[3,\,e^{i\,x}]))$$

```
Plot[N[%] // Chop, {x, 0, Pi}];
```



Of course this is no longer possible for a more complicated inversion. The commands for the Finite Fourier Cosine transform and its inverse are similar. We now demonstrate the simultaneous use of the Laplace and Finite Fourier transforms in the following examples.

Example. Let us consider the forced vibration of a string, fixed at $x = 0$ and $x = \ell$, with a general loading $q(x, t)$. The model is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} + \frac{1}{\rho c^2}\,q(x,t) = 0, \qquad \frac{1}{c^2} = \frac{\rho}{T}, \quad (0 < x < \ell,\ t > 0),$$

$$y(0, t) = y(\ell, 0) = 0,$$

$$y(x, 0) = h(x), \quad \frac{\partial y(x, 0)}{\partial t} = g(x),$$

where T is the tension of the string and ρ the mass density (see [Graff (1991)]). We first apply the Laplace transform with $y(x, t) \,\square\!\!-\!\!\blacksquare\, Y(x, s)$, $q(x, t) \,\square\!\!-\!\!\blacksquare\, Q(x, s)$ and obtain

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s^2}{c^2}\,Y(x,s) = -\frac{1}{\rho c^2}\,Q(x,s) - \frac{s\,h(x)}{c^2} - \frac{g(x)}{c^2},$$

$$Y(0, s) = Y(\ell, s) = 0.$$

Now apply the Finite Fourier Sine transform with $Y(x, s) \overset{S}{\to} \overline{Y}(n, s)$ $Q(x, s) \overset{S}{\to} \overline{Q}(n, s)$ and $h(x) \overset{S}{\to} \overline{h}(n)$, $g(x) \overset{S}{\to} \overline{g}(n)$. The Differentiation Rule for the Finite Fourier Sine transform with the boundary condition $Y(0, s) = Y(\ell, s) = 0$ implies

$$\left[-\left(\frac{\pi n}{\ell}\right)^2 - \frac{s^2}{c^2}\right]\overline{Y}(n, s) = -\frac{1}{\rho c^2}\overline{Q}(n, s) - \frac{s\,\overline{h}(n)}{c^2} - \frac{\overline{g}(n)}{c^2},$$

and finally,

$$\overline{Y}(n, s) = \frac{\overline{Q}(n, s)}{\rho\,(s^2 + \lambda^2 n^2)} + \frac{s\,\overline{h}(n)}{(s^2 + \lambda^2 n^2)} + \frac{\overline{g}(n)}{(s^2 + \lambda^2 n^2)}, \quad \left(\lambda = \frac{\pi c}{\ell}\right).$$

A) If for example $q(x, t) = 0$, $h(x) = h_0\, x(\ell - x)$, $g(x) = 0$, we obtain with

```
FiniteFourierSineTransform[h0 x (ℓ - x), x, n, ℓ]
```

$$-\frac{2\,(-1 + (-1)^n)\,\ell^3\,h_0}{n^3\,\pi^3}$$

$$\overline{Y}(n, s) = \frac{2\ell^3 h_0}{\pi^3}\,\frac{(1 - (-1)^n)}{n^3}\,\frac{s}{(s^2 + \lambda^2 n^2)}.$$

The inverse Laplace transform yields

$$\overline{y}(n, t) = \frac{2\ell^3 h_0}{\pi^3}\,\frac{(1 - (-1)^n)}{n^3}\cos(\lambda n t).$$

The inverse Finite Fourier Sine transform yields

$$y(x, t) = \frac{4\ell^2 h_0}{\pi^3}\sum_{n=1}^{\infty}\frac{(1 - (-1)^n)}{n^3}\cos(\lambda n t)\sin\left(\frac{\pi}{\ell}x n\right)$$

$$= \frac{8\ell^2 h_0}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\cos\left(\frac{\pi c}{\ell}(2k-1)t\right)\sin\left(\frac{\pi}{\ell}x(2k-1)\right)$$

$$= \frac{4\ell^2 h_0}{\pi^3}\sum_{k=1}^{\infty}\frac{\sin[\frac{\pi}{\ell}(2k-1)(x - ct)] + \sin[\frac{\pi}{\ell}(2k-1)(x + ct)]}{(2k-1)^3}$$

.

B) Let for example $q(x, t) = \delta(x - \ell/2)\cos t$, $h(x) = 0$, $g(x) = 0$. We then have

$$\overline{Q}(n, s) = \sin\left(\frac{\pi}{\ell}\frac{\ell}{2}\right)\frac{s}{s^2 + 1} = \frac{s}{s^2 + 1},$$

$$\overline{Y}(n, s) = \frac{s}{\rho\,(s^2 + \lambda^2 n^2)(s^2 + 1)}.$$

If $\lambda = \pi c/\ell \neq 1$, i.e. in a non-resonance case,

```
InverseLaplaceTransform[s / ((s^2 + (λ n)^2) (s^2 + 1)), s, t]
```

$$2\left(\frac{1}{4\,(-1 + n\lambda)} - \frac{1}{4\,(1 + n\lambda)}\right)\text{Cos}[t] + 2\left(-\frac{1}{4\,(-1 + n\lambda)} + \frac{1}{4\,(1 + n\lambda)}\right)\text{Cos}[n\,t\,\lambda]$$

and we obtain

$$\bar{y}(n, t) = \frac{\cos t - \cos(n\lambda t)}{\rho\,[(n\lambda)^2 - 1]}.$$

The inverse Finite Fourier Sine transform gives

$$y(x, t) = \frac{2}{\ell\rho} \sum_{n=1}^{\infty} \frac{\cos t - \cos(\frac{\pi c}{\ell} n t)}{(\frac{\pi c}{\ell} n)^2 - 1} \sin\left(\frac{\pi}{\ell} x n\right).$$

C) Let us study the case with a general loading $q(x, t)$ and $h(x) = 0$, $g(x) = 0$.

$$\bar{Y}(n, s) = \frac{\bar{Q}(n, s)}{\rho\,(s^2 + \lambda^2 n^2)}, \quad \left(\lambda = \frac{\pi c}{\ell}\right).$$

By the convolution theorem for the Laplace transformation we can write

$$\bar{y}(n, t) = \frac{1}{\rho\lambda n} \int_0^t \bar{q}(n, \tau)\, \sin[\lambda n(t - \tau)]\, d\tau$$

and by taking the inverse Finite Fourier Sine transform

$$y(x, t) = \frac{2}{\ell\rho\lambda} \sum_{n=1}^{\infty} \frac{\int_0^t \bar{q}(n, \tau)\, \sin[\lambda n(t - \tau)]\, d\tau}{n} \sin\left(\frac{\pi}{\ell} x n\right),$$

$$\bar{q}(n, \tau) = \int_0^{\ell} q(\xi, \tau) \sin\left(\frac{\pi}{\ell} \xi n\right) d\xi$$

$$\Longrightarrow \int_0^t \int_0^{\ell} q(\xi, \tau) \sin\left(\frac{\pi}{\ell} \xi n\right) \sin[\lambda n(t - \tau)]\, d\xi\, d\tau$$

$$= \int_0^{\ell} \sin\left(\frac{\pi}{\ell} \xi n\right) \int_0^t q(\xi, \tau)\, \sin[\lambda n(t - \tau)]\, d\tau\, d\xi\,.$$

Therefore, we obtain

$$y(x, t) = \frac{2}{\rho\pi c}$$

$$\times \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_0^{\ell} \sin\left(\frac{\pi}{\ell} \xi n\right) \int_0^t q(\xi, \tau)\, \sin\left[\frac{\pi c}{\ell} n(t - \tau)\right] d\tau\, d\xi \right\} \sin\left(\frac{\pi}{\ell} x n\right).$$

Example. Let us consider a finite, prestressed, visco-elastically supported beam. The governing equation for the transversal displacement $y(x, t)$ of a beam, including the effects of a visco-elastic foundation, and prestressed by a tension T is derived in [Graff (1991)] p. 171 f. The visco-elastic foundation reacts on the beam via the force

$$f(x, t) = k\,y + c\,\frac{\partial y}{\partial t}.$$

The final form for the transverse equation of motion is

$$E I \frac{\partial^4 y}{\partial x^4} - T \frac{\partial^2 y}{\partial x^2} + k\, y(x, t) + c \frac{\partial y}{\partial t} + \rho A \frac{\partial^2 y}{\partial t^2} = q(x, t)$$
$$(0 < x < \ell,\ t > 0).$$

The case of axial compression is merely obtained by changing the sign of T. We assume the loading to be of the form $q(x, t) = q(x)\, p(t)$, and that the beam is pinned at both ends, implying the boundary conditions

$$y(0, t) = \frac{\partial^2 y(0, t)}{\partial x^2} = 0, \quad y(\ell, t) = \frac{\partial^2 y(\ell, t)}{\partial x^2} = 0.$$

The initial conditions are assumed to be

$$y(x, 0) = h(x), \quad \frac{\partial y(x, 0)}{\partial t} = g(x).$$

With the Laplace transforms $y(x, t) \circ\!-\!\bullet\, Y(x, s)$, $p(t) \circ\!-\!\bullet\, P(s)$, we obtain the first image equation

$$E I \frac{\partial^4 Y}{\partial x^4} - T \frac{\partial^2 Y}{\partial x^2} + (k + c s + \rho A s^2)\, Y(x, s)$$
$$= q(x)\, P(s) + (c + \rho A s)\, h(x) + \rho A\, g(x)$$

with the transformed boundary conditions

$$Y(0, s) = \frac{\partial^2 Y(0, s)}{\partial x^2} = 0, \quad Y(\ell, s) = \frac{\partial^2 Y(\ell, s)}{\partial x^2} = 0.$$

These boundary conditions call for the Finite Fourier Sine transform. With the transform $Y(x, s) \overset{S}{\to} \overline{Y}(n, s)$ and the Differentiation Rule for the Finite Fourier Sine transform, we obtain the second image equation

$$\left[E I \left(\frac{\pi}{\ell} n\right)^4 + T \left(\frac{\pi}{\ell} n\right)^2 + (k + c s + \rho A s^2)\right] \overline{Y}(n, s)$$
$$= \overline{q}(n)\, P(s) + (c + \rho A s)\, \overline{h}(n) + \rho A\, \overline{g}(n)$$

with the Finite Fourier Sine transforms $q(x) \overset{S}{\to} \overline{q}(n)$, $h(x) \overset{S}{\to} \overline{h}(n)$, $g(x) \overset{S}{\to} \overline{g}(n)$.

A) Consider the case $q(x, t) = 0$, $h(x) = h_0\, x(\ell - x)$, $g(x) = 0$. The latter equation then gives

$$\left[E I \left(\frac{\pi}{\ell} n\right)^4 + T \left(\frac{\pi}{\ell} n\right)^2 + (k + c s + \rho A s^2)\right] \overline{Y}(n, s)$$
$$= \frac{2 \ell^3 h_0}{\pi^3} (c + \rho A s) \frac{(1 - (-1)^n)}{n^3}$$

where we used the Finite Fourier Sine transform

$$\overline{h}(n) \overset{S}{\to} \frac{2 \ell^3 h_0}{\pi^3} \frac{(1 - (-1)^n)}{n^3}$$

from above. We finally obtain

$$\overline{Y}(n, s) = \frac{2 \ell^3 h_0}{\pi^3} \frac{(c + \rho A s)\,(1 - (-1)^n)}{n^3 \left[E I\, (\frac{\pi}{\ell} n)^4 + T (\frac{\pi}{\ell} n)^2 + (k + c s + \rho A s^2)\right]}.$$

With

$$\kappa := \frac{c}{2\rho A}, \quad \Omega(n) := \sqrt{\frac{EI}{\rho A}\left(\frac{\pi}{\ell}n\right)^4 + \frac{T}{\rho A}\left(\frac{\pi}{\ell}n\right)^2 + \frac{k}{\rho A} - \kappa^2},$$

where we assume that the visco-friction constant c is so small that

$$\frac{EI}{\rho A}\left(\frac{\pi}{\ell}\right)^4 + \frac{T}{\rho A}\left(\frac{\pi}{\ell}\right)^2 + \frac{k}{\rho A} - \kappa^2 > 0.$$

Then we can write $\overline{Y}(n, s)$ in the form

$$\overline{Y}(n, s) = \frac{2\ell^3 h_0}{\pi^3}\left\{\frac{(1-(-1)^n)(\kappa+s)}{n^3\,[(s+\kappa)^2 + \Omega^2(n)]} + \frac{(1-(-1)^n)\kappa}{n^3\,[(s+\kappa)^2 + \Omega^2(n)]}\right\}.$$

Applying the inverse Laplace transform yields

$$\overline{y}(n, t) = \frac{2\ell^3 h_0}{\pi^3} e^{-\kappa t}\left\{\frac{(1-(-1)^n)}{n^3}\cos[\Omega(n)\,t] + \frac{(1-(-1)^n)}{n^3\,\Omega(n)}\sin[\Omega(n)\,t]\right\}.$$

Finally, taking the inverse Finite Fourier Sine transform gives

$$y(x, t) = \frac{8\ell^2 h_0}{\pi^3} e^{-\kappa t}\sum_{k=1}^{\infty}\left\{\frac{\cos[\Omega(2k-1)\,t]}{(2k-1)^3} + \frac{\sin[\Omega(2k-1)\,t]}{(2k-1)^3\,\Omega(2k-1)}\right\} \times \sin\left[\frac{\pi}{\ell}x(2k-1)\right].$$

B) Consider now an impulse load of the form $q(x, t) = \delta(x - \ell/2)\,\delta(t)$ and zero initial conditions $h(x) = 0$, $g(x) = 0$. The above second image equation then becomes

$$\left[EI\left(\frac{\pi}{\ell}n\right)^4 + T\left(\frac{\pi}{\ell}n\right)^2 + (k + cs + \rho A s^2)\right]\overline{Y}(n, s) = \overline{q}(n)\,P(s) = \sin\left(\frac{\pi n}{2}\right)\cdot 1.$$

Using the same abbreviations as in A), we can write

$$\overline{Y}(n, s) = \frac{1}{\rho A}\,\frac{1}{[(s+\kappa)^2 + \Omega^2(n)]}\sin\left(\frac{\pi n}{2}\right).$$

Taking the inverse Laplace transform yields

$$\overline{y}(n, t) = \frac{1}{\rho A\,\Omega(n)} e^{-\kappa t}\sin[\Omega(n)\,t]\,\sin\left(\frac{\pi n}{2}\right)$$

and by taking the inverse Finite Fourier Sine transform, we obtain

$$y(x, t) = \frac{2}{\rho A\,\ell}\sum_{n=1}^{\infty}\frac{(-1)^{k-1}\sin[\Omega(2k-1)\,t]}{\Omega(2k-1)}\sin\left[\frac{\pi}{\ell}x(2k-1)\right].$$

Chapter 12
Numerical Inversion of Laplace Transforms

In this chapter we present some procedures for numerically computing inverse Laplace transforms. Several methods based on different approaches have been proposed during the past fifty years. The numerical computation of an inverse Laplace transform is not readily done. It is a so-called ill-conditioned or ill-posed problem. In Chapter 6 we saw that the inverse Laplace transform is essentially determined by the poles and branch points of the image function, hence by the singularities of the image. Thus, as it was pointed out, asymptotic analysis is a naturally adapted tool that can help to find information about an inverse Laplace transform. It is now clear that if the singularities, i.e. the behavior of the image function near the singularities, essentially determine the corresponding original function, numerical methods basically are ill adapted for the computation of the original. Nevertheless, several methods were proposed. Some methods are working astonishingly well for certain image functions, while giving poor results for other examples. Because there is no method that is the best in all situations, we propose to use several different methods for a given inversion problem. If two or more methods give about the same result, you can be more confident and accept the numerically computed inverse Laplace transform. It is a feature common to the different numerical inversion methods that they work the better, the smoother the original $f(t)$ is. Singularities of $F(s)$ at infinity have, in general, an unfavourable influence on the precision. Image functions which are readily analytically inverted can offer tough problems for a numerical inversion, on the other hand, there are image functions that are very hard or impossible to transform back analytically, but can be numerically inverted very efficiently. Many useful informations about the numerical inversion of Laplace transforms can be found in Peter Valko's Website: http://pumpjack.tamu.edu/~valko/.

We shall only give some short explanations about the underlying analysis for a given inversion procedure and refer the reader to the corresponding papers where the method is developed. As a general rule the numerical inversion methods work best for problems where the essential behavior of the original function is concentrated on a finite interval $[0, T]$, i.e. the original should be decaying for increasing values of its independent variable. You cannot expect good results from a numerical method for problems, where the original has an essential non-vanishing behavior over large parts of the positive real axis, such as a periodic original function, or one that is increasing. Mathematically speaking, this amounts to a request that the original function $f(t)$ be of class $L^1[0, \infty)$, or $L^2[0, \infty)$, or both. Often one numerically first finds the original of $e^{-\sigma t} f(t/b)$, for some constant $\sigma > 0$, and some constant $b \geq 1$, rather than of $f(t)$ itself.

It is also assumed that the image function $F(s)$ is given by a closed expression so that for any s the function value can be computed. Most numerical methods contain one or more tuning parameters that affect the numerical computation and the precision. These tuning or scaling parameters can be used to provide an accuracy check. Since the computation is different with different tuning parameters, the accuracy of the result can be estimated by doing the same computations with different tuning or scaling parameters.

■ 12.1 Inversion by the Use of Laguerre Functions

This method is based on the papers by William T. Weeks "Numerical Inversion of Laplace Transforms Using Laguerre Functions", *Journal of the Association for Computing Machinery, 13, No 3 (July 1966)*, pp. 419–426, and Joseph Abate, Gagan L. Choudhury and Ward Whitt "On the Laguerre Method for Numerically Inverting Laplace Transforms", 1995, available on the site www.research.att.com/ ~wow/abate.html . These methods are based on the expansion of the original function in Laguerre functions. Laguerre functions form an orthogonal system of functions on the positive part of the real axis. Other authors, such as Lanczos and Papoulis, have earlier exploited this idea, but Weeks' contribution is that the coefficients in the expansion of the inverse transform are obtained by a trigonometric interpolation applied to the Laplace transform. In the paper of Abate–Choudhury–Whitt a detailed analysis is presented. Weeks assumes that there is some constant c_0 such that the original $f(t)$ satisfies $\exp(-c\,t)\,f(t) \in L^1[0,\infty) \cap L^2[0,\infty)$ whenever $c \geq c_0$. The constant c_0 is the abscissa of absolute convergence for $f(t)$. The original is represented by

$$f(t) = \sum_{n=0}^{\infty} a_n \Phi_n(t) := \sum_{n=0}^{\infty} a_n e^{-t/2} L_n(t)$$

where $L_n(t)$ is the Laguerre polynomial of degree n. From the completeness property of the orthonormal Laguerre functions it follows that the original $f(t)$ can be approximated by a function

$$f_N(t) = e^{ct} \sum_{n=0}^{N} a_n \Phi_n\left(\frac{t}{T}\right)$$

where $T > 0$ is a scaling factor and

$$a_n = \frac{1}{T} \int_0^{\infty} e^{-ct} f(t) \Phi_n\left(\frac{t}{T}\right) dt\,.$$

The approximation is in the sense of the square mean, i.e.,

$$\int_0^{\infty} e^{-2ct} \, | f(t) - f_N(t) | \, dt \to 0$$

as $N \to \infty$. It is then shown (see the paper for details) that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| F(c + i\omega) - \sum_{n=0}^{N} \frac{a_n (i\omega - 1/(2T))^n}{(i\omega + 1/(2T))^{n+1}} \right| dw \to 0$$

as $N \to \infty$ which after a change of variable, leads to

$$\left(\frac{1}{2T} + \frac{i}{2T} \cot\frac{\theta}{2}\right) F\left(c + \frac{i}{2T} \cot\frac{\theta}{2}\right) \approx \sum_{n=0}^{N} a_n e^{in\theta}.$$

From this formula the coefficients a_n can be computed from the given image function $F(s)$ which is therefore evaluated on the vertical straight line $\Re s = c$. However, in contrast to Weeks, we compute the Fourier coefficients by making use of the FFT algorithm. Thus N

should be a power of 2, for example 32. We have two tuning parameters c and T. It is shown that a good choice for T is

$$T \approx \frac{t_{\max}}{N}$$

if $f(t)$ should be approximated on the interval $[0, t_{\max}]$. Furthermore, Weeks proposes to choose c such that

$$c = \max\left\{0, c_0 + \frac{1}{t_{\max}}\right\}$$

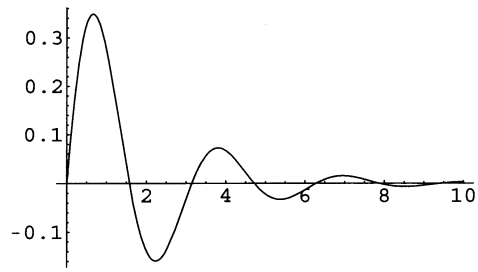
where the abscissa of convergence c_0 is most readily obtained by examining $F(s)$ to find the position of the right-most singularity.

The original $f(t)$ is then computed by using the command

`NInverseLaplaceLaguerre2[F[s],s,t,N,`$t_{\max}$`,`c_0`]`.

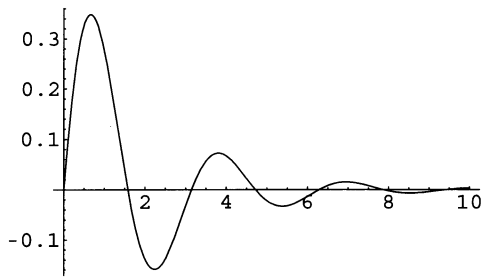
Example 1.

```
F = 1 / (s^2 + s + 4.25);

fapprox = NInverseLaplaceLaguerre2[F, s, t, 32, 10, 0];
Plot[fapprox, {t, 0, 10}, PlotRange → All];
```



Comparison with the exact inverse Laplace transform shows a fairly good approximation

```
f = InverseLaplaceTransform[F, s, t]

0.5 e^(-t/2) Sin[2. t]

Plot[f, {t, 0, 10}, PlotRange → All];
```



Example 2. Try yourself the following

```
F = 1 / (s (√s + 1));

fapprox = NInverseLaplaceLaguerre2[F, s, t, 32, 10, 0];
Plot[fapprox, {t, 0, 10}, PlotRange → All];
```

Comparison with the exact inverse Laplace transform shows the quality of the approximation (do not choose a larger N).

```
f = InverseLaplaceTransform[F, s, t]
```

$$1 - e^{t}\,\text{Erfc}\left[\sqrt{t}\right]$$

```
Plot[f, {t, 0, 10}, PlotRange → All];
```

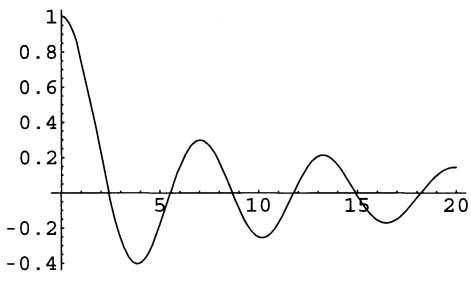
Also compute the inverse Laplace transform with this method for

$$F(s) = \frac{1}{\sqrt{s^2+1}}, \quad F(s) = \frac{e^{-5s}}{s}$$

and compare the approximation with the exact inverse Laplace transforms. You will encounter the limitations of this method.

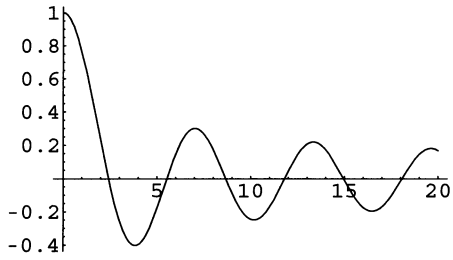
By use of the command `NInverseLaplaceLaguerre1[F[s],s,t,N,σ,b]` the more recent method of Abate–Choudhury–Whitt is used, with the two optional scaling parameters σ and b. Their default values are 0 and 1 (no scaling). With a scale the method is applied to $f_1(t) = \text{Exp}(-\sigma t)\, f(t/b)$ and $f(t)$ is then recovered by $f(t) = \text{Exp}(\sigma b t)\, f_1(b t)$. σ is important for eliminating the bad effect of a singularity of $F(s)$ at 0. Sometimes the result can be improved by choosing b greater than 1. Only the simplest method is programmed. For more sophisticated refinements of the metho, see the paper of Abate–Choudhury–Whitt.Test the method with the following examples:

```
F = 1 / s^2;

fapprox = NInverseLaplaceLaguerre1[F, s, t, 20, 0.1];
Plot[fapprox, {t, 0, 10}, PlotRange → All];

F = 1 / (s^2 + 1);

fapprox = NInverseLaplaceLaguerre1[F, s, t, 20, 0.1];
Plot[fapprox, {t, 0, 10}, PlotRange → All];

F = 1 / Sqrt[s^2 + 1];

fapprox = NInverseLaplaceLaguerre1[F, s, t, 20, 0.1, 2];
Plot[fapprox, {t, 0, 20}, PlotRange → All];
```



Compare this with the exact Bessel $J_0(t)$ function.

```
Plot[BesselJ[0, t], {t, 0, 20}, PlotRange → All];
```



■ 12.2 Inversion by Use of Fourier Analysis

The use of Fourier series for computing inverse Laplace transforms was explored by Dubner and Abate in the paper H. Dubner and J. Abate, "Numerical Inversion of Laplace Transform by Relating Them to the Finite Fourier Cosine Transform", 1968, *Journal ACM 15*, 115–123. Since then, several authors proposed variations and improvements, for example, Kenny S. Crump, "Numerical Inversion of Laplace Transforms Using a Fourier Series Approximation", *Journal of the Association for Computing Machinery, 23, No. 1, January 1976*, pp. 89–96. A more recent detailed discussion of the method is J.Abate and W. Whitt, 1992, "The Fourier-Series Method for Inverting Transforms of Probability Distributions", 1992, *Queueing Systems* 10, 5–88. See also the shorter paper of J. Abate and W. Whitt, "Numerical Inversions of Laplace Transforms of Probability Distributions", *ORSA Journal on Computing 7 (1995)*, pp. 36–43,also available on the site www.research.att.com/~wow/abate.html.

These methods compute the original function $f(t)$ for a given single value of t. The fast Fourier transform (FFT) algorithm can often be exploited. All methods of the Dubner–Abate family start with the complex inversion formula.

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} F(s)\, ds,$$

where it is assumed that $F(s)$ is holomorphic in a right-half plane such that a can be any positive number. Assuming that $F(s)$ is real in the sense that $F(\bar{s}) = \overline{F(s)}$, the formula is transformed into

$$f(t) = \frac{2e^{at}}{\pi} \int_0^{\infty} \Re[F(a + iu)] \cos ut\, du$$

$$\left(= -\frac{2e^{at}}{\pi} \int_0^{\infty} \mathrm{I}[F(a + iu)] \sin ut\, du \right).$$

The cosine-transform integral is numerically evaluated by means of the trapezoidal rule. With step size h, we obtain

$$f(t) \approx f_h(t) := h\left\{ \frac{e^{at}}{\pi} \Re[F(a)] + \frac{2e^{at}}{\pi} \sum_{k=1}^{\infty} \Re[F(a + ikh)] \cos(kht) \right\}.$$

For the given t and a positive number A, let us choose the step $h = \pi/(2t)$ and the integration abscissa $a = A/(2t)$, such that a nearly alternating series results:

$$f_h(t) := \frac{e^{A/2}}{2t} F\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k \Re\left[F\left(\frac{A + 2k\pi i}{2t}\right)\right].$$

This is an important formula. It is possible to get some insight of the discretization error that results from passing from $f(t)$ to $f_h(t)$. In order to establish this result, we use the formula derived in Sections 2.1.1.1 and 7.1.3.2, i.e.,

$$\sum_{k=0}^{\infty} f(kT + \tau)\, e^{-kTs} = \text{pv}\, \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(s + i\frac{2\pi}{T}k\right) \exp\left[\left(s + i\frac{2\pi}{T}k\right)\tau\right].$$

The formula holds for $0 < \tau < T$. Substituting t for τ, b for s and $T = 2\pi/h$, we obtain

$$\sum_{k=0}^{\infty} f\left(t + k\frac{2\pi}{h}\right) e^{-k2\pi b/h} = \text{pv}\, \frac{h}{2\pi} \sum_{k=-\infty}^{\infty} F(b + ihk) \exp[(b + ihk)t]\ ,$$

and with $h = \pi/t$, $b = A/(2t)$,

$$\sum_{k=0}^{\infty} f((2k+1)t)\, e^{-kA} = \text{pv}\, \frac{e^{A/2}}{2t} \sum_{k=-\infty}^{\infty} F\left(\frac{A + 2\pi ik}{2t}\right)(-1)^k\ .$$

This can be written as

$$f(t) + \sum_{k=1}^{\infty} f((2k+1)t)\, e^{-kA}$$
$$= \frac{e^{A/2}}{2t} F\left(\frac{A}{2t}\right) + \text{pv}\, \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k \,\text{Re}\left[F\left(\frac{A + 2\pi ik}{2t}\right)\right].$$

Therefore

$$f(t) = f_h(t) - \sum_{k=1}^{\infty} f((2k+1)t)\, e^{-kA}.$$

This shows that the sampling error, associated with the trapezoidal rule, is given by

$$\text{err}_d = \sum_{k=1}^{\infty} f[(2k+1)t]\, e^{-kA}.$$

If we know, for example, that $|f(t)|$ is bounded by a constant K, we obtain the estimate

$$\left|\text{err}_d\right| \le K \frac{e^{-A}}{1 - e^{-A}} \approx K e^{-A}$$

when e^{-A} is small (for example $A = 19$). The remaining problem is to evaluate the above infinite sum. Since we have a nearly alternating series, an acceleration method for alternating series is proposed, for example the Euler summation method. With

$$s_n(t) := \frac{e^{A/2}}{2t} F\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{n} (-1)^k \Re\left[F\left(\frac{A + 2k\pi i}{2t}\right)\right]$$

the Euler summation is applied to m terms after an initial n (typically $m = 11$ and $n = 15$):

$$f(t) \approx f_h(t) \approx E(m, n, t) := \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(t) .$$

This method is used by the `NInverseLaplaceFourier1[F[s],s,t,σ,{A,m,n}]` command. A governs the discretization error, m and n are parameters for the Euler summation to accelerate the convergence of the approximating series. You can use the set {19, 11, 10}. $\sigma > 0$ is used for applying the method on $e^{-\sigma t} f(t)$ rather than to $f(t)$ itself. $f(t)$ is computed for a single value t. The following shows a table of the computed values (second column) for the inversion of the image $s/(s^2 + 1)$ with the exact values (third column) of the cosine function.

```
F = s / (s^2 + 1);

f[t_] = NInverseLaplaceFourier1[F, s, t, 0.1, {19, 11, 10}];

Table[{t, f[t], Cos[t] // N}, {t, 8}];

TableForm[%]
```

1	0.540302	0.540302
2	-0.416147	-0.416147
3	-0.989992	-0.989992
4	-0.653643	-0.653644
5	0.283662	0.283662
6	0.960171	0.96017
7	0.753902	0.753902
8	-0.1455	-0.1455

Another example:

```
F = Exp[-Sqrt[s]] / Sqrt[s];

InverseLaplaceTransform[F, s, t]
```

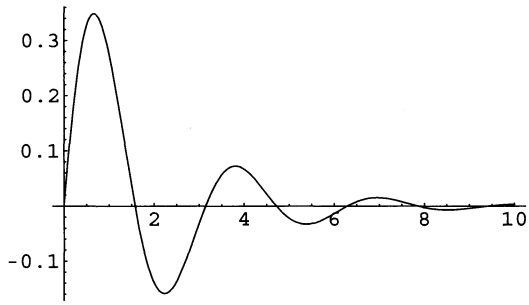
$$\frac{e^{-\frac{1}{4t}}}{\sqrt{\pi}\,\sqrt{t}}$$

```
f[t_] = NInverseLaplaceFourier1[F, s, t, 0.1, {19, 11, 10}];

Table[{t, f[t], N[Exp[-1 / (4 t)] / Sqrt[Pi t]]}, {t, 8}];

TableForm[%]
```

1	0.439391	0.439391
2	0.352065	0.352065
3	0.299691	0.299691
4	0.265004	0.265004
5	0.240008	0.240008
6	0.22093	0.22093
7	0.205762	0.205762
8	0.193334	0.193334

The following procedure is after Crump's paper, but the Fourier coefficients are computed via the FFT algorithm

```
? NInverseLaplaceFourier2

   NInverseLaplaceFourier2[F, s, t, NN, alpha, T]
     computes numerically the inverse Laplace transform of F[
     s] using the method of K.S.Crump. The computed original
     f[t] will be a periodic function with period 2T. The
     integer NN should be power of 2 because FFT is used.
     The real parameter alpha is chosen slightly larger than
     the singularity with the greatest real part of F[s].

F = 1 / (s^2 + s + 4.25);
NInverseLaplaceFourier2[F, s, t, 256, 0, 12]

InterpolatingFunction[{{0., 23.9063}}, <>][t]

Plot[%, {t, 0, 10}, PlotRange → All];
```



Another member of this family, at least partially, is the important *Post–Widder–Stehfe* method. We shall give an outline of this method, for details see the second paper of Abate and Whitt mentioned above. We start with the *Post–Widder Inversion formula* mentioned briefly in Section 1.1.4.2. The formula says

$$f(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n+1}{t}\right)^{n+1} F^{(n)}\left(\frac{n+1}{t}\right)$$

if $f(t) \circ\!\!-\!\!\bullet F(s)$. By denoting

$$f_n(t) := \frac{(-1)^n}{n!}\left(\frac{n+1}{t}\right)^{n+1} F^{(n)}\left(\frac{n+1}{t}\right)$$

we can calculate $f_n(t)$ via its generating function

$$G(z) := \left(\frac{n+1}{t}\right) F\left(\frac{n+1}{t}(1-z)\right) = \sum_{n=0}^{\infty} f_n(t)\, z^n$$

as can readily be verified by Taylor's theorem. By using the formula for the coefficients of a Laurent series, one can express $f_n(t)$ by the Cauchy contour integral

$$f_n(t) = \frac{1}{2\pi i}\oint_C \frac{G(z)}{z^{n+1}}\, dz,$$

where the contour C is a circle of radius r centered at the origin. This radius r will later be determined as a function of the given value of t. A change of variables $z = r\,e^{iu}$ leads to

$$f_n(t) = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} G(r\,e^{iu})\, e^{-inu}\, du$$
$$= \frac{n+1}{t} \frac{1}{\pi r^n} \int_0^{\pi} \mathrm{Re}\Big[F\Big(\frac{n+1}{t}(1 - r\,e^{iu})\Big) e^{-inu}\Big]\, du,$$

where it is assumed that $F(s)$ is real in the sense that $F(\bar{s}) = \overline{F(s)}$. The integral over a periodic function is most efficiently evaluated with the trapezoidal rule. Step-size $h = \pi/n$ gives

$$\tilde{f}_n(t) = \frac{n+1}{t} \frac{1}{2nr^n} \left\{ F\Big(\frac{n+1}{t}(1-r)\Big) + (-1)^n F\Big(\frac{n+1}{t}(1+r)\Big) \right.$$
$$\left. + 2\sum_{k=1}^{n-1} (-1)^k \mathrm{Re}\Big[F\Big(\frac{n+1}{t}(1 - r\,e^{i\pi k/n})\Big)\Big] \right\}.$$

In the paper of Abate and Whitt mentioned above, it is shown that the discretization error amounts to

$$f_n(t) - \tilde{f}_n(t) = -\sum_{j=1}^{\infty} f_{n+j2n}\Big(t + \frac{t\,j2n}{n+1}\Big) r^{2jn}.$$

Therefore, if again we suppose that $|f(t)| \le K$ for all t, we have the estimate

$$\Big| f_n(t) - \tilde{f}_n(t) \Big| \le K \frac{r^{2n}}{1 - r^{2n}} \approx K r^{2n}.$$

We can make the error in the calculation of $f_n(t)$ suitably small by a proper choice of r for a given value of t (for example $r = 10^{-\gamma/2}$, $\gamma = 8$). It is known that $f_n(t)$ converges quite slowly to $f(t)$. It was Stehfest's contribution to propose a convergence acceleration. He found that the expression

$$(f^{\dagger})_{j,m}(t) = \sum_{k=1}^{m} (-1)^{m-k} \frac{k^m}{k!\,(m-k)!} \tilde{f}_{jk}(t)$$

with, for example, $j = 10$, $m = 6$, has the desired property.

```
? NInverseLaplaceWPS

   NInverseLaplaceWPS[F[s],s,t,{γ,j,m}] computes numerically the
     inverse Laplace transform of F[s] by making use of the Widder-
     Post inversion formula and the Stehfest modification. γ governs
     the discretisation error while j and m are the parameters for
     the Stehfest convergence acceleration. Use the set {8,10,6}.

f[t_] = NInverseLaplaceWPS[1 / Sqrt[s^2 + 1], s, t, {8, 10, 6}];
Table[{t, f[t], BesselJ[0, t] // N}, {t, 10}]
```

```
TableForm[%]
```

1	0.765198	0.765198
2	0.223891	0.223891
3	-0.260052	-0.260052
4	-0.397153	-0.39715
5	-0.177586	-0.177597
6	0.150731	0.150645
7	0.300094	0.300079
8	0.171002	0.171651
9	-0.0913232	-0.0903336
10	-0.244372	-0.245936

■ 12.3 The Use of Gaussian Quadrature Formulas

Another family of approaches to the problem of the numerical inversion of a Laplace transform is based on the use of a quadrature formula for Bromwich's integral (the complex inversion integral). In his paper "Gaussian Quadrature Formulas for the Numerical Integration of Bromwich's Integral and the Inversion of the Laplace Transform", *Journal of Engineering Mathematics, 5, No. 1, January 1971*, R. Piessens uses a Gaussian quadrature applied to the complex inversion formula to determine $f(t)$. The method turns out to be exact whenever the image function $F(s)$ is a linear combination of inverse powers

$$\frac{1}{s^{\gamma+k}}, \quad k = 0, 1, 2, \ \ldots, \ 2N-1, \ \ \gamma > 0,$$

and N is chosen to be the number of terms in the approximating sum. The approximating formula for the original is expressed in the form

$$f(t) = t^{\gamma-1} \sum_{k=1}^{N} A_k \left(\frac{u_k}{t}\right)^{\gamma} F\left(\frac{u_k}{t}\right),$$

where γ is a positive real number chosen in order that $s^{\gamma} F(s)$ is holomorphic at infinity. Hence we can write

$$s^{\gamma} F(s) = \sum_{k=0}^{\infty} \frac{a_k}{s^k}, \qquad |s| > \sigma > 0.$$

Therefore, $F(s)$ itself can have a branch point at infinity, a situation where other methods often fail. The method gives formulas for the coefficients A_k and the abscissas u_k. Both will depend on the values of γ and N. If the vertical integration path $\Re s = c$ is chosen on the right of all singularities of the given image function $F(s)$, we have

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds.$$

By a change of variables $u = s\,t$ ($t > 0$ is fixed), we obtain

$$f(t) = \frac{1}{2\pi i} \int_{ct-i\infty}^{ct+i\infty} e^{u} F\left(\frac{u}{t}\right) \frac{1}{t}\, du$$

$$\Longrightarrow t\, f(t) = \frac{1}{2\pi i} \int_{ct-i\infty}^{ct+i\infty} e^{u}\, u^{-\gamma}\, G(u)\, du$$

where γ is a positive parameter and $G(u) := u^{\gamma} F(u/t)$. The Gaussian integration formula is found in the form

$$t f(t) = \frac{1}{2\pi i} \int_{c t - i\infty}^{c t + i\infty} e^{u} u^{-\gamma} G(u)\, du \approx \sum_{k=1}^{N} A_k G(u_k)$$

where the coefficients $A_k = A_k(\gamma, N)$ and the abscissas $u_k = u_k(\gamma, N)$ are determined such that the formula is exact for polynomials of degree up to $2N - 1$ in $1/s$. Going back to the original variables, we obtain

$$f(t) = t^{\gamma - 1} \sum_{k=1}^{N} A_k \left(\frac{u_k}{t}\right)^{\gamma} F\left(\frac{u_k}{t}\right).$$

We shall not go into the rather complicated analysis of determining the abscissas u_k and the coefficients A_k, but refer the reader to Piessens' paper.

```
? NInverseLaplaceGauss
```

```
NInverseLaplaceGauss[F, s, t, NN, γ] computes numerically the
  inverse Laplace transform of F[s] using the gaussian quadrature
  formula (Method of R. Piessens). NN is the number of terms
  in the approximating sum. The positive parameter γ has to
  be chosen such that s^γ F(s) is a holomorhic function at ∞.
```

The method is exact for inverse powers of s, for example,

```
F = 24 / s^5
```

$$\frac{24}{s^5}$$

```
NInverseLaplaceGauss[F, s, t, 3, 1]
```

```
1. t^4
```

The following image function has a branch point at infinity.

```
F = 1 / Sqrt[s] / (s - 1)
```

$$\frac{1}{(-1 + s)\sqrt{s}}$$

The exact inversion yields

```
InverseLaplaceTransform[F, s, t]
```

```
e^t Erf[√t]
```

which can be compared with the numerical inversion.

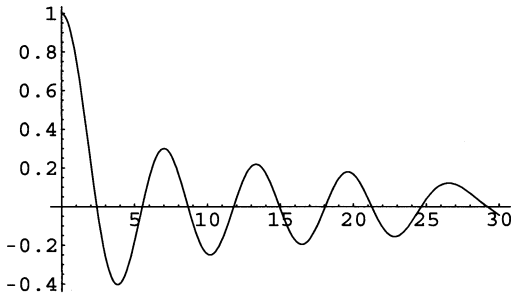
```
Table[{k, NInverseLaplaceGauss[F, s, k, 10, 1.5], N[Exp[k] Erf[√k]]},
  {k, 10}];
```

```
TableForm[%]
```

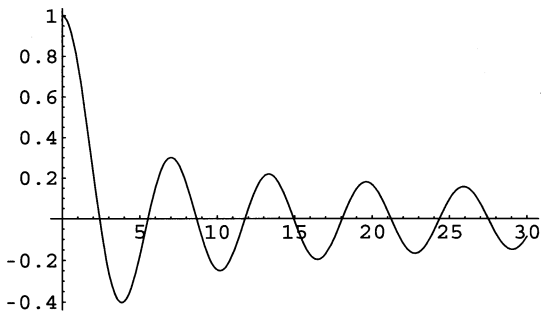
```
1       2.2907      2.2907
2       7.05285     7.05285
3       19.7982     19.7982
4       54.3428     54.3428
5       148.181     148.181
6       403.214     403.214
7       1096.43     1096.43
8       2980.75     2980.77
9       8102.24     8102.9
10      22006.4     22026.3
```

Comparing the inversion of $1/\sqrt{s^2+1}$ with the exact Bessel function $J_0(t)$ yields:

```
fapprox = NInverseLaplaceGauss[1 / Sqrt[1 + s^2], s, t, 14, 0.5];
Plot[fapprox, {t, 0, 30}, PlotRange → All];
```




```
Plot[BesselJ[0, t], {t, 0, 30}, PlotRange → All];
```



■ 12.4 The Method of Gaver and Stehfest

This is a popular method based on the paper by H. Stehfest, "Algorithm 386, Numerical Inversion of Laplace Transforms", *Communications of the ACM, 13, Number 1, Jan. 1970.* Stehfest uses for his work the paper by D.P. Gaver, "Observing stochastic processes, and approximate transform inversion", *Operation Research, 14, Number 3, 1966.*

```
?NInverseLaplaceGaverStehfest
```

```
NInverseLaplaceGaverStehfest[F[s],s,t,n] computes numerically the
  inverse Laplace transform using the method of Gaver-Stehfest.
  The integer n must be even, try 10 or 16, for example.
```

```
F = Exp[-1 / s] / s^(3 / 2);
```

The exact inversion is

```
InverseLaplaceTransform[F, s, t]
```

$$\frac{\text{Sin}\left[2\sqrt{t}\right]}{\sqrt{\pi}}$$

```
Table[{t, N[Sin[2 Sqrt[t]]] / Sqrt[Pi],
   NInverseLaplaceGaverStehfest[F, s, t, 20]}, {t, 1, 10}];

TableForm[%]
```

1	0.513016	0.513013
2	0.173811	0.173803
3	-0.178818	-0.178812
4	-0.42698	-0.426993
5	-0.547985	-0.547988
6	-0.554397	-0.55439
7	-0.472197	-0.472182
8	-0.330715	-0.330706
9	-0.157643	-0.157662
10	0.0233339	0.0233092

For larger n the rounding errors become predominant and the result worsens. For the Bessel function, for example, the result is worse than with Piessens' method. You can see this by using

```
fapprox = NInverseLaplaceGaverStehfest[1 / Sqrt[1 + s^2], s, t, 16];
Plot[fapprox, {t, 0, 30}, PlotRange → All];
```

■ 12.5 Example

Let us demonstrate a real problem. Consider the Laplace transform

$$F(s) = \frac{1}{s}\,\text{Exp}[s^2]\,\text{Erfc}[s]\;\text{Exp}\left[-a\sqrt{s}\right] \quad (a > 0)$$

where it is probably impossible to find a closed-form expression for $f(t)$. However

```
InverseLaplaceTransform[Exp[s^2] Erfc[s] / s, s, t]
```

$$\text{Erf}\left[\frac{t}{2}\right]$$

```
InverseLaplaceTransform[Exp[-a Sqrt[s]], s, t]
```

$$\frac{a\, e^{-\frac{a^2}{4t}}}{2\sqrt{\pi}\, t^{3/2}}$$

which implies that $f(t)$ can be written as a convolution integral

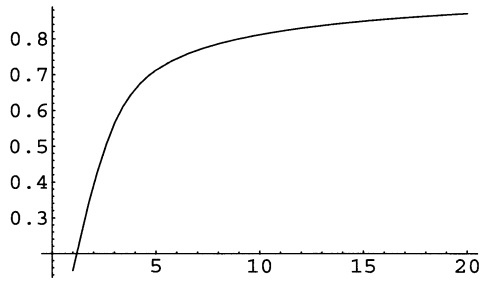
$$f(t) = \frac{a}{2\sqrt{\pi}} \int_0^t \tau^{-3/2}\, e^{-\frac{a^2}{4\tau}}\, \text{Erf}\left(\frac{t-\tau}{2}\right) d\tau\,.$$

For a given numerical value of the parameter a, the integral can numerically be evaluated, for example, for $a = 1$:

```
g[τ_, t_, a_] := a / 2 / Sqrt[Pi] * Exp[-a^2 / (4 τ)] τ^(-3 / 2) Erf[(t - τ) / 2] ;

Table[{t, NIntegrate[g[τ, t, 1.0], {τ, 10^-8, t}]}, {t, 1, 20}]
```

```
{{1, 0.151443}, {2, 0.392606}, {3, 0.564141}, {4, 0.659768},
 {5, 0.712376}, {6, 0.745194}, {7, 0.768357}, {8, 0.785998},
 {9, 0.800069}, {10, 0.81165}, {11, 0.821408}, {12, 0.82978},
 {13, 0.837068}, {14, 0.843488}, {15, 0.849202}, {16, 0.85433},
 {17, 0.858966}, {18, 0.863185}, {19, 0.867046}, {20, 0.870596}}
```

```
Plot[Interpolation[%][x], {x, 1, 20}];
```



Let us use now a numerical inversion of the Laplace transform.

$$F(s) = \frac{1}{s}\,\mathrm{Exp}[s^2]\,\mathrm{Erfc}[s]\;\mathrm{Exp}\left[-a\sqrt{s}\right] \quad (a > 0)$$

```
F[s_, a_] = N[Exp[s^2 - a Sqrt[s]] Erfc[s] / s];
```

We first take the method of Abate–Whitt:

```
Table[
 {t, NInverseLaplaceFourier1[F[s, 1], s, t, 0, {19, 11, 10}]}, {t, 1, 20}]
```

```
{{1, 0.151443}, {2, 0.392606}, {3, 0.564141}, {4, 0.659768},
 {5, 0.712376}, {6, 0.745194}, {7, 0.768357}, {8, 0.785998},
 {9, 0.800069}, {10, 0.81165}, {11, 0.821408}, {12, 0.82978},
 {13, 0.837068}, {14, 0.843488}, {15, 0.849202}, {16, 0.85433},
 {17, 0.858966}, {18, 0.863185}, {19, 0.867046}, {20, 0.870596}}
```

The computation is considerably more time consuming, but we have a perfect agreement. Next, we take the method of Widder–Post–Stehfest

```
Table[{t, NInverseLaplaceWPS[F[s, 1], s, t, {8, 10, 6}]}, {t, 1, 20}]
```

```
{{1, 0.151443}, {2, 0.392606}, {3, 0.564141}, {4, 0.659768},
 {5, 0.712377}, {6, 0.745193}, {7, 0.768357}, {8, 0.785999},
 {9, 0.800069}, {10, 0.81165}, {11, 0.821408}, {12, 0.82978},
 {13, 0.837068}, {14, 0.843488}, {15, 0.849202}, {16, 0.85433},
 {17, 0.858966}, {18, 0.863185}, {19, 0.867046}, {20, 0.870596}}
```

Again a perfect agreement and it is a bit quicker than the method of Abate–Whitt. The reader may verify that Piessens' method Gaver–Stehfest's method compute more rapidly, but they are slightly less accurate:

```
Table[{t, NInverseLaplaceGauss[F[s, 1], s, t, 10, 0.5]}, {t, 1, 20}]

Table[{t, NInverseLaplaceGaverStehfest[F[s, 1], s, t, 20]}, {t, 1, 20}]
```

It is clear that the first approach, which by analytical methods leads to a simple integral, is more preferable than a numerical inversion of the Laplace transform. Nevertheless the numerical methods work very well for this example.

Appendix: Package Commands by Subjects

One simple example of each command is given with the reference to the section in the book where the command is introduced. The reference starts with the number of the chapter: (*3.2.1*) means Chapter 3, Section 2, Subsection 1. On the CD of the book is the complete Reference Guide of all commands. We strongly recommand printing this Reference Guide. It shall be very helpful when working with the Package. A short explanation for all Package commands is obtained with the ?-operator, for example,

```
? TwoSidedLaplaceTransform

   TwoSidedLaplaceTransform[f[t],t,s] computes the two-sided or
     bilateral Laplace transform of f[t].The image function
     has the form {F[s],Interval[{a,b}]}, where F[s] is the
     image function with the strip of convergence a < Re[s] < b.
```

■ Loading the Package

The Package is loaded with the commands

```
<< LaplaceAndzTransforms`LZSession`;
InitLZPackage[]

InitOK!
```

■ Laplace Transforms

```
LaplaceTransform[t Cos[a t],t,s]  (*3.1.3*)

LaplaceTransform[Cos[t]//RightShift[c,t],t,s]  (*3.2.4*)

LaplaceTransform[Cos[t]//LeftShift[c,t],t,p]  (*3.2.4*)

LaplaceTransform[Exp[x]//ScaleChange[c,x],x,s]  (*3.2.5*)

LaplaceTransform[BesselJ[4,t]//IntegrationTo[t],t,s]  (*3.2.2*)

LaplaceTransform[Sin[t]//DivisionBy[t],t,s]  (*3.2.7*)

DirectLaplaceTransform[1 / (1 + t^3), t, s]  (*3.4 .4*)

FiniteLaplaceTransform[t/(2 Pi),{t,0,2 Pi},s]  (*3.3.1*)

DiscreteLaplaceTransform[k Cos[k], k, T, s] (*Reference Guide*)

TwoSidedLaplaceTransform[Exp[-2 t^2 + b t + c],t,s] (*Reference Guide*)
```

■ Inverse Laplace Transform

```
InverseLaplaceTransform[s/(s^2 + 2 s + 17)^2,s,t]  (*3.1.3*)

InverseLaplaceTransform[1/(a s^2 + b s + c)^2,s,t,Assumptions->b^2-4 a
c > 0]  (*3.5.1*)
```

```
InverseLaplaceTransform[(s/(s^2+2s+5)^2)//ImageShift[c,s],s,t]
(*3.2.3*)

InverseLaplaceTransform[(1/(s^2 + 2 s + 17)^2)//ScaleChange[c,s],s,t]
(*3.2.5*)

InverseLaplaceTransform[1/Sqrt[1+s^2]//DivisionBy[s],s,t] (*Reference
Guide*)

InverseLaplaceTransform[Exp[-a s + b]/Sqrt[s^2+a^2],s,t] /. ImageRules
(*6.1.2.1*)

InverseLaplaceTransform[(F[s]//Subst[s,s+1/s])/s,s,t]  (*3.2.9*)

InverseLaplaceTransform[1/((s^2 + 2 s + 17)(1-Exp[-2
s])),s,t,Transient->True]  (*3.5.2*)

InverseLaplaceTransform[Sinh[x s] / (s^5 Cosh[a s]), s, t] /.
 TruncateSummation[5] (*3.5 .2*)

ResidueLaplace[1/(s^2 + ω^2)^3,{ω I,-ω I},s,t]  (*6.1.2.1*)

ExpansionTheorem[s + 1, s^3 (s^4 - 1) (s^2 + 2 s + 5),
 {1, -1}, {I, -1 + 2 I}, s, t] (*6.1 .2 .2*)

ExpansionTheorem[Cosh[x s], Cosh[a s] (s^2 - b^2), {b, -b},
 Hold[Table[I Pi (1 / 2 + k) / a , {k, 0, Infinity}]], s, t]  (*6.1 .2 .2*)

NFiniteInverseLaplaceTransform[(Exp[-s]-Exp[-2 s])/s,{s,0,3}]
(*3.3.2*)

InitialValues[(a s + b) / (c s^2 + s + 1) ^2, s, 5] (*3.2 .10 .1*)

InverseTwoSidedLaplaceTransform[{s^5 Exp[s^2],Interval[{-
Infinity,Infinity}]},s,t]  (*11.2.1.3*)
```

■ Functions and Sequences

```
PeriodicFunction[#/(2 Pi)&,2 Pi]  (*3.4.2.1*)

PeriodicEvenFunction[#^2&,0.5]  (*3.4.2.2*)

PeriodicOddFunction[#^2&,1]     (*3.4.2.3*)

Periodic3dsFunction[#^2&,1]  (*3.4.2.4*)

DiracDelta[t - a]; UnitStep[t - a];

PiecewiseFunction[{{0,1,0&},{1,2,(#-1)&},{2,4,1&},{4,5,(1-(#-4))&},
{7,8,2&}}]   (*3.4.1*)

StepFunction[#^2&,0.5]  (*3.4.2.5*)

StepFunction[(#^2) &, Pi / 2, Cos[#] &]

DiracDeltaRow[(# Exp[-#])&,T]  (*3.4.3*)

GenFac[5,3,a][k] /. ExpandGenFac (*4.1.2*)

PfPair1[1+I,2,3+I,s] /. ExpandPfPair  (*Reference Guide*)

PfPair2[1+I,2,3+I,z] /. ExpandPfPair  (*Reference Guide*)

PartialFraction[(s^6+s+1)/((s^2 + 2 s + 26)^2 (s+a)^3),s]  (*Reference
Guide*)
```

```
GeneralPartialFraction[(s^6+s+1)/((s^2 + 2 s + 26)^2 (s+3)^3),s,c,α]
(*Reference Guide*)

SpecialPartialFraction[(s^7 + a)/((s^2 + 2 s + 17)^2 (s+b)),s]
(*Reference Guide*)

SpecialPartialFraction[z^2/((z^2 + 2 z + 17)(z-3)^2),z,Type ->
ZTransformation]  (*Reference Guide*)

IncompletePartialFraction[(s^2 + s + 1)/((3 s^2 + 2 s + 5)^3
(s+a)^2),h,k,s] (*3.5.1*)

SimplifyRImageFunction[(s+a)^2/(s^2 + a s^2 - b s + c s - d s + e),s]
(*3.2.10.2*)

NormalizeRImageFunction[(a0 + a1 s + a2 s^2)/(b0 + b1 s + b2 s^2 + b3
s^3),s]  (*3.2.10.2*)

(s+Sqrt[s^2+a^2])/Sqrt[a^2+s^2]//AmplifyBy[-s+Sqrt[s^2+a^2]]
(*3.2.10.2*)

Convolution[Cos[u],Exp[u],u][t] /. EvaluateConvolution  (*Reference
Guide*)

Convolution[Cos[u],Exp[u],u][t] /. ComputeConvolution (*Reference
Guide*)

DiscreteSignal[k^4 2^-k, {k,0,15}]//Show; (*4.1.2*)

DiscreteConvolution[a^i,i^2,i][k] /. ComputeDiscreteConvolution
(*4.2.7*)

DiscreteConvolution[a^i,i^2,i][k] /. EvaluateDiscreteConvolution
(*4.2.7*)

DiscreteComposition[x,Sin[x],x,T][t,τ] /. ComputeDiscreteComposition
(*7.3.2*)

DiscreteComposition[x,Sin[x],x,T][t] /. EvaluateDiscreteComposition
(*7.3.2*)

Resolvent[s, {{1, -1}, {3, -4}}] (*3.5 .1 .4*)

Vieta[2 + s + k (s + 1) + k s^2 - 7 s^3, s, σ] (*ReferenceGuide*)
```

■ Differential Equations

```
LinearIDSolve[y''[t] + 2 y'[t] == -5 y[t] + Exp[-t] Sin[t],{y[0]->0,
y'[0]->1},y[t],t]  (*3.5.1.1*)

LinearIDSolve[{x''[t] - 2 x'[t] == -3 y'[t],y'[t] - y[t] == -2 x[t] +
t},{x[0]->0,x'[0]->1},{x[t],y[t]},t] (*3.5.1.1*)

LinearIDSolve[{x''[t] - 2 x'[t] == -3 y'[t],y'[t] - y[t] == -2 x[t] +
t},{x[0]->0,x'[0]->1},{x[t],y[t]},t,GoFor->Image[{Y[2]},s]]
(*3.5.1.1*)

LinearIDSolve[y[t] == t^2 + Convolution[y[u],Sin[u],u][t],y[t],t]
(*3.5.1.1*)

LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p+3)[Exp[-t] Cos[2 t]],p,y[t],t]
(*3.5.1.2*)

MDOp1 = {{17 + 2 p + p^2, -5 + p, -3}, {-1 + 2 p, p, -p^2}, {3,4 - p,
-p}}; MDOp2 = {{-1 + 3 p, p, 1}, {0, p^3, 0}, {p, 0, -5}};
LinearDOpSolve[MDOp1[{y1,y2,y3}] == MDOp2[{Sin[t], Exp[-t],
t}],p,{y1[t],y2[t],y3[t]},t]//Chop  (*3.5.1.2*)

LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p+3)[Exp[-t] Cos[2 t]],{y[0]->3,
y'[0]->-4},p,y[t],t,GoFor->Image[Y,s]] (*3.5.1.2*)

SpecialDOpSolve[(p^4 + a^2/s^2)[y] == 0,p,y[x],x] (*3.5.1.3*)
```

```
SpecialDOpSolve[(p^4 + a^2/s^2)[y] == x^3,p,y[x],x, GoFor ->
OnlyParticular[]]  (*3.5.1.3*)

ToDOpForm[a0 y[t] + a1 y'[t] + a2 y''[t] + a3 y'''[t] == Cos[t],p,y,t]

AnomalousSystemQ[{{p+p^2,1},{2+p^2,-p}}[{y1,y2}] ==
{10,t},p,{y1[t],y2[t]},t]

CompatibilityConditions[{{p+p^2,1},{2+p^2,-p}}[{y1,y2}] ==
{10,t},p,{y1[t],y2[t]},t]

VariableDSolve[t^2 y''[t] + t y'[t] + (t^2-n^2)y[t] == 0,t,Y,s]

A0 = {{1, 1}, {0, -1}}; A1 = {{1, -1}, {0, -2}};
A2 = {{0, 1}, {2, -1}}; B0 = {{1}, {2}}; B1 = {{3}, {4}};
MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}]

ReduceLagrangian[-g l m Cos[α[t]] + M x'[t]^2 / 2 +
  m (3 x'[t]^2 - 6 l Cos[α[t]] x'[t] α'[t] + 4 l^2 α'[t]^2) / 6,
 {{x[t], 0}, {α[t], 0}, {x'[t], 0}, {α'[t], 0}}] (*9.2 .2 .1*)
```

■ Asymptotics

```
AsymptoticSeries[1 / Sqrt[t] / (1 + Sqrt[t]) ^ 2, {t, 0, 4}] (*6.2 .5 .1*)

AsymptoticSeries[1 / Sqrt[t] / (1 + Sqrt[t]) ^ 2, {t, 0, 4}]  /.
 RemoveLandauO  (*6.2 .5 .1*)

AsymptoticSeries[LaplaceIntegral[1 / Sqrt[t] / (1 + Sqrt[t]) ^ 2, {t, 0}][s],
 {s, Infinity, 5}] (*6.2 .5 .2*)

AsymptoticSeries[HankelIntegral[Exp[-a Sqrt[s]] / (s^2 + ω^2), {s, 0}][t],
 {t, Infinity, 5}]  (*6.2 .5 .3*)

AsymptoticSeries[
 LaplaceTypeIntegral1[c + t^3 (1 + t^2 + 3 t^4), Cos[t], {t, 0}][z],
 {z, Infinity, 6}] (*6.2 .5 .4*)

AsymptoticSeries[
 LaplaceTypeIntegral2[1 + (Pi / 2 - t) ^ 2 Sin[t], Cos[t], {t, Pi / 2}][z],
 {z, Infinity, 4}] (*6.2 .5 .4*)

AsymptoticSeries[
 LaplaceTypeIntegral3[(t - 1) ^ 2 Exp[t], (t - 1) ^ 3 Cos[t], {t, 1}][z],
 {z, Infinity, 6}] (*6.2 .5 .4*)

LaplaceMethodCoefficients[{p0, p1, p2}, {q0, q1, q2}, μ, λ] (*6.2 .5 .4*)
```

■ Numerical Inversion of Laplace Transforms

```
NInverseLaplaceLaguerre1[1 / s^2, s, t, 20, 0.1] (*12.1*)

F = 1 / (s^2 + s + 4.25);
NInverseLaplaceLaguerre2[F, s, t, 32, 10, 0]  (*12.1*)

F = Exp[-Sqrt[s]] / Sqrt[s];
f[t_] = NInverseLaplaceFourier1[F, s, t, 0.1, {19, 11, 10}]
(*12.2*)

F = 1 / (s^2 + s + 4.25);
NInverseLaplaceFourier2[F, s, t, 256, 0, 12] (*12.2*)

f[t_] = NInverseLaplaceWPS[1 / Sqrt[s^2 + 1], s, t, {8, 10, 6}]  (*12.2*)
```

```
F = 1 / Sqrt[s] / (s - 1);
NInverseLaplaceGauss[F, s, t, 10, 1.5]  (*12.3*)

F =  Exp[-1 / s] / s^(3 / 2);
NInverseLaplaceGaverStehfest[F, s, t, 20] (*12.3*)
```

■ Finite Fourier Transforms

```
FiniteFourierSineTransform[x (Pi - x), x, n] (*11.4 .2*)

InverseFiniteFourierSineTransform[1 / n^3, n, x] (*11.4 .1*)

FiniteFourierCosineTransform[x^2, x, n]   (*11.4 .1*)

FFCosineConvolution[x^2, (x - L)^2, L]
```

■ Z–Transforms and Advanced Z–Transforms

```
ZTransform[k(k-1)Exp[-a k] + 5 k^3 -b k(k-1)a^-k,k,z] (*4.1.2*)

ZTransform[GenFac[5,3,a][k],k,z]      (*4.1.2*)

ZTransform[k^2//RightShift[5,k],k,z]  (*4.2.2*)

ZTransform[k^2//LeftShift[5,k],k,z]  (*4.2.2*)

AdvancedZTransform[t^2 Cos[a t],t,T,τ,z]  (*7.3.1*)

AdvancedZTransform[Sin[t]//RightShift[4,t],t,T,τ,z] (*7.3.2*)

AdvancedZTransform[Sin[t]//LeftShift[4,t],t,T,τ,z]  (*7.3.2*)

AdvancedZTransform[Cos[t]//IntegrationTo[t],t,T,τ,z]  (*7.3.2*)

AdvancedZTransform[(t^5)'',t,T,τ,z]  (*7.3.2*)

LaplaceToZImage[(s+1)/((s+2)(s^2 + 2 s + 17)), s,T,τ,z]  (*7.3.4*)

ZLaplace[(s+1)/((s+2)(s^2 + 2 s + 17)), s,T,τ,z]  (*7.3.4*)
```

■ Inverse Z–Transforms

```
InverseZTransform[(z(3 z +1)/(z^2+1)/(z-1))//ScaleChange[c,z],z,k]
(*4.2.3*)

InverseZTransformInitialData[(z^3-z)/((z^2 + 0.1 z + 0.95)^2 (z^2 -
0.3 z + 0.8)),z,20]

F = AdvancedZTransform[t^3,t,T,τ,z];
InverseAdvancedZTransform[F,z,f,k,T,τ]  (*7.3.1*)

F = AdvancedZTransform[t Sin[t],t,T,τ,z]; InitialBehavior[F,z,T,τ,3]
(*7.3.3*)

InverseZTransformInitialData[(z^3-z)/((z^2 + 0.1 z + 0.95)^2 (z^2 -
0.3 z + 0.8)),z,20]  (*4.2.5*)

F = AdvancedZTransform[t Cos[t],t,T,τ,z];
EvaluateToFunction[InverseAdvancedZTransform[F,z,g,k,T,τ],
Pi/2]//Short; Plot[%[t],{t,0,10}]; (*7.3.1*)
```

■ Difference Equations

```
DiscreteSystemSolve[y[k + 2] + 2 y[k + 1] + 5 y[k] == k^2,
  {y[0] → 0, y[1] → 1}, y[k], k]  (*4.3 .1 .1*)

sys = {a[k+1] == 3 a[k] + 4 c[k],b[k+1] == 3 b[k] + 4 d[k], c[k+1] ==
a[k] + 3 c[k],d[k+1] == b[k] + 3 d[k]};
iv = {a[0]->1, b[0]->0,c[0]->0,d[0]->1};
DiscreteSystemSolve[sys,iv,{a[k],b[k],c[k],d[k]},k] (*4.3.1.1*)

DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[k^2],{y[0]->1,y[1]-
>-5},T,y[k],k]  (*4.3.1.2*)

dop1 = {{2 T - 1, T - 2},{T + 3,T + 4}};
dop2 = {{T,1},{T^2,0}};
DiscreteDOpSolve[dop1[{x,y}] == dop2[{k^2, 2^k}],T,{x[k],y[k]},k]
(*4.3.1.2*)

DiscreteDOpSolve[dop1[{x,y}] == dop2[{k^2, 2^k}],T,{x[k],y[k]},k,GoFor-
>Image[{Y[2]},z]]  (*4.3.1.2*)

ToDiscreteDOpForm[c2 y[k+2] + c1 y[k+1] + c0 y[k] == f[k],T,y,k]
(*4.3.1.2*)

ToShiftedForm[(3 T^3 -b T^2 + a T + c)[y] == (2 T - 4)[k a^k],T,y,k]
(*4.3.1.2*)

deqn = y[t - 2 T] == 4 y[t - T] - 3 y[t] + t;
BackwardDifferenceEquationSolve[deqn,y[t],t,T,τ,k]  (*7.3.5.2*)

deqn = y[t + 2 T] == - 2 y[t + T] - y[t] == Sin[t];
ForwardDifferenceEquationSolve[deqn, {y[τ]->τ, y[τ+T]->T-τ},y[t],t,T,τ
,k]  (*7.3.5.1*)
```

■ Control and Systems

```
FrequencyResponse[s/(s^2 + 2 s + 5),s,a Cos[ω t + φ],t]  (*5.5.2.1*)

FrequencyResponsePlot[s/(s^2 + 0.2 s + 1),{s,0,3}]; (*5.5.2.1)

BodePlot[s/(s^2 + 0.2 s + 1),{s,0.5,3}]; (*5.5.2.3*)

PolarPlot[s/(s^2 + 0.2 s + 1),{s,0.5,3}]; (*5.5.2.4*)

StabilityQ[s^3+s^2+s+1,s]  (*5.4.2.1*)

StabilityQ[a + b s + 5 s^2 + c s^3,s]  (*5.4.2.1*)

HurwitzTest[a + b s + 5 s^2 + c s^3,s]  (*5.4.2.1*)

ZStabilityTest[Product[(s+k/10),{k,9}],s] (*5.4.2.2*)

JuryTest[z^3 - 3 z^2 + 2 z + 1 / 2, z]

ButterworthFilter[6,s]  (*5.5.3.2*)

ChebyshevFilter[6,0.3,s] (*5.5.3.2*)
```

Controller Configurations (*5.1*):

```
cc1 = StandardCC[-1, {1 / (s + 1)}, {(s - 1) / (s + 2)}, s]
```

```
cc2 = CascadeCC[-1, {s / (s + 2)}, {s / (s^2 + s + 1), 1 / s}, s]

cc3 = FeedbackCC[-1, {s / (s^2 + s + 1)}, {s / (s + 2)}, s]

cc4 = SeriesFeedbackCC[-1, {1 / s}, {s / (s^2 + s + 1)}, {(s + 1) / (s + 2)}, s]

cc5 = FeedForwardCC[-1, {1 / s}, {1 / (s - 2)}, {1 / (s^2 + s + 1)}, s]

cc6 = PerturbationCascadeCC[-1, {s + 2}, {1 / s}, {1 / (s^2 + 2 s + 17)}, s]

A0 = {{1, 1}, {0, -1}}; A1 = {{1, -1}, {0, -2}};
A2 = {{0, 1}, {2, -1}}; B0 = {{1}, {2}};
B1 = {{3}, {4}};
MdiffSys = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}];
cc7 = FullFeedbackCC[MdiffSys, {{{a, b}}, {{c, d}}}, s];

Am = {{0, 1}, {-2, -3}};
Bm = {{0}, {1}};
Cm = IdentityMatrix[2];
Dm = NullMatrix[2, 1];
Km = {{-k, -5}};
sys = StateControlSystem[Am, Bm, Cm, Dm];
cc8 = StateFeedbackCC[sys, Km, s]
```

ClosedLoopTransferFunction and Transfer Matrices (*5.1*):

```
ClosedLoopTransferFunction[cc1] (*5.1 .1 .1*)

ClosedLoopCharacteristicFunction[cc4] (*5.1 .1 .1*)

ClosedLoopCharacteristicFunction[cc7] (*5.1 .1 .1*)

ClosedLoopCharacteristicFunction[cc8] (*5.1 .1 .1*)

ClosedLoopTransferMatrix[cc8] (*5.2 .2 .3*)

Hm={{1},{-2}};
ClosedLoopTransferMatrix[cc7,Hm]

Em = {{1 / 10}, {0}}; Hm = {{1 / 5}, {1 / 4}};
ClosedLoopTransferMatrices[cc8, Em, Hm] (*5.2 .2 .3*)
```

Transfer Matrices (*5.2*):

```
Am = {{0, 1}, {-2, -3}};
Bm = {{0}, {1}};
Cm = IdentityMatrix[2];
Dm = NullMatrix[2, 1];
Em = {{1 / 10}, {0}}; Hm = {{1 / 5}, {1 / 4}};
sys = StateControlSystem[Am, Bm, Cm, Dm];
TransferMatrices[sys, Em, Hm, s] (*5.2 .2 .2*)

A2 = {{1,0},{0,-1}}; A1 = {{1,1},{1,-1}}; A0 = {{0,1},{1,0}}; B2 =
{{0},{1}}; B1 ={{0},{-1}}; B0 = {{1},{1}}; DSys =
MatrixDifferentialSystem[{A0,A1,A2},{B0,B1,B2}];
Hm={{1},{-2}};
TransferMatrices[DSys,Hm,s]

A0 = {{1, 1}, {0, -1}}; A1 = {{1, -1}, {0, -2}};
A2 = {{0, 1}, {2, -1}}; B0 = {{1}, {2}}; B1 = {{3}, {4}};
SYS = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}];
Hm = {{0}, {-3}};
TransferMatrices[SYS, Hm, s]  (*5.3 .1 .1*)
```

DynamicSystemResponse (*5.2*):

```
Am = {{1, 2}, {0, -1}}; Bm = {{1}, {4}};
Em = {{1}, {3}}; Cm = {{1, 0}}; Dm = {{1}}; Hm = {{3}};
StC = StateControlSystem[Am, Bm, Cm, Dm];
DynamicSystemResponse[StC, Em, Hm][
 {UnitStep[t]}, {DiracDelta[t]}, t, s]  (*5.2 .2 .1*)

A0 = {{1, 1}, {0, -1}}; A1 = {{1, -1}, {0, -2}}; A2 = {{0, 1}, {2, -1}};
B0 = {{1, 0}, {2, 0}}; B1 = {{3, 0}, {4, 1}}; Hm = {{a}, {3}};
MdiffSys = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}];
DynamicSystemResponse[MdiffSys, Hm,
 {IdentityMatrix[2], NullMatrix[2]}, {1, 0}, {0, 1}, {Sin[t]}, t, s]

DynamicSystemResponse[SYS, Hm, {K0m, K1m},
 {0, 0}, {0, 0}, {DiracDelta[t]}, t, s] (*5.3 .1 .2*)

Am = {{0, 1}, {-5, -2}};
Bm = {{0}, {1}};
Cm = IdentityMatrix[2];
Em = NullMatrix[2, 1];
SYS = ToDiscreteStateControlSystem[Am, Bm, Cm, Em, 0.5];
DiscreteDynamicSystemResponse[SYS[[1]], SYS[[2]], SYS[[3]]][
  {UnitStep[k]}, {0}, k, z] // Chop      (*5.6 .4 .2*)

A0 = {{1, 1}, {0, -1}}; A1 = {{1, -1}, {0, -2}};
A2 = {{0, 1}, {2, -1}}; B0 = {{1, 0}, {2, 0}};
B1 = {{0, 3}, {1, 4}}; Hm = {{1, 0}, {2, 0}};
K0 = {{1, 2}, {0, 3}}; K1 = {{1, 0}, {0, 1}};
MSys = MatrixDifferentialSystem[{A0, A1, A2}, {B0, B1}];
DynamicSystemResponse[MSys, Hm,
 {IdentityMatrix[2], NullMatrix[2]}, {1, 0}, {0, 1}, {Sin[t], 0}, t, s]

A = {{0, 1}, {-2, -3}};
StateTransitionMatrix[A, t]  (*5.2 .5 .1*)

EvansForm[1 + k s + (1 - k) s^2 + s^3, s, k]

Vieta[1 + k s + (1 - k) s^2 + s^3, s, σ]

Am = {{0, 1, 0}, {1, 0, 0}, {1, 1, -1}}; Cm = {{1, 0, 0}, {1, -1, 2}};
ObservabilityQ[Am, Cm]

Am = {{0, 1, 0}, {1, 0, 0}, {1, 1, -1}}; Bm = {{1}, {0}, {0}};
ControllabilityQ[Am, Bm]

Am = {{0, 1, 0}, {1, 0, 0}, {1, 1, -1}};
Bm = {{1}, {0}, {0}}; Cm = {{1, 2, 3}};
OutputControllabilityQ[Am, Bm, Cm]
```

■ Definitions (*3.1.3.1*)

```
DefEven[x]; DefOdd[x]; DefReal[x]; DefInteger[x];
DefPositive[x]; DefNegative[x]; DefNonNegative[x]

DefGreater[a, b]; DefLess[a, b]; DefLessEqual[a, b]
```

■ Named Rules

```
GenFac[5,3,a][k] /. ExpandGenFac  (*4.1.2*)
```

```
PfPair1[1+I,2,3+I,s] /. ExpandPfPair

PfPair2[1+I,2,3+I,z] /. ExpandPfPair

(a b Sin[b t] + a c Cos[2 b t] - 4 a) /. FactorOutOfSum[a]

Sqrt[(a + b I)^2 + (a - b I)^2] /. ExpandUnderSqrt

Exp[c (a + b I) + c (a - b I)] /. SimplifyExp

(Exp[(x-a) s]-Exp[-(x+a) s])/(s^2 (1+Exp[-2 a s])) /.
ToHyperbolicFunctions

Summation[(-1)^k Cos[Pi k t]/k, {k,Infinity}] /. TruncateSummation[10]

InverseLaplaceTransform[s Exp[-2 s + b] / (s^2 + a^2), s, t] /. ImageRules

f[x] DiracDelta[ x + b] /. DiracDeltaRules

(Exp[I t] (A + B I) - Exp[-I t] (A - B I)) / 2 / I /. EulerRule

ArcTan[-1,-3] /. ToPositiveAngle

a(Cos[φ] Cos[ω t] - Sin[φ] Sin[ω t]) /. AdditionTheorems

Sqrt[a + b I] /. CartesianSqrt

AsymptoticSeries[1 / Sqrt[t] / (1 + Sqrt[t])^2, {t, 0, 4}] /.
 RemoveLandauO (*6.2 .5 .1*)

A Cos[ω t] + B Sin[ω t] /. ToAmplitudePhaseForm

a (Cos[φ] Cos[ω t] - Sin[φ] Sin[ω t]) /. AdditionTheorems
```

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